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VARIOUS REPRESENTATIONS FOR THE SYSTEM OF NON-MODEL HYPERBOLIC EQUATIONS WITH REGULAR COEFFICIENTS

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Abstract. The authors have got new integral representations for the linear system of hyperbolic equations with regular coefficients in infinite regions. Furthermore, they used these integral representations to investigate new boundary value problems in infinite regions.

1. Introduction

Hyperbolic differential equations with singular coefficients or singular surfaces possess importance in diverse areas of mathematical physics and mathematical engineering, including elasticity, hydrodynamics, thermodynamics and other problems [3, 5, 4]. Furthermore, it is also well known that hyperbolic differential equations with one or more singular lines occur in engineering and physical processes. For example, the non-model hyperbolic equation of second order with two singular lines is employed to describe the transformation spectrum of electric signals on long lines with variable parameters in the theory of the electric flail [4, 7, 8, 6, 1, 2]. The problem of obtaining various solutions for equations of hyperbolic type with singular coefficients is described in a number of works (see the references). In [7], the equation

$$\frac{\partial^2 U}{\partial t^2}=\sum_{j=1}^{n}\left[a_j\frac{\partial^2 U}{\partial x_j^2}+\frac{\eta_j}{X_j}\frac{\partial U}{\partial x_j}\right]+g(x,t),$$

where a positive constant, η_j positive constant, possesses various solutions which may be represented in terms of solutions to the regular equation (see [7]). Depending on the obtained results, a solution of the Cauchy problem follows when boundary conditions are specified on the initial surface $t=0$. In [4] for the equation

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$$\frac{\partial^2 U}{\partial t^2}+\frac{\eta}{t}\frac{\partial U}{\partial t}-\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)=f(x,y,t),$$

where $\eta>0$ in region $x>0, t>0, -\infty<y<\infty$ the problem is solved subject to

$$U(x,y)=0, \left.\frac{\partial U(x,y,t)}{\partial t}\right|_{t=0}=0, U(0,y,t)=g(y,t),$$

x, y, t are arbitrary variables.

2. Main Results

Let D be the following infinite regions:

$$D=\{(x,y):-\infty<x<\infty,-\infty<y<\infty\},$$

with the boundary:

$$\begin{aligned}\Gamma_1^+ =&\{(x,y): y=0,-\infty\leq x\leq\infty\},\\ \Gamma_2^+ =&\{(x,y): x=0,-\infty\leq y\leq\infty\}.\end{aligned}$$

In D we consider the following system:

$$\frac{\partial^2 U_s}{\partial x\partial y}+\sum_{j=1}^{n}\{a_{js}(x,y)\frac{\partial U_j}{\partial x}+b_{js}(x,y)\frac{\partial U_j}{\partial y}+c_j(x,y)U_j\}=f_s(x,y), \tag{1}$$

$(1\leq s\leq n)$, where $a(x,y), b(x,y), c(x,y), f_s(x,y)$ are given continuous functions.

Case 1.

Theorem 1. *Let the coefficients in system (1) satisfy: $a_{ss}(x,y)\in C_x^1(D)$, $b_{ss}(x,y), c_s(x,y)\in C(D)$, $1\leq s\leq n, a_{js}(x,y)\in C_x^1(D)$, $b_{js}(x,y)\in C_y^1(D)$ at $j\neq s, j,s=1,2,\ldots,n$. Then any solution of system (1) within the class $C^2(D)\cap C(\overline{D})$ is:*

$$\begin{aligned}U_s(x,y)-&\int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau\int_x^\infty c_1^s(t,\tau)U_s(t,\tau)e^{w_1^s(t,\tau)-w_1^s(x,\tau)}dt\\ &+\int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau\int_x^\infty e^{w_1^s(t,\tau)-w_1^s(x,\tau)}\times\\ &\times[-\sum_{\substack{j=1\\ j\neq s}}^{n}(a_{js}(t,\tau)\frac{\partial U_j}{\partial t}+b_{js}(t,\tau)\frac{\partial U_j}{\partial \tau}+c_j(t,\tau)U_j(t,\tau))]dt\\ =&g_s(x,y), 1\leq s\leq n,\end{aligned}\tag{2}$$

where $g_s(x,y)$ is the Volterra system integral equation of the second type in the form:

$$
\begin{aligned}
g_s(x,y) =& \Psi_s(x)e^{-w_2^s(x,y)} + \int_y^{\infty} e^{w_2^s(x,\tau)-w_2^s(x,y)-w_1^s(x,\tau)}\Phi_s(\tau)d\tau \\
&+\int_y^{\infty} e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau \int_x^{\infty} e^{w_1^s(t,\tau)-w_1^s(x,\tau)} f_s(t,\tau)dt,
\end{aligned}
\tag{3}
$$

$$
w_1^s(x,y) = \int_x^{\infty} b_{ss}(t,y)dt, \;\; w_2^s(x,y) = \int_y^{\infty} a_{ss}(x,\tau)d\tau, \tag{4}
$$

and $\Psi_s(x), \Phi_s(y)$ are given continuous functions on Γ_1, Γ_2.

Proof. Let the coefficients in system (1) satisfy: $a_{ss}(x,y) \in C_x^1(D)$, $b_{ss}(x,y)$, $c_s(x,y) \in C(D)$, $s = 1, 2, \ldots, n$, $a_{js}(x,y) \in C_x^1(D)$, and $b_{ss}(x,y) \in C_y^1(D)$ at $j \neq s$. Then the system (1) can be written in the form:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial x} + b_{ss}(x,y)\right]\left[\frac{\partial}{\partial y} + a_{ss}(x,y)\right] U_s(x,y) = f_s(x,y) + c_s^1(x,y)U_s(x,y) \\
&-\sum_{j=s+1}^{n}\left[a_{js}(x,y)\frac{\partial U_j(x,y)}{\partial x} + b_{js}(x,y)\frac{\partial U_j(x,y)}{\partial y} + c_j(x,y)U_j(x,y)\right] \\
&-\sum_{j=s+1}^{n}\left[a_{js}(x,y)\frac{\partial U_j(x,y)}{\partial x} + b_{js}(x,y)\frac{\partial U_j(x,y)}{\partial y} + c_j(x,y)U_j(x,y)\right] \\
\equiv& F_s(x,y), 1 \leq s \leq n,
\end{aligned}
\tag{5}
$$

$$
c_s^1(x,y) = -c_s(x,y) + \frac{\partial a_{ss}(x,y)}{\partial x} + a_{ss}(x,y)b_{ss}(x,y), \tag{6}
$$

then we can put

$$
\frac{\partial U_s(x,y)}{\partial y} + a_{ss}(x,y)U_s(x,y) = g_s(x,y). \tag{7}
$$

Substituting from equation (7) into equation (5), we get

$$
\frac{\partial g_s(x,y)}{\partial x} + b_{ss}(x,y)g_s(x,y) = F_s(x,y). \tag{8}
$$

Solving equation (8), (7) and substituting the obtained results into equation (5), we get the solutions of equations (2), (3) where

$$\int_x^\infty e^{w_1^s(t,\tau)-w_1^s(x,\tau)}\sum_{\substack{j\neq s\\ j=1}}^n a_{js}(t,\tau)\frac{\partial U_j}{\partial t}dt =$$
$$=\sum_{\substack{j=1\\ j\neq s}}^n[(a_{js}(x,\tau)U_j(x,\tau)-e^{-w_1^s(x,\tau)}a_{js}(0,\tau)U_j(0,\tau))$$
$$-\int_x^\infty(\frac{\partial}{\partial t}(e^{w_1^s(t,\tau)-w_1^s(x,\tau)}a_{js}(t,\tau)))U_j(t,\tau)dt],$$

$$\int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau\int_x^\infty e^{w_1^s(t,\tau)-w_1^s(x,\tau)}b_{js}(t,\tau)\frac{\partial U_j}{\partial \tau}dt =$$
$$=\int_x^s e^{w_1^s(t,y)-w_1^s(x,y)}b_{js}(t,y)U_j(t,y)dt$$
$$-e^{-w_2^s(x,y)}\int_x^\infty e^{w_1^s(t,0)-w_1^s(x,0)}b_{js}(t,0)U_j(t,0)dt$$
$$-\int_y^s d\tau\left[\frac{\partial}{\partial\tau}(e^{w_2^s(x,t)-w_2^s(x,y)}))e^{w_1^s(t,\tau)-w_1^s(x,\tau)}b_{js}(t,\tau)\right.$$
$$\left.+\frac{\partial}{\partial\tau}(e^{w_1^s(t,\tau)-w_1^s(x,\tau)}b_{js}(t,\tau)e^{w_2^s(x,\tau)-w_2^s(x,y)})\right]U_j(t,\tau)dt.$$

Then we can get

$$U_s(x,y)-\int_y^\infty d\tau\int_x^\infty\{e^{w_2^s(x,\tau)-w_2^s(x,y)+w_1^s(t,\tau)-w_1^s(x,\tau)}c_1^s(t,\tau)U_s(t,\tau)$$
$$+\sum_{\substack{j\neq s\\ j=1}}^n(c_j(t,\tau)U_j(t,\tau)e^{w_2^s(x,\tau)-w_2^s(x,y)+w_1^s(t,\tau)-w_1^s(x,\tau)}$$
$$-\frac{\partial}{\partial t}(e^{w_1^s(t,\tau)-w_1^{s"}(x,\tau)}a_{js}(t,\tau))e^{w_2^s(x,\tau)-w_2^s(x,y)U_j(t,\tau)}$$
$$-[\frac{\partial}{\partial t}(e^{w_2^s(x,\tau)-w_2^s(x,y)})e^{w_1^s(t,\tau)-w_1^s(x,\tau)}b_{js}(t,\tau)$$
$$+e^{w_2^s(x,\tau)-w_2^s(x,y)}\frac{\partial}{\partial t}(e^{w_1^s(t,\tau)-w_1^s(x,0)}b_{js}(t,\tau))]U_j(t,\tau)\}d\tau$$

$$-\int_{y}^{\infty} e^{w_2^s(x,\tau)-w_2^s(x,y)} \sum_{\substack{j\neq s\\ j=1}}^{n}(a_{js}(x,\tau)U_j(x,\tau)-e^{-w_1^s(x,\tau)}a_{js}(0,\tau)U_j(0,\tau))]d\tau$$

$$-\int_{x}^{\infty}\sum_{\substack{j\neq s\\ j=1}}^{n} e^{w_1^s(t,y)-w_1^s(x,y)}b_{ks}(t,y)U_j(t,y)-e^{-w_2^s(x,y)+w_1^s(t,0)-w_1^s(x,0)}b_{js}(t,0)U_j(t,0))dt$$

$$=g_s(x,y),\ 1\leq s\leq n.$$

We get

$$K_s(x,y;t,\tau)=e^{w_2^s(x,\tau)-w_2^s(x,y)+w_1^s(t,\tau)-w_1^s(x,\tau)}c_1^s(t,\tau), \tag{9}$$

$$\begin{aligned} K_{js}(x,y;&t,\tau)=e^{w_2^s(x,\tau)-w_2^s(x,y)+w_1^s(t,\tau)-w_1^s(x,\tau)}c_j(t,\tau)\\ &-\frac{\partial}{\partial t}(e^{w_1^s(t,\tau)-w_1^s(x,\tau)}a_{js}(t,\tau))e^{w_2^s(x,\tau)-w_2^s(x,y)}\\ &-\frac{\partial}{\partial \tau}(e^{w_2^s(x,\tau)-w_2^s(x,y)})e^{w_1^s(t,\tau)-w_1^s(x,\tau)}b_{js}(t,\tau)\\ &+e^{w_2^s(x,\tau)-w_2^s(x,y)}\frac{\partial}{\partial\tau}(b_{js}(t,\tau))\\ &=e^{w_2^s(x,\tau)-w_2^s(x,y)+w_1^s(t,\tau)-w_1^s(x,\tau)}\times\\ &\times\left[c_j(t,\tau)-\frac{\partial a_{js}(t,\tau)}{\partial t}-a_{js}(t,\tau)b_{js}(t,\tau)-a_{ss}(x,\tau)b_{js}(t,\tau)\right.\\ &\left.+\frac{\partial b_{js}(t,\tau)}{\partial t}+\left(\frac{\partial w_1^s(t,\tau)}{\partial\tau}-\frac{\partial w_1^s(x,\tau)}{\partial\tau}\right)b_{js}(t,\tau)\right], \end{aligned} \tag{10}$$

$$K_{js}^{(1)}(x,y;\tau)=e^{w_2^s(x,\tau)-w_2^s(x,y)}a_{js}(x,\tau), \tag{11}$$

$$K_{js}^{(2)}(x,y;\tau)=e^{w_2^s(x,\tau)-w_2^s(x,y)-w_1^s(x,\tau)}a_{js}(0,\tau), \tag{12}$$

$$K_{js}^{(3)}(x,y;\tau)=e^{w_1^s(t,y)-w_1^s(x,y)}b_{js}(t,y), \tag{13}$$

$$K_{js}^{(4)}(x,y;\tau)=e^{w_1^s(t,0)-w_2^s(x,y)-w_1^s(x,0)}b_{js}(t,0). \tag{14}$$

Then the problem takes the solution of Volterra system integral equation of the second type in the form:

(15)
$$\begin{aligned} g_s(x,y) =&U_s(x,y)\\ &-\int_y^\infty d\tau\int_x^\infty[K_s(x,y;t,\tau)U_s(t,\tau)+\sum_{\substack{j=1\\j\neq s}}^n(K_{js}(x,y;t,\tau)U_j(t,\tau)]dt\\ &-\int_y^\infty(\sum_{\substack{j=1\\j\neq s}}^n(K_{ks}^{(1)}(x,y;\tau)U_j(x,\tau))-K_{js}^{(2)}(x,y;\tau)U_j(0,\tau))d\tau-\\ &-\int_x^\infty[\sum_{\substack{j=1\\j\neq s}}^n(K_{js}^{(3)}(x,y;t)U_j(t,y)-K_{js}^{(4)}(x,y;t)U_j(t,0))]dt,\ 1\leq s\leq n \end{aligned}$$

The proof is complete. □

Remark 1. The solution $g_s(x,y)$ of equation (15) in the neighborhood of Γ_1^+ can be represented in the form:

$$U_s(0,y)-\int_y^\infty[\sum_{\substack{j=1\\j\neq s}}^n(K_{js}^{(1)}(0,y;\tau)U_s(0,\tau)-K_{js}^{(2)}(0,y;\tau)U_j(0,\tau))]d\tau=g_s(0,y),$$

(16) $$g_s(0,y)=\Psi_s(0)e^{-w_1^s(0,\tau)}+\int_y^\infty e^{w_2^s(0,\tau)-w_2^s(0,y)}\Phi_s(\tau)d\tau.$$

From equation (11), (12) we get

$$K_{js}^{(1)}(0,y;\tau)=K_{js}^{(2)}(0,y;\tau),$$

then

$$U_s(0,y)=g_s(0,y)$$

Similarly, the solution $g_s(x,y)$ of equation (15) in the neighborhood of Γ_2^+ can be represented in the form:

(17) $$U_s(x,0)-\sum_{\substack{j=1\\j\neq s}}^n\int_x^\infty[K_{js}^{(3)}(x,0;t)-K_{js}^{(4)}(x,0;t)]U_j(t,0)dt=g_s(x,0),$$

(18) $$g_s(x,0)=\Psi_s(x),\ 1\leq s\leq n.$$

Then we can put

$$\begin{aligned} K_{gs}^{(3)}(x,0;t) =&e^{w_1^s(t,0)-w_1^s(x,0)}b_{js}(x,0),\\ K_{gs}^{(4)}(x,0;t) =&e^{w_1^s(t,0)-w_1^s(x,0)}b_{js}(t,0)=K_{js}^{(3)}(x,0;t). \end{aligned}$$

We get

$$U_s(x,0)=g_s(x,0)=\Psi_s(x).$$

Problem P_1. Find a solution of system (1) within the class $C^2(D) \cap C(D \cup \Gamma_1 \cup \Gamma_2)$ under the boundary conditions:

$$
\begin{aligned}
U_s(0, y) =& a_s(y), y \in \Gamma_2, \\
U_s(x, 0) =& b_s(x), x \in \Gamma_1, 1 \leq s \leq n, \\
a_s(0) =& b_s(0).
\end{aligned} \tag{19}
$$

Solution of Problem P_1. Substitute the conditions $U_s(x,0), U_s(0,y)$ in the integral representations (17), (16) we can get

$$
\begin{aligned}
\Psi_s(x) =& b_s(x), \\
\int_y^\infty e^{w_2^s(0,\tau)-w_2^s(0,y)} \Phi_s(\tau) d\tau =& a_s(y) - b_s(0) e^{-w_2^s(0,y)}, \\
\int_y^\infty e^{w_2^s(0,\tau)} \Phi_s(\tau) d\tau =& e^{w_2^s(0,y)} - b_s(0),
\end{aligned}
$$

then

$$
\begin{aligned}
\Phi_s(y) =& e^{-w_2^s(0,y)} \frac{d}{dt} [e^{w_2^s(0,y)} a_s(y) - b_s(0)] \text{ or} \\
\Phi_s(y) =& e^{-w_2^s(0,y)} [e^{w_2^s(0,y)} a_{ss}(0,y)(a_s(y) - b_s(0)) + e^{w_2^s(0,y)} a_s^1(y)], \\
\Phi_s(y) =& a_{ss}(0,y)[a_s(y) - b_s(0)] + a_s(y).
\end{aligned} \tag{20}
$$

We get

$$
\Phi_s(0) = a_{ss}(0,0) a_s(0) + a_s^1(0), \ a_s(0) = b_s(0).
$$

Substituting the obtained functions $\Psi_s(x), \Phi_s(y)$ into the integral representation (15) and using the condition (19) of problem P_1 we get the solution of Volterra system integral equation of the second type in the form:

$$
\begin{aligned}
U_s(x,y) - \int_y^\infty d\tau \int_x^\infty [K_s(x,y;t,\tau) U_s(x,t) + \sum_{\substack{j=1 \\ j \neq s}}^n K_{js}(x,y;t,\tau) U_j(t,\tau)] dt \\
- \int_y^\infty \sum_{\substack{j=1 \\ j \neq s}}^n K_{js}^{(1)}(x,y;\tau) U_j(x,\tau) d\tau - \int_x^\infty \sum_{\substack{j=1 \\ j \neq s}}^n K_{js}^{(3)}(x,y;t) U_j(t,y) dt \\
= G_s(x,y), 1 \leq s \leq n,
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
G_s(x,y) =& g_s(x,y) \\
& - \sum_{\substack{j=1 \\ j \neq s}}^n \left(\int_y^\infty K_{js}^{(2)}(x,y;\tau) a_j(\tau) d\tau + \int_x^\infty b_j(t) K_{js}^{(4)}(x,y;t) dt \right).
\end{aligned} \tag{22}
$$

The system (21) is solvable and its uniqueness solution can be obtained by using the kernels (9), (10), (11).

The proof of the following Theorem is completed.

Theorem 2. *Let the coefficients of system (1) satisfy the conditions of Theorem 1, $a_{ss}(0,y) \in C^2(\Gamma_2), a_s(y) \in C^2(\Gamma_2)$ and $b_s(x) \in C^1(\Gamma_1)$. Then* Problem P_1 *has a unique solution in the form:*

$$\begin{aligned} U_s(x,y) =& g(x,y) + \int_y^\infty \int_x^\infty \Gamma_1(x,y;t,\tau)g(t,\tau)dtd\tau \\ &+ \int_y^\infty \Gamma_2(x,y;\tau)g(x,\tau)d\tau + \int_x^\infty \Gamma_3(x,y;t)g(t,y)dt \\ &- \int_y^\infty \Gamma_4(x,y;\tau)g(0,\tau)d\tau - \int_x^\infty \Gamma_5(x,y;\tau)g(t,0)dt \end{aligned} \tag{23}$$

where $g(x,y) = (g_1(x,y), g_2(x,y), g_3(x,y), \ldots, g_n(x,y))$ *and* $U(x,y) = (U_1(x,y), U_2(x,y), U_3(x,y), \ldots, U_n(x,y)), \Gamma_1, \Gamma_2$, *are the kernels of the formula (21).*

The formula (23) of the functions $G_s(x,y)$ can be obtained by using the inequalities (22), (3), (4), (20), (19).

Problem P_2. Find a solution of system (1) within the class $C^2(D) \cap C(D \cup \Gamma_1 \cup \Gamma_2)$ under the boundary conditions:

$$\begin{aligned} U_s(x,0) =& g_s(x), \\ \left.\frac{\partial U_s}{\partial y}\right|_{x=0} =& f_s(y) \end{aligned} \tag{24}$$

where $f_s(y), g_s(x)$ are continuous functions on Γ_2^+, Γ_1^+.

Solution of Problem P_2. From equation (2), we have

$$\begin{aligned} \left.\frac{\partial U_s}{\partial y}\right|_{x=0} =& \left.\frac{\partial g_s(x,y)}{\partial y}\right|_{x=0}, \\ \frac{\partial g_s(x,y)}{\partial y} =& -e^{-w_2^s(x,y)} a_{ss}(x,y) \left[\Psi_s(x) + \int_y^\infty e^{w_2^s(x,0) - w_1^s(x,\tau)} \Phi_s(\tau) d\tau \right. \\ &\left. + \int_y^\infty e^{w_2^s(x,\tau)} \int_x^\infty e^{w_1^s(t,\tau) - w_1^s(x,\tau)} f_s(t,\tau) dt \right] \\ &+ e^{-w_2^s(x,y)} \left[\Phi_s(y) e^{w_2^s(x,y) - w_1^s(x,y)} + e^{w_2^s(x,y)} \int_x^\infty e^{w_1^s(t,y) - w_1^s(x,y)} f_s(t,y) dt \right]. \end{aligned}$$

We get

$$\left.\frac{\partial g_s(x,y)}{\partial y}\right|_{x=0} = -e^{-w_2^s(0,y)} a_{ss}(0,y)[\Psi_s(0) + \int_y^\infty e^{w_2^s(0,\tau)} \Phi_s(\tau) d\tau] + \Phi_s(y) = f_s(y),$$

or

$$-a_{ss}(0,y)[\Psi_s(0) + \int_y^\infty e^{w_2^s(0,\tau)} \Phi_s(\tau) d\tau)] + e^{w_2^s(0,y)} + \Phi_s(y) = f_s(y) e^{w_2^s(0,y)}.$$

Also we get

$$\begin{aligned} U_s(x,0) =& \Phi_s(x) = g_s(x), \\ \Psi_s(x) =& g_s(x). \end{aligned}$$

We can get the function $\Phi_s(y)$ from the integral equation

$$
\begin{aligned}
e^{w_2^s(0,y)}\Phi_s(y) - a_{ss}(0,y)\int_y^\infty e^{w_2^s(0,\tau)}\Phi_s(\tau)d\tau) =& f_s(y)e^{w_2^s(0,y)} - a_{ss}(0,y)g_s(0),\\
e^{w_2^s(0,y)}\Phi_s(y) =& \Phi_s^1(y),\\
f_s(y)e^{w_2^s(0,y)} - a_{ss}(0,y)g_s(0) =& F_s(y), 1 \le s \le n,
\end{aligned}
$$

$$
\Phi_s^1(y) - a_{ss}(0,y)\int_y^\infty \Phi_s^1(\tau)d\tau = F_s(y), \tag{25}
$$

$$
\begin{aligned}
&\Phi_s^1(0) = F_s(0) = f_s(0) - a_{ss}(0,0)g_s(0), \text{ or}\\
&\Phi_s(0) = f_s(0) - a_{ss}(0,0)g_s(0).
\end{aligned}
$$

By solving equation (25), we can get

$$
\begin{aligned}
\Phi_s(y) =& f_s(y) - a_{ss}(0,y)g_s(0)e^{-w_2^s(0,y)}\\
&+a_{ss}(0,y)\int_y^\infty \frac{f_s(\tau)}{a_{ss}(0,\tau)} - e^{-w_2^s(0,\tau)}g_s(0)d\tau.
\end{aligned} \tag{26}
$$

Substituting the obtained functions $\Psi_s(x), \Phi_s(y)$ of equations (25), (26) into the integral representation (15) and then solving the obtained system, we get the solution of *Problem* P_2.

Theorem 3. *Let in system (1) the coefficients* $a(x,y), b(x,y), c(x,y), f(x,y)$ *satisfy the conditions of Theorem 1 and in Problem* P_2*:* $f_s(y) \in C^1(\Gamma_2), g_s(x) \in C^2(\Gamma_1)$. *Then problem* P_1 *has a unique solution which is given by the formulae (23), (3), (25), (26).*

The values of the kernels $\Gamma_1 - \Gamma_5$ can be obtained from equation (23) and then substituting into the integral representation (15).

Cases II.

Theorem 4. *Let the coefficients in system (1) satisfy:* $b_{ss}(x,y) \in C_y^1(D)$, $a_{ss}(x,y), c_s(x,y) \in C(D), s = 1,2,\dots,n$; $a_{js}(x,y) \in C_x^1(D)$, $b_{js}(x,y) \in C_y^1(D)$ *at* $j \ne s, j,s = 1,2,\dots,n$. *Then any solution of the system (1) within the class*

$C^2(D) \cap C(\overline{D})$ *is:*

$$
\begin{aligned}
&U_s(x,y) - \sum_{\substack{j=1\\ j\neq s}}^{n} \int_y^\infty d\tau \int_x^\infty K_{js}^{(1)}(x,y;t,\tau)U_j(t,\tau)dt \\
&+\sum_{\substack{j=1\\ j\neq s}}^{n} \int_x^\infty [K_{js}^{(2)}(x,y;t)U_j(t,y) - -K_{js}^{(3)}(x,y;t)U_j(t,0)]dt \\
&+\sum_{\substack{j=1\\ j\neq s}}^{n} \int_y^\infty [K_{js}^{(4)}(x,y;\tau)U_j(x,\tau) - K_{js}^{(5)}(x,y;\tau)U_j(0,\tau)]d\tau \\
&=g_s^{(1)}(x,y),
\end{aligned} \tag{27}
$$

where $g_s^{(1)}(x,y)$ *is the Volterra system integral equation of the second type, and the Kernels of equation (27) are:*

$$
\begin{aligned}
K_{js}^{(1)}(x,y;t,\tau) =& \frac{\partial}{\partial t}[e^{w_1^s(t,y)-w_1^s(x,y)+w_2^s(t,\tau)-w_2^s(t,y)}] \\
&+e^{w_1^s(t,y)-w_1^s(x,y)-w_2^s(t,y)}\frac{\partial}{\partial \tau}[e^{w_2^s(t,\tau)}b_{js}(t,\tau)] \\
&+e^{w_1^s(t,y)-w_1^s(x,y)+w_2^s(t,\tau)-w_2^s(t,y)}c_{js}(t,\tau), \\
K_{js}^{(2)}(x,y;t) =& e^{w_1^s(t,y)-w_1^s(x,y)}b_{js}(t,y), \\
K_{js}^{(3)}(x,y;t) =& e^{w_1^s(t,y)-w_1^s(x,y)}b_{js}(t,0), \\
K_{js}^{(4)}(x,y;\tau) =& e^{w_2^s(t,\tau)-w_2^s(x,y)}a_{js}(x,\tau), \\
K_{js}^{(5)}(x,y;\tau) =& e^{w_2^s(0,\tau)-w_2^s(0,y)-w_1^s(x,y)}a_{js}(0,\tau).
\end{aligned}
$$

Proof. Let the coefficients in system (1) satisfy: $b_{ss}(x,y) \in C_y^1(d)$, $a_{ss}(x,y)$, $c_s(x,y) \in CD, 1 \leq s \leq n, a_{js}(x,y) \in C_x^1(D)$, and $b_{js}(x,y) \in C_y^1(D)$ at $j \neq s$. Then the system (1) can be written in the form:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial y} + a_{ss}(x,y)\right]\left[\frac{\partial}{\partial x} + b_{ss}(x,y)\right]U_s(x,y) = \\
=&f_s(x,y) + c_2^s(x,y)U_s(x,y) \\
&-\sum_{j=1}^{s-1}\left[a_{js}(x,y)\frac{\partial U_j(x,y)}{\partial x} + b_{js}(x,y)\frac{\partial U_j(x,y)}{\partial y} + c_j(x,y)U_j(x,y)\right] \\
&-\sum_{j=s+1}^{n}\left[a_{js}(x,y)\frac{\partial U_j(x,y)}{\partial x} + b_{js}(x,y)\frac{\partial U_j(x,y)}{\partial y} + c_j(x,y)U_j(x,y)\right] \\
\equiv& F^{(1)}(x,y),
\end{aligned} \tag{28}
$$

$$c_2^s(x,y) = -(x,y)^{-1}c_s(x,y) + x^{-1}\frac{\partial b_{ss}(x,y)}{\partial y} + (x,y)^{-1}a_{ss}(x,y)b_{ss}(x,y) \tag{29}$$

Solving the system (28) we get

$$\begin{aligned} U_s(x,y) - &\int_x^\infty \{e^{w_1^s(t,y)-w_1^s(x,y)}dt \int_y^\infty c_2^s(t,\tau)e^{w_2^s(t,\tau)-w_2^s(t,y)}U_s(t,\tau)d\tau \\ &+ \int_x^\infty e^{w_1^s(t,y)-w_1^s(x,y)}dt \int_y^\infty e^{w_2^s(t,\tau)-w_2^s(t,\tau)} \times \\ &\times \left[\sum_{\substack{j=1 \\ j\neq s}}^n \left(a_{js}(t,\tau)\frac{\partial U_j}{\partial t} + b_{js}(t,\tau)\frac{\partial U_j}{\partial t} + c_j(t,\tau)U_j(t,\tau)\right)\right] d\tau \\ &= g_s^{(1)}(x,y), \end{aligned} \tag{30}$$

$$\begin{aligned} g_s^{(1)}(x,y) =& \Phi_s^1(y)e^{-w_1^s(x,y)} + \int_x^\infty e^{w_1^s(t,y)-w_1^s(x,y)-w_2^s(t,y)}\Psi_s^{(1)}(t)dt + \\ &+ \int_x^\infty e^{w_1^s(t,y)-w_1^s(x,y)}dt \int_y^\infty e^{w_2^s(t,\tau)-w_2^s(t,y)} f_s(t,\tau)d\tau, 1 \le s \le n \end{aligned} \tag{31}$$

Also, we have

$$\begin{aligned} &\int_y^\infty d\tau \left[\int_x^\infty (e^{w_1^s(t,y)-w_1^s(x,y)+w_2^s(t,\tau)-w_2^s(t,y)}) \sum_{\substack{j=1 \\ j\neq s}}^n a_{js}(t,\tau)\frac{\partial U_j}{\partial t}dt\right] = \\ &= \int_y^\infty \{\sum_{j=s}^n e^{-w_1^s(x,y)}[(e^{w_1^s(x,y)+w_2^s(x,\tau)-w_2^s(x,y)}a_{js}(x,\tau)U_j(t,\tau) \\ &-e^{w_2^s(0,\tau)-w_2^s(0,y)}a_{js}(0,\tau)U_j(0,\tau)) \\ &- \int_x^\infty \frac{\partial}{\partial t}(e^{w_1^s(t,y)+w_2^s(t,\tau)-w_2^s(t,y)}a_{js}(t,\tau))U_j(t,\tau)dt]\}d\tau, \end{aligned}$$

and

$$\int_y^\infty e^{w_2^s(t,\tau)}(\sum_{\substack{j=1 \\ j\neq s}}^n b_{js}(t,\tau)\frac{\partial U_j}{\partial \tau})d\tau =$$

$$= \sum_{\substack{j=1 \\ j\neq s}}^n [e^{w_2^s(t,y)}b_{js}(t,y)U_j(t,y) - b_{js}(t,0)U_j(t,0) - \int_y^\infty U_j(t,\tau)\frac{\partial}{\partial \tau}[e^{w_2^s(t,\tau)b_{js}(t,\tau)d\tau}].$$

Substituting the obtained integral values into equation (30), we get the solution of the integral system in the form (27). □

Remark 2. From equation (27) we get the following integral equations

$$U_s(x,0)+\sum_{\substack{j=1\\ j\neq s}}^{\infty}\int_x^{\infty} K_{js}^{(2)}(x,0;t)U_j(t,0)-K_{js}^{(3)}(x,0,t)U_j(t,0)dt = g_s^{(1)}(x,0), \tag{32}$$

$$U_s(0,y)+\sum_{\substack{j=1\\ j\neq s}}^{\infty}\int_y^{\infty} K_{js}^{(4)}(0,y;\tau)-K_{js}^{(5)}(0,y;\tau)U_j(0,\tau)d\tau = g_s^{(1)}(0,y), \tag{33}$$

where

$$\begin{aligned}
K_{js}^{(2)}(x,0;t) =& e^{w_1^s(t,0)-w_1^s(x,0)}b_{js}(t,0) = K_{js}^{(3)}(x,0,t),\\
K_{js}^{(4)}(0,y;\tau) =& e^{w_2^s(0,\tau)-w_2^s(0,y)}a_{js}(0,\tau) = K_{js}^{(5)}(0,y;\tau),\\
U_s(x,0) =& g_s^{(1)}(x,0),\\
U_s(0,y) =& g_s^{(1)}(0,y).
\end{aligned}$$

From equation (31), we get

$$\begin{aligned}
g_s^{(1)}(0,y) =& \Phi_s^1(y),\\
g_s^{(1)}(x,0) =& \Phi_s^1(0)e^{-w_1^s(x,0)} + \int_x^{\infty} e^{w_1^s(t,0)-w_1^s(x,0)}\Psi_s^{(1)}(t)dt.
\end{aligned}$$

Theorem 5. *Let the coefficients of system (1) satisfy the conditions of Theorem 1. Then any solution of system (1) within the class $C^2(D)\cap C(\overline{D})$ contains $2n$ arbitrary functions of one variable (n-functions of the variable x and n-functions of the variable y) can be represented in the form:*

$$\begin{aligned}
U(x,y) =& g(x,y) + \int_y^{\infty}\int_x^{\infty}\Gamma_1^{(1)}(x,y;t,\tau)g(t,\tau)dtd\tau\\
&+\int_x^{\infty}\Gamma_2^{(1)}(x,y;t)g(t,y)dt + \int_x^{\infty}\Gamma_3^{(1)}(x,y;\tau)g(t,\tau)d\tau\\
&+\int_x^{\infty}\Gamma_4^{(1)}(x,y;t)g(t,0)dt + \int_y^{\infty}\Gamma_5^{(1)}(x,y;\tau)g(0,\tau)d\tau,
\end{aligned} \tag{34}$$

where $\Phi_1(y),\Phi_2(y),\dots,\Phi_n(y);\Psi_1(x),\Psi_2(x),\dots,\Psi_n(x)$ are arbitrary continuous functions of the variables x and y, $\Gamma_1^{(1)}-\Gamma_5^{(1)}$ are the kernels of system (15), $g=g(\Phi_1,\Phi_2,\dots,\Phi_n;\Psi_1,\Psi_2,\dots,\Psi_n)=(g_1,g_2,\dots,g_n), U=(U_1,U_2,\dots,U_n)$.

Moreover $\Phi_s(y)\in C^1(\Gamma_2), \Psi_n(x)\in C^2(\Gamma_1)$.

Theorem 6. *Let the coefficients of system (1) satisfy the conditions of Theorem 4. Then any solution of system (1) within the class $C^2(D)\cap C(\overline{D})$ contains*

2n arbitrary functions of one variable can be represented in the form:

$$
\begin{aligned}
U(x,y) =& g^{(1)}(x,y) + \int_x^\infty dt \int_y^\infty \Gamma_1^{(2)}(x,y;t,\tau) g^{(1)}(t,\tau) dt d\tau \\
& + \int_x^\infty \Gamma_2^{(2)}(x,y;t) g^{(1)}(t,y) dt + \int_y^\infty \Gamma_3^{(2)}(x,y;\tau) g^{(1)}(x,\tau) d\tau \\
& + \int_x^\infty \Gamma_2^{(2)}(x,y;t) g^{(1)}(t,0) dt + \int_y^\infty \Gamma_5^{(2)}(x,y;\tau) g^{(1)}(0,\tau) d\tau,
\end{aligned} \tag{35}
$$

where $\Phi_s^{(1)}(y), \Psi_s^{(1)}(x), 1 \leq s \leq n$ are arbitrary continuous functions of the Variables x and y, $\Gamma_1^{(1)} - \Gamma_5^{(1)}$ are the Kernels of system (27), $g^1 = g(\Phi_1^{(1)}, \Phi_2^{(1)}, \ldots, \Phi_n^{(1)}; \Psi_1^{(1)}, \Psi_2^{(1)}, \ldots, \Psi_n^{(1)}) = (g_1^{(1)}, g_2^{(1)}, \ldots, g_n^{(1)}), U = (U_1, U_2, \ldots, U_n)$.

Moreover, $\Phi_s^{(1)}(y) \in C^2(\Gamma_2), \Psi_s^{(1)}(x) \in C^1(\Gamma_1)$.

Remark 3. The obtained integral representations can be used to solve various boundary value problems.

For solving Problem P_1, we use the representation of the form (35), and the functions $\Phi_s^{(1)}(y), \Psi_s^{(1)}(x)$ can be obtained from equation (33). From (33) we get

$$
\begin{aligned}
U_s(x,0) =& g_s^{(1)}(x,0), \\
U_s(0,y) =& g_s^{(1)}(0,y).
\end{aligned}
$$

Then

$$
\begin{aligned}
g_s^{(1)}(x,0) =& \Phi_s^1(0) e^{-w_1^s(x,0)} + \int_x^\infty e^{w_1^s(t,0) - w_1^s(x,0)} \Psi_s^{(1)} dt, \\
g_s^{(1)}(0,y) =& \Phi_s^1(y),
\end{aligned} \tag{36}
$$

$$
\Phi_s^1(y) = a_s(y) \tag{37}
$$

$$
a_s(0) e^{-w_1^s(x,0)} + \int_x^\infty e^{w_1^s(t,0) - w_1^s(x,0)} \Psi_s^{(1)}(t) dt = b_s(x).
$$

We get

$$
\int_x^\infty e^{w_1^s(t,0)} \Psi_s^{(1)}(t) dt = b_s(x) e^{w_1^s(x,0)} + a_s(0).
$$

Solving the above equation, we get

$$
\Psi_s^{(1)}(t) = [b_s^1(x) + b_{ss}(x,0) b_s(x)]. \tag{38}
$$

Then the proof of the following Theorem is complete.

Theorem 7. *Let the coefficients of system (1) satisfy the conditions of Theorem 4. The functions $a_s(y), b_s(y)$ satisfy the conditions of Theorem 3, $b_{ss}(x,0) \in C^1(\Gamma_1)$. Then Problem P_1 has a unique solution which is given by the formulas (35), (31), (36), (37).*

Problem P_3. Find a solution of system (1) within the class $C^2(D)\cap C(D\cup \Gamma_1\cup\Gamma_2)$ under the boundary conditions:

$$U_s(0,y) = f_s^{(1)}(y),$$
$$\left.\frac{\partial U_s}{\partial x}\right|_{x=0} = g_s^{(1)}(x),$$

where $f_s^{(1)}(y), g_s(1)(x)$ are given continuous functions on Γ_2, Γ_1.

Solution of Problem P_3. From equation (30), we have

$$U_s(0,y) = \Phi_s^{(1)}(y),$$
$$\left.\frac{\partial U_s}{\partial x}\right|_{y=0} = \left.\frac{\partial g_s^{(1)}(x,y)}{\partial x}\right|_{y=0},$$

but

$$\frac{\partial g_s^{(1)}(x,y)}{\partial x}|_{y=0} = - b_{ss}(x,0)e^{-w_1^s(x,0)}[\Phi_s^{(1)}(0) + \int_x^\infty e^{w_1^s(t,0)}\Psi_s^{(1)}(t)dt] + \Psi_s^{(1)}(x)$$
$$= g_s^{(1)}(x).$$

Then

$$\Phi_s^{(1)}(y) = f_s^{(1)}(y), \tag{39}$$
$$e^{w_1^s(x,0)}\Psi_s^{(1)}(x) - b_{ss}(x,0)\int_x^\infty e^{w_1^s(t,0)}\Psi_s^{(1)}(t)dt = g_s^{(1)}(x)e^{w_1^s(x,0)} + f_s^{(1)}(0)b_{ss}(x,0).$$

Solving this integral equation we get

$$\begin{aligned}\Psi_s^{(1)}(x) =& g_s^{(1)}(x) + f_s^{(1)}(0)b_{ss}(x,0)e^{-w_1^s(x,0)}\\ &+ b_{ss}(x,0)e^{-w_1^s(x,0)}\int_x^\infty (g_s^{(1)}(t) + f_s^{(1)}(0)b_{ss}(t,0)e^{-w_1^s(x,0)})dt.\end{aligned} \tag{40}$$

Theorem 8. *Let in the system (10) the functions $a_{js}(x,y)$, $b_{js}(x,y)$, $c_j(x,y)$, $f_s(x,y)$ satisfy the conditions of Theorem 4 and in Problem P_3: $f_s^{(1)}(x,y)\in C^2(\Gamma_2)$, $g_s^{(1)}\in C^1(\Gamma_1)$. Then problem P_3 has a unique solution which can be obtained by using the formulas (35), (31), (38), (39).*

Remark 4. Let in system (1)

$$a_{js}(x,y) = \frac{a_{js}^0(x,y)}{|x-y|^{\alpha_{js}}}, b_{js}(x,y) = \frac{b_{js}^0(x,y)}{|x-y|^{\beta_{js}}},$$
$$c_j(x,y) = \frac{c_j^0(x,y)}{|x-y|^{\gamma_j}}, f_s(x,y) = \frac{f_s^0(x,y)}{|x-y|^{\delta_j}},$$

where $\alpha_{js}<1, \beta_{js}<1, \gamma_j<1, \delta_j<1, 1\le s,j\le n$.

From equation (4), we get the continuous functions $w_1^s(x,y), w_2^s(x,y)$. The continuous functions $\frac{\partial w_1^s(t,\tau)}{\partial\tau}, \frac{\partial w_2^s(x,\tau)}{\partial x}$ satisfy the condition $b_{ss}(x,y) = 0$ ($|x-$

$y|^{\alpha_s})$, $\alpha_s > \beta_{ss}$ when $x \to y$. Then the integral equation (15) is Volterra integral equation of the second type.

Theorem 9. *Let in the system*

$$\frac{\partial^2 U_s}{\partial x \partial y} + \sum_{j=1}^{n} \left\{ \frac{a_{js}^0(x,y)}{|x-y|^{\alpha_{js}}} \frac{\partial U_j}{\partial x} + \frac{b_{js}^0(x,y)}{|x-y|^{\beta_{js}}} \frac{\partial U_j}{\partial y} + \frac{c_{js}^0(x,y)}{|x-y|^{\gamma_j}} U_j \right\} = \frac{f_s^0(x,y)}{|x-y|^{\delta_j}},$$

where $\alpha_{js} < 1, \beta_{js} < 1, \gamma_j < 1, \delta_j < 1, 1 \le s \le n, 1 \le s \le n$, *the coefficients satisfy:*

(1) $b_{js}(x,y)$ *of the variable* y *have continuous derivative of the first order, continuous of the variable* x.
(2) $a_{js}(x,y)$ *at* $j \ne s$ *have continuous derivative of the variable* y, *continuous of the variable* x.
(3) $c_s^0(x,y), f_s^0(x,y)$ *are continuous functions in* $\bar{D}$.

Then any solution of system (1) within the class $C^2(D)$ *can be written in the form (34), where*

$$a_{js}(x,y) = \frac{a_{js}^0(x,y)}{|x-y|^{\alpha_{js}}}, b_{js}(x,y) = \frac{b_{js}^0(x,y)}{|x-y|^{\beta_{js}}},$$
$$c_j(x,y) = \frac{c_j^0(x,y)}{|x-y|^{\gamma_j}}, f_s(x,y) = \frac{f_s^0(x,y)}{|x-y|^{\delta_s}}.$$

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