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RULED SURFACES IN LORENTZ-MINKOWSKI 3-SPACES WITH NON-DEGENERATE SECOND FUNDAMENTAL FORM

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Abstract. In this paper, we study some properties of ruled surfaces with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space related to its the Gaussian curvature, the second Gaussian curvature and the mean curvature.

1. Introduction

The inner geometry of the second fundamental form has been a popular research topic for ages. It is readily seen that the second fundamental form of a surface is non-degenerate if and only if a surface is non-developable.

On a non-developable surface M, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a new Riemannian metric. The curvature K_{II} will be called the second Gaussian curvature of the surface M. The curvature K_{II} of M will be defined and discussed in the section 2.

Several authors studied the second Gaussian curvature (see [2], [4], [10], [11], [12], [14], [15]). D. Koutroufiotis ([12]) has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$, where K is the Gaussian curvature. Th. Koufogiorgos and T. Hasanis ([11]) proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, W. Kühnel ([13]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. In [2] C. Baikoussis and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, D.E. Blair and

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Th. Koufogiorgos ([4]) investigated a non-developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ such that $aK_{II} + bH, 2a + b \neq 0$, is constant along each ruling.

Recently, Y.H. Kim and the present author ([10]) studied a non-developable ruled surface in a 3-dimensional Lorentz-Minkowski space satisfying the conditions

$$aH + bK = \text{constant}, \quad a \neq 0, \tag{1.1}$$

$$aK_{II} + bK = \text{constant}, \quad a \neq 0, \tag{1.2}$$

$$aK_{II} + bH = \text{constant}, \quad 2a - b \neq 0, \tag{1.3}$$

along each ruling. In particular, if it satisfies the condition (1.1), then a surface is called a linear Weingarten surface (cf. [7]). In [15] W. Sodsiri studied a non-developable ruled surface with non-null rulings in a 3-dimensional Lorentz-Minkowski space such that the linear combination $aK_{II} + bH + cK, a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0$ is constant along ruling.

In this article, we investigate a ruled surface with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space satisfying the condition

$$K_{II} = K^m H^n, \tag{1.4}$$

along each ruling, where m, n are natural numbers.

2. Preliminaries

Let $\mathbb{L}^3$ be a 3-dimensional Lorentz-Minkowski space with the scalar product of index 1 given by $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{L}^3$. A vector x of $\mathbb{L}^3$ is said to be space-like if $\langle x, x \rangle > 0$ or $x = 0$, time-like if $\langle x, x \rangle < 0$ and light-like or null if $\langle x, x \rangle = 0$ and $x \neq 0$. A time-like or null vector in $\mathbb{L}^3$ is said to be causal. A curve in $\mathbb{L}^3$ is called space-like, time-like or null if its tangent vector field is space-like, time-like or null, respectively.

We denote a surface M in $\mathbb{L}^3$ by

$$x(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)).$$

Then the first fundamental form I of the surface M is defined by

$$I = Eds^2 + 2Fdsdt + Gdt^2,$$

where $E = < x_s, x_s >, F = < x_s, x_t >, G = < x_t, x_t >, x_s = \frac{\partial x(s,t)}{\partial s}$. We define the second fundamental form II of M by

$$
\begin{aligned}
II &= e ds^2 + 2f ds dt + g dt^2, \\
e &= \frac{1}{\sqrt{|EG - F^2|}} \det(x_s\ x_t\ x_{ss}), \\
f &= \frac{1}{\sqrt{|EG - F^2|}} \det(x_s\ x_t\ x_{st}), \\
g &= \frac{1}{\sqrt{|EG - F^2|}} \det(x_s\ x_t\ x_{tt}).
\end{aligned}
$$

Quite similarly to the case of Gaussian curvature of a surface in Euclidean space (see, [16, p. 112]), the Gaussian curvature K of M in $\mathbb{L}^3$ is defined by

$$
\begin{aligned}
K = & \frac{1}{(EG-F^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s & F_s - \frac{1}{2}E_t \\ F_t - \frac{1}{2}G_s & E & F \\ \frac{1}{2}G_t & F & G \end{vmatrix} \right. \\
& \left. - \begin{vmatrix} 0 & \frac{1}{2}E_t & \frac{1}{2}G_s \\ \frac{1}{2}E_t & E & F \\ \frac{1}{2}G_s & F & G \end{vmatrix} \right\}.
\end{aligned} \tag{2.1}
$$

At this stage we are able to compute the second Gaussian curvature K_{II} of a surface with non-degenerate second fundamental form in $\mathbb{L}^3$ by replacing E, F, G by the components of the second fundamental form e, f, g respectively in (2.1). Thus, the second Gaussian curvature K_{II} is given by (cf. [2])

$$
\begin{aligned}
K_{II} = & \frac{1}{(eg-f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} \right. \\
& \left. - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}.
\end{aligned} \tag{2.2}
$$

It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

Now, we define a ruled surface M in $\mathbb{L}^3$. Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve of J into $\mathbb{L}^3$ and $\beta = \beta(s)$ a vector field along α. Then, a ruled surface M is defined by the parametrization given as follows:

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

For such a ruled surface, α and β are called the base curve and the director vector field, respectively.

According to the causal character of α' and β, there are four possibilities:

(1) α' and β are non-null and linearly independent.
(2) α' is null and β is non-null with $< \alpha', \beta > \neq 0$.
(3) α' is non-null and β is null with $< \alpha', \beta > \neq 0$.
(4) α' and β are null with $< \alpha', \beta > \neq 0$.

It is easily to see that, with an appropriate change of the curve α, cases (2) and (3) reduce to (1) and (4), respectively (For the details, see [1]).

First of all, we consider the ruled surface of the case (1). In this case, the ruled surface M is said to be cylindrical if the director vector field β is constant and non-cylindrical otherwise.

Let the base curve α and the director vector field β be non-null. Then, the base curve α can be chosen to be orthogonal to the director vector field β and β can be normalized satisfying $\langle \beta(s), \beta(s) \rangle = \varepsilon(= \pm 1)$ for all $s \in J$. In this case, according to the character of vector fields α' and β, we have ruled surfaces of five different kinds as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. If the vector field β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When the vector field β is time-like, β' is space-like because of the causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, the director vector field is always space-like. According as its derivative β' is non-null or null, it is also said to be of type M_-^1 or M_-^2, respectively (cf. [9]).

The ruled surface M of the case (4) is called a null scroll (see [8]). One of typical examples of null scrolls is B-scroll which is defined as follows:

Let $\alpha(s)$ be a null curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields along α in $\mathbb{L}^3$ satisfying the following conditions:

$$< A, A > = < B, B > = 0, \qquad < A, B > = -1,$$
$$< A, C > = < B, C > = 0, \qquad < C, C > = 1,$$

and

$$\alpha' = A,$$
$$C' = -aA - k(s)B,$$

where a is a constant and $k(s)$ a function vanishing nowhere.

Then the map

$$x : M \longrightarrow \mathbb{L}^3$$
$$(s, t) \rightarrow \alpha + tB(s)$$

defines a Lorentz surface M in $\mathbb{L}^3$ that L. K. Graves ([8]) called a B-scroll.

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion ([5]). Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map ([6]). In this regards, Y.H. Kim and the present author defined pointwise finite type Gauss map ([9]). In particular, the Gauss map G on a submanifold M of a pseudo-Euclidean space

$\mathbb{E}^m_s$ of index s is said to be of pointwise 1-type if $\Delta G = fG$ for some smooth function f on M where Δ denotes the Laplace operator defined on M. They showed that minimal non-cylindrical ruled surfaces in a 3-dimensional Lorentz-Minkowski space have pointwise 1-type Gauss map ([9]). Based on this fact, they proved the following theorem:

Theorem 2.1 ([9])**.** *Let M be a non-cylindrical ruled surface with space-like or time-like base curve in a 3-dimensional Lorentz-Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following spaces: the space-like or time-like helicoid of the 1st, the 2nd and the 3rd kind, the space-like or time-like conjugate of Enneper's surface of the 2nd kind.*

This theorem will be useful to prove our theorems in this paper.

3. Main Theorems

In this section we study a ruled surface with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ satisfying the condition (1.4). It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature K is identically zero. Therefore, the second fundamental form is degenerate. Thus, non-cylindrical ruled surfaces are meaningful for our study.

Theorem 3.1. *Let m, n be natural numbers. A non-cylindrical ruled surface with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space satisfying the condition $K_{II} = K^m H^n$ along each ruling is an open part of one of the following surfaces:*

1. *the helicoid of the 1st kind as space-like or time-like surface,*
2. *the helicoid of the 2nd kind as space-like or time-like surface,*
3. *the helicoid of the 3rd kind as space-like or time-like surface,*
4. *the conjugate of Enneper's surfaces of the 2nd kind as space-like or time-like surface.*

Proof. We consider two cases separately.

Case 1. Let M be a non-cylindrical ruled surface of the three types M^1_+, M^3_+ or M^1_-. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = \varepsilon_1(=\pm 1), \langle \beta', \beta' \rangle = \varepsilon_2(=\pm 1)$ and $\langle \alpha', \beta' \rangle = 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the (pseudo-)spherical curve β. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the first fundamental form of the surface is given by $E = \langle \alpha', \alpha' \rangle + \varepsilon_2 t^2, F = \langle \alpha', \beta \rangle$ and $G = \varepsilon_1$. For later use, we define the smooth functions Q, J and D as follows :

$$Q = \langle \alpha', \beta \times \beta' \rangle \neq 0, \quad J = \langle \beta'', \beta' \times \beta \rangle, \quad D = \sqrt{|EG - F^2|}.$$

In terms of the orthonormal basis $\{\beta, \beta', \beta\times\beta'\}$ we obtain

$$\alpha' = \varepsilon_1 F\beta - \varepsilon_1\varepsilon_2 Q\beta\times\beta', \quad \beta'' = \varepsilon_1\varepsilon_2(-\beta + J\beta\times\beta'), \quad \alpha'\times\beta = \varepsilon_2 Q\beta'.$$

On the other hand, one obtains $EG - F^2 = -\varepsilon_2 Q^2 + \varepsilon_1\varepsilon_2 t^2$. And, the unit normal vector N is written as $N = \frac{1}{D}(\varepsilon_2 Q\beta' - t\beta\times\beta')$. Then, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(\varepsilon_1 Q(F - QJ) - Q't + Jt^2), \quad f = \frac{Q}{D} \neq 0, \quad g = 0.$$

Therefore, using the data described above and (2.2), we obtain

$$\begin{aligned} K_{II} &= \frac{1}{f^4}\left(ff_t(f_s - \frac{1}{2}e_t) - f^2(-\frac{1}{2}e_{tt} + f_{st})\right) \\ &= \frac{1}{2Q^2D^3}\left(Jt^4 + \varepsilon_1 Q(F - 2QJ)t^2 + 2\varepsilon_1 Q^2Q't + Q^3(F + QJ)\right). \end{aligned} \tag{3.1}$$

Furthermore, the mean curvature H and the Gaussian curvature K are given respectively by

$$\begin{aligned} H =& \frac{1}{2}\frac{Eg - 2Ff + Ge}{|EG - F^2|} \\ =& \frac{1}{2D^3}\left(\varepsilon_1 Jt^2 - \varepsilon_1 Q't - Q(F + QJ)\right), \end{aligned} \tag{3.2}$$

and

$$K = \langle N, N\rangle \frac{eg - f^2}{EG - F^2} = \frac{Q^2}{D^4}. \tag{3.3}$$

Suppose that the ruled surface M satisfies the equation $K_{II} = K^mH^n$ for some natural numbers m and n. Then we have by using (3.1), (3.2) and (3.3)

$$\begin{aligned} &2^{2n-2}D^{8m+6n-6}(Jt^4 + \varepsilon_1 Q(F - 2QJ)t^2 + 2\varepsilon_1 Q^2Q't + Q^3(F + QJ))^2 \\ &\quad = Q^{4m+4}(\varepsilon_1 Jt^2 - \varepsilon_1 Q't - Q(F + QJ))^{2n}. \end{aligned} \tag{3.4}$$

Thus, the coefficient of the highest order $t^{8m+6n+2}$ of the equation (3.4) is

$$(-1)^{4m+3n-3}2^{2n-2}J^2 = 0,$$

which implies $J = 0$. So, we can rewrite (3.4) in the form

$$\begin{aligned} &2^{2n-2}D^{8m+6n-6}(\varepsilon_1 QFt^2 + 2\varepsilon_1 Q^2Q't + Q^3F)^2 \\ &\quad = Q^{4m+4}(-\varepsilon_1 Q't - QF)^{2n}. \end{aligned} \tag{3.5}$$

In this case, we can show that the coefficient of the highest order $t^{8m+6n-2}$ of the equation (3.5) is

$$(-1)^{4m+3n-3}2^{2n-2}Q^2F^2 = 0.$$

Since $Q \neq 0$, we infer that $F = 0$. Therefore, (3.5) becomes

$$2^{2n}Q^4Q'^2D^{8m+6n-6}t^2 = Q^{4m+4}Q'^{2n}t^{2n}, \tag{3.6}$$

from which $Q' = 0$. Thus, by (3.2) the mean curvature H is identically zero, that is, the surface M is minimal. Hence the surface is a helicoid of the 1st kind, the 2nd kind and the 3rd kind according to Theorem 2.1.

Case 2. Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2. Then, the surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = 1, \langle \alpha', \beta \rangle = 0, \langle \beta', \beta' \rangle = 0$ and $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$. We have put the non-zero smooth functions q and S as follows :

$$q = ||x_s||^2 = \varepsilon \langle x_s, x_s \rangle = \varepsilon(\varepsilon_1 + 2St), \quad S = \langle \alpha', \beta' \rangle,$$

where ε denotes the sign of x_s. We note that $\beta \times \beta' = \beta'$. Then, the components of the induced pseudo-Riemannian metric on M are obtained by $E = \varepsilon q, F = 0$ and $G = 1$. For the moving frame $\{\alpha', \beta, \alpha' \times \beta\}$ we can calculate

$$\beta' = \varepsilon_1 S(\alpha' - \alpha' \times \beta), \quad \alpha'' = -S\beta - \varepsilon_1 R \alpha' \times \beta, \tag{3.7}$$

where $R = \langle \alpha'', \alpha' \times \beta \rangle$. Furthermore, using (3.7) we have

$$\langle \beta'', \alpha' \times \beta \rangle = S' + \varepsilon_1 SR, \quad \langle \alpha', \beta'' \rangle = S' + \varepsilon_1 SR.$$

The unit normal vector N is given by

$$N = \frac{1}{\sqrt{q}}(\alpha' \times \beta - t\beta'), \tag{3.8}$$

from which the coefficients of the second fundamental form are given by

$$e = \frac{1}{\sqrt{q}}(R + (S' + 2\varepsilon_1 SR)t), \quad f = \frac{S}{\sqrt{q}}, \quad g = 0.$$

On the other hand, the second Gaussian curvature K_{II}, the mean curvature H and the Gaussian curvature K are obtained respectively by

$$K_{II} = \frac{\varepsilon_1 S'}{2Sq^{\frac{3}{2}}}, \tag{3.9}$$

$$H = \frac{1}{2q^{\frac{3}{2}}}((S' + 2\varepsilon_1 SR)t + R) \tag{3.10}$$

and

$$K = \frac{S^2}{q^2}. \tag{3.11}$$

Suppose that the ruled surface M satisfies the equation $K_{II} = K^m H^n$ for some natural numbers m and n. Then, from (3.9), (3.10) and (3.11) we obtain

$$2^{2n-2} S'^2 q^{4m+3n-3} = S^{4m+2}((S' + 2\varepsilon_1 SR)t + R)^{2n}. \tag{3.12}$$

Thus, the coefficient of the highest order $t^{4m+3n-3}$ of the equation (3.12) is

$$(-1)^{4m+3n-3} 2^{4m+5n-5} S'^2 S^{4m+3n-3} = 0,$$

which implies $S' = 0$. So, (3.12) becomes

$$S^{4m+2}(2\varepsilon_1 SRt + R)^{2n} = 0.$$

Thus, $R = 0$ and by (3.10) the mean curvature H is identically zero. Consequently, the surface is a conjugate of Enneper's surface of the 2nd kind according to Theorem 2.1. This completes the proof. $\square$

Combining the results of Theorems 3.1, 2.1 and Theorem 4.3 in [10] we have

Theorem 3.2. *Let M be a non-cylindrical ruled surface with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space. Then, the following are equivalent :*

1. *M has pointwise 1-type Gauss map.*
2. *M satisfies the equation $aK_{II} + bH = costant$, $a, b \in \mathbb{R} - \{0\}, 2a - b \neq 0$, along each ruling.*
3. *M satisfies the equation $aH + bK = costant$, $a \neq 0, b \in \mathbb{R}$ along each ruling.*
4. *M satisfies the equation $K_{II} = K^m H^n$ along each ruling, for some natural numbers m, n.*

Finally, we investigate the relations between the second Gaussian curvature, the Gaussian curvature and the mean curvature of null scrolls M with non-degenerate second fundamental form in $\mathbb{L}^3$.

Let $\alpha = \alpha(s)$ be null curve in $\mathbb{L}^3$ and $B = B(s)$ be null vector field along α. Then, the null scroll M is parametrized by

$$x = x(s,t) = \alpha(s) + tB(s)$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle B, B \rangle = 0$ and $\langle \alpha', B \rangle = -1$. Furthermore, without loss of generality, we may choose α as a null geodesic of M. We then have $\langle \alpha'(s), B'(s) \rangle = 0$ for all s. By putting, $C = \alpha' \times B$, then $\{\alpha', B, C\}$ is an orthonormal basis along α in $\mathbb{L}^3$. In terms of the basis, we have

$$\begin{aligned} &\alpha'' =< \alpha'', C > C, \\ &B' = -uC, \\ &C' = -u\alpha' + < \alpha'', C > B \end{aligned}$$

u being the function defined by $u =< B, C' >$. The induced Lorentz metric on M is given by $E = \langle B', B' \rangle t^2$, $F = -1$, $G = 0$ and the unit normal vector N is obtained by

$$N = C + tB' \times B.$$

Thus, the component functions of the second fundamental form are given by

$$e = \langle \alpha'' + tB'', N \rangle, \quad f = \langle B', C \rangle = -u, \quad g = 0,$$

which imply $H = u$ and $K = u^2$.

In the orthonormal basis $\{\alpha', B, C\}$, the vector B'' can be reconstructed from

$$B'' = u^2\alpha' - \langle \alpha', B'' \rangle B + \langle B'', C \rangle C,$$

from which
$$e_{tt} = 2\langle B'', N_t\rangle = 2\langle B'', B' \times B\rangle = 2u^3.$$
Therefore, using (2.2) and the above equation the second Gaussian curvature K_{II} is given by
$$K_{II} = \frac{1}{2u^2} e_{tt} = u.$$

Suppose that the null scroll M satisfies the equation $K_{II} = K^m H^n$ for some natural numbers m and n. Then we obtain
$$u(u^{2m+n-1} - 1) = 0.$$
Thus, u is non-zero constant because of $u \neq 0$. Consequently, we have

Theorem 3.3. *Let m, n be natural numbers. B-scrolls over null curves are the only null scrolls with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space satisfying the condition $K_{II} = K^m H^n$ along each ruling.*

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