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QUASI-SUMS IN SEVERAL VARIABLES

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Abstract. In this note we introduce the notions of quasi-sums and of the local quasi-sums in several variables, respectively. We prove that the local quasi-sums are also quasi-sums. We show how this result can be applied to find the continuous solutions of the functional equation

$$g(u_{11}+\cdots+u_{1N},\ldots,u_{M1}+\cdots+u_{MN}) = f(g_1(u_{11},\ldots,u_{M1}),\ldots,g_N(u_{1N},\ldots,u_{MN}))$$

that are strictly monotonic in each variable. Finally we give a proof of a known result on the aggregation equation shorter than that is given in [3].

1. Introduction

By an interval we mean a connected subset of $\mathbb{R}$ (the reals) containing at least two different elements. For a fixed positive integer n, an n-dimensional interval is a set $X_1 \times \cdots \times X_n$ where $X_k \subset \mathbb{R}$ is an interval ($k = 1,\ldots,n$). A CM function is a continuous real-valued function defined on an n-dimensional interval and strictly monotonic in each variable. The notion of quasi-sum is the following. Let $n > 1$ be a fixed integer, $X_1,\ldots,X_n$ be intervals, and $X_1 \times \cdots \times X_n \subset R \subset \mathbb{R}^n$ be an n-dimensional interval. A function $Q\colon R \to \mathbb{R}$ is quasi-sum on the n-dimensional interval $X_1 \times \cdots \times X_n$ if there exist CM functions

$$\alpha_k\colon X_k \to \mathbb{R} \text{ and } \varphi\colon \sum_{k=1}^{n} \alpha_k(X_k) \to \mathbb{R}$$

such that

$$Q(x_1,\ldots,x_n) = \varphi(\alpha_1(x_1)+\cdots+\alpha_n(x_n)) \qquad (x_k \in X_k, k = 1,\ldots,n).$$

The $(n+1)$-tuple $(\varphi,\alpha_1,\ldots,\alpha_n)$ is a generator of Q on $X_1 \times \cdots \times X_n$. The function $Q\colon R \to \mathbb{R}$ is local quasi-sum on R if for each point $(x_1,\ldots,x_n)$ of

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R there exists an open n-dimensional interval S containing $(x_1, \ldots, x_n)$ such that Q is quasi-sum on the n-dimensional interval $R \cap S$. Important examples for quasi-sums are associative CM functions

$$x \circ y = \phi^{-1}(\phi(x) + \phi(y))$$

(Aczél [1]), quasi-arithmetic means

$$Q(x_1, \ldots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \varphi(x_k)\right) \qquad ((x_1, \ldots, x_n) \in I^n)$$

where $I \subset \mathbb{R}$ is an interval and $\varphi\colon I \to \mathbb{R}$ is continuous and strictly monotonic (see Hardy-Littlewood-Pólya [2]), and the CM solutions of equation of aggregation

$$\begin{aligned} \text{(1.1)} \quad & G(F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots, x_{mn})) \\ & \qquad = F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})) \end{aligned}$$

(see Maksa [3] and its references). In Maksa [4] we have proved the following two theorems.

Theorem 1. *If $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ are intervals and $Q\colon X \times Y \to \mathbb{R}$ is local quasi-sum on $X \times Y$ then Q is quasi-sum on $X \times Y$.*

Theorem 2. *If $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ are intervals and the CM function $Q\colon X \times Y \to \mathbb{R}$ is local quasi-sum on $X^\circ \times Y^\circ$ then Q is quasi-sum on $X \times Y$. (Here X° and Y° denote the interior of X and Y, respectively.)*

These results can be applied to find the CM solutions of the generalized associativity equation

$$F(G(x, y), z) = H(x, K(y, z))$$

and of the generalized bisymmetry equation

$$\text{(1.2)} \qquad G(F_1(x_{11}, x_{12}), F_2(x_{21}, x_{22})) = F(G_1(x_{11}, x_{21}), G_2(x_{12}, x_{22}))$$

(see [4] and [3]). In this note we extend the results on two variable quasi-sums discussed in [4] to several variable quasi-sums and apply them to find the CM solutions of the particular aggregation equation

$$\begin{aligned} \text{(1.3)} \quad & g(u_{11} + \cdots + u_{1N}, \ldots, u_{M1} + \cdots + u_{MN}) \\ & \qquad = f(g_1(u_{11}, \ldots, u_{M1}), \ldots, g_N(u_{1N}, \ldots, u_{MN})). \end{aligned}$$

Having this result and the results on equation (1.2) (see [3]), we present a way to find the CM solutions of the general equation (1.1) of aggregation, shorter than that is given in [3]. On the other hand, we hope that the quasi-sum method, developed in this paper, can help to find the CM solutions of other associative type or bisymmetry type functional equations, too.

2. Some basic properties of CM functions

Throughout the paper n denotes a fixed integer greater then one. We begin with three lemmata.

Lemma 1. *Let $1 \le k \le n$ be a fixed integer, $X_1, \dots, X_n, X_k^* \subset \mathbb{R}$ be intervals such that $X_k \cap X_k^* \ne \emptyset$. Let further $\alpha_i \colon X_i \to \mathbb{R}$, $1 \le i \le n$, $i \ne k$ and $\alpha_k \colon X_k \cup X_k^* \to \mathbb{R}$ be CM functions. Then*

$$\begin{aligned}(2.1)\qquad \alpha_1(X_1) + \dots + \alpha_k(X_k \cap X_k^*) + \dots + \alpha_n(X_n) \\ = (\alpha_1(X_1) + \dots + \alpha_k(X_k) + \dots + \alpha_n(X_n)) \\ \cap (\alpha_1(X_1) + \dots + \alpha_k(X_k^*) + \dots + \alpha_n(X_n)).\end{aligned}$$

Proof. It is clear that the set on the left-hand side is a subset of the set on the right-hand side. Thus we only prove the reverse inclusion. Suppose that ξ is an element of the set of the right-hand side of (2.1). Then

$$(2.2)\quad \xi = \alpha_1(\xi_1)+\dots+\alpha_k(\xi_k)+\dots+\alpha_n(\xi_n) = \alpha_1(\eta_1)+\dots+\alpha_k(\eta_k)+\dots+\alpha_n(\eta_n)$$

holds for some $\xi_i, \eta_i \in X_i$, $i \in \{1, \dots, n\} \setminus \{k\}$ and $\xi_k \in X_k, \eta_k \in X_k^*$. If $\xi_k \in X_k \cap X_k^*$ or $\eta_k \in X_k \cap X_k^*$ then there is nothing to prove. Suppose that $\xi_k \in X_k \setminus X_k^*$ and $\eta_k \in X_k^* \setminus X_k$. Let furthermore ω_k be a fixed element of $X_k \cap X_k^*$. Since α_k is strictly monotonic the value $\alpha_k(\omega_k)$ lies between $\alpha_k(\xi_k)$ and $\alpha_k(\eta_k)$. Thus

$$(2.3)\qquad \alpha_k(\omega_k) = \lambda\alpha_k(\xi_k) + (1-\lambda)\alpha_k(\eta_k)$$

for some $0 < \lambda < 1$. On the other hand the numbers $\lambda\alpha_i(\xi_i)+(1-\lambda)\alpha_i(\eta_i)$, $i = 1, \dots, n$, $i \ne k$ lie between $\alpha_i(\xi_i)$ and $\alpha_i(\eta_i)$ for all $1 \le i \le n$, $i \ne k$. Thus there are $\omega_i \in X_i$, $i = 1, \dots, n$, $i \ne k$ such that

$$(2.4)\qquad \alpha_i(\omega_i) = \lambda\alpha_i(\xi_i) + (1-\lambda)\alpha_i(\eta_i)$$

for some $\omega_i \in X_i$, $i = 1, \dots, n$, $i \ne k$. Therefore equations (2.2), (2.4), and (2.3) imply that

$$\begin{aligned}\xi &= \lambda\xi + (1-\lambda)\xi \\ &= \lambda(\alpha_1(\xi_1) + \dots + \alpha_n(\xi_n)) + (1-\lambda)(\alpha_1(\eta_1) + \dots + \alpha_n(\eta_n)) \\ &= \lambda\alpha_1(\xi_1) + (1-\lambda)\alpha_1(\eta_1) + \dots + \lambda\alpha_n(\xi_n) + (1-\lambda)\alpha_n(\eta_n) \\ &= \alpha_1(\omega_1) + \dots + \alpha_n(\omega_n).\end{aligned}$$

Hence

$$\xi \in \alpha_1(X_1) + \dots + \alpha_k(X_k \cap X_k^*) + \dots + \alpha_n(X_n).$$

□

In this section, we use the following property of CM functions frequently, mostly without explicit references.

Lemma 2. *Let $Q\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ be a CM function and $k \in \{1, \ldots, n\}$ be a fixed integer. If Q is strictly increasing (resp. strictly decreasing) in each variable but the k^{th} one for some $x_i \in X_i$, $i \in \{1, \ldots, n\} \setminus \{k\}$ then Q has the same property for any $x_i \in X_i$, $i \in \{1, \ldots, n\} \setminus \{k\}$, too.*

Proof. Suppose that, for example, $Q(x_1, \ldots, x_{k-1}, \xi_k, x_{k+1} \ldots, x_n)$ strictly increasing while $Q(x_1, \ldots, x_{k-1}, \xi'_k, x_{k+1} \ldots, x_n)$ is strictly decreasing in the variables x_i, $i \in \{1, \ldots, n\} \setminus \{k\}$ for some fixed $\xi_k, \xi'_k \in X_k$. Let $x_i, x'_i \in X_i$, $x_i < x'_i$, $i \in \{1, \ldots, n\} \setminus \{k\}$. Then

$$Q(x_1, \ldots, x_{k-1}, \xi_k, x_{k+1} \ldots, x_n) - Q(x'_1, \ldots, x'_{k-1}, \xi_k, x'_{k+1} \ldots, x'_n) < 0$$

and

$$Q(x_1, \ldots, x_{k-1}, \xi'_k, x_{k+1} \ldots, x_n) - Q(x'_1, \ldots, x'_{k-1}, \xi'_k, x'_{k+1} \ldots, x'_n) > 0.$$

Therefore, because of the continuity,

$$Q(x_1, \ldots, x_{k-1}, \eta_k, x_{k+1} \ldots, x_n) - Q(x'_1, \ldots, x'_{k-1}, \eta_k, x'_{k+1} \ldots, x'_n) = 0$$

for some η_k lying between ξ_k and ξ'_k. This contradicts to the strict monotonicity. The other statements of the lemma can be proved similarly. □

In the following (as before in Theorem 2) we denote the set of all inner points of $A \subseteq \mathbb{R}$ by A°.

Lemma 3. *Let $Q\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ be a CM function. Then*

$$\begin{aligned} Q(X_1, X_2^\circ, \ldots, X_n^\circ) &= Q(X_1^\circ, X_2, X_3^\circ, \ldots, X_n^\circ) = \cdots = Q(X_1^\circ, \ldots, X_{n-1}^\circ, X_n) \\ &= Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ) = Q(X_1, X_2, \ldots, X_n)^\circ. \end{aligned}$$

Proof. Suppose that Q is strictly increasing in each variable. This can be done without loss of generality. Indeed, if Q were strictly decreasing in its first variable and strictly increasing in the others (say) then we would consider the function Q_1 defined by

$$Q_1(x_1, \ldots, x_n) = Q(-x_1, \ldots, x_n) \qquad ((x_1, \ldots, x_n) \in (-X_1, \ldots, X_n))$$

instead of Q. (See also Lemma 2.) First we prove that

$$Q(X_1, X_2^\circ, \ldots, X_n^\circ) \subset Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ). \tag{2.5}$$

Let $(x, y_2, \ldots, y_n) \in X_1 \times X_2^\circ \times, \ldots, \times X_n^\circ$. If $x \in X_1^\circ$ then obviously

$$Q(x, y_2, \ldots, y_n) \in Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ).$$

If $x \in X_1 \setminus X_1^\circ$ then first suppose that $x \in \min X_1$. In this case choose an element $(y'_2, \ldots, y'_n) \in X_2^\circ \times \ldots, \times X_n^\circ$ so that $y'_i < y_i$, $i = 2, \ldots, n$ and let $\varepsilon = Q(x, y_2, \ldots, y_n) - Q(x, y'_2, \ldots, y'_n)$. Then $\varepsilon > 0$ and, because of the continuity of Q, there exists $(x_1, y_2, \ldots, y_n) \in X_1^\circ \times X_2^\circ \times \cdots \times X_n^\circ$ such that

$$Q(x_1, y_2, \ldots, y_n) - Q(x, y_2, \ldots, y_n) < \varepsilon = Q(x, y_2, \ldots, y_n) - Q(x, y'_2, \ldots, y'_n)$$

whence

$$\frac{Q(x_1, y_2, \ldots, y_n) + Q(x, y'_2, \ldots, y'_n)}{2} < Q(x, y_2, \ldots, y_n) \tag{2.6}$$

follows. Define the function q on $[0, 1]$ by

$$q(t) = Q((1-t)x + tx_1, (1-t)y_2 + ty'_2, \ldots, (1-t)y_n + ty'_n).$$

Then $q\colon [0, 1] \to \mathbb{R}$ is continuous. Thus, for some $t_0 \in [0, 1]$, we get that

$$q(t_0) = \frac{q(0) + q(1)}{2} = \frac{Q(x, y_2, \ldots, y_n) + Q(x_1, y'_2, \ldots, y'_n)}{2}. \tag{2.7}$$

If $t_0 \in \{0, 1\}$ then $q(0) = q(1)$. Thus, by (2.6),

$$Q(x_1, y_2, \ldots, y_n) < Q(x, y_2, \ldots, y_n).$$

However this is impossible, since $x = \min X_1$ and Q is strictly increasing in each variable. Hence $t_0 \in]0, 1[$ therefore

$$((1-t_0)x + t_0x_1, (1-t_0)y_2 + t_0y'_2, \ldots, (1-t_0)y_n + t_0y'_n) \in X_1^\circ \times X_2^\circ \times, \ldots, \times X_n^\circ$$

and, by (2.6) and (2.7),

$$Q((1-t_0)x + t_0x_1, (1-t_0)y_2 + t_0y'_2, \ldots, (1-t_0)y_n + t_0y'_n) < Q(x, y_2, \ldots, y_n).$$

On the other hand $Q(x, y_2, \ldots, y_n) < Q(x_2, y_2, \ldots, y_n)$ if $x < x_2$, $x_2 \in X_1^\circ$. Thus $Q(x, y_2, \ldots, y_n)$ is an intermediate value of Q on $X_1^\circ \times X_2^\circ \times \ldots, \times X_n^\circ$. Therefore $Q(x, y_2, \ldots, y_n) \in Q(X_1^\circ, X_2, X_3^\circ, \ldots, X_n^\circ)$ which implies (2.5), in case $x = \min X_1$. The case $x = \max X_1$ can be handled similarly. Since the inclusion $Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ) \subset Q(X_1, X_2^\circ, \ldots, X_n^\circ)$ is obvious, we get that

$$Q(X_1, X_2^\circ, \ldots, X_n^\circ) = Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ). \tag{2.8}$$

Interchanging the role of the variables we have that

$$\begin{aligned} Q(X_1, X_2^\circ, \ldots, X_n^\circ) &= Q(X_1^\circ, X_2, X_3^\circ, \ldots, X_n^\circ) = \ldots \\ &= Q(X_1^\circ, \ldots, X_{n-1}^\circ, X_n) \\ &= Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ). \end{aligned} \tag{2.9}$$

It remains only to prove that

$$Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ) = Q(X_1, X_2, \ldots, X_n)^\circ. \tag{2.10}$$

The inclusion $Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ) \subset Q(X_1, X_2, \ldots, X_n)^\circ$ is obvious. For the proof of the reverse inclusion let $z \in Q(X_1, X_2, \ldots, X_n)^\circ$. Thus

$$z = Q(x_1, \ldots, x_n)$$

for some $(x_1, \ldots, x_n) \in (X_1 \times \cdots \times X_n)$. If $x_k \in X_k^\circ$ for some $1 \le k \le n$ then, by (2.8)-(2.9), $z \in Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ)$. In the opposite case we have that x_k is boundary point of X_k for all $1 \le k \le n$. However neither $x_k = \min X_k$ for all $1 \le k \le n$ nor $x_k = \max X_k$ for all $1 \le k \le n$ are valid. (Otherwise $z \notin Q(X_1, X_2, \ldots, X_n)^\circ$ would follow.) Therefore z is an intermediate value of Q on $X_1^\circ \times X_2^\circ \times \ldots, \times X_n^\circ$. Thus (2.10) is proved. □

Finally, in this section, we prove an extension theorem which says that, if a CM function is quasi-sum on the interior of its domain then it is quasi-sum on its entire domain, as well.

Theorem 3. *Let the CM function $Q\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ be quasi-sum on $X_1^\circ \times \cdots \times X_n^\circ$ with generator $(\varphi_0, \alpha_{10}, \ldots, \alpha_{n0})$. Then Q is quasi-sum on its domain. Moreover Q has a generator $(\varphi, \alpha_1, \ldots, \alpha_n)$ so that*

$$\alpha_{k0} = \alpha_k | X_k^\circ,\ 1 \le k \le n$$

and

$$\varphi_0 = \varphi | \alpha_{10}(X_1^\circ) + \cdots + \alpha_{n0}(X_n^\circ).$$

Proof. First we prove that, if $x_1^* \in X_1 \setminus X_1^\circ$ then α_{10} has finite limit at x_1^*. Indeed, let (y_m) be a sequence in X_1° that converges to x_1^*. Let further $x_k \in X_k^\circ$, $2 \le k \le n$ be arbitrary. By Lemma 3,

$$Q(x_1^*, x_2, \ldots, x_n) \in Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ).$$

On the other hand $\varphi_0^{-1}\colon Q(X_1^\circ, X_2^\circ, \ldots, X_n^\circ) \to \mathbb{R}$ and Q are continuous functions. Thus

$$\begin{aligned} \alpha_{10}(y_m) = {} & \varphi_0^{-1}(Q(y_m, x_2, \ldots, x_n)) - \alpha_{20}(x_2) - \cdots - \alpha_{n0}(x_n) \\ \to {} & \varphi_0^{-1}(Q(x_1^*, x_2, \ldots, x_n)) - \alpha_{20}(x_2) - \cdots - \alpha_{n0}(x_n) \end{aligned} \tag{2.11}$$

as $m \to \infty$. Therefore the definition

$$\alpha_1(x_1) = \begin{cases} \alpha_{10}(x_1) & \text{if } \ x_1 \in X_1^\circ \\ \lim\limits_{t \to x_1^*} \alpha_{10}(t) & \text{if } \ x_1 = x_1^* \end{cases} \tag{2.12}$$

is correct, $\alpha_1\colon X_1 \to \mathbb{R}$ is CM function, $\alpha_{10} = \alpha_1 | X_1^\circ$, and, by (2.11),

$$Q(x_1, x_2, \ldots, x_n) = \varphi_0(\alpha_1(x_1) + \alpha_{20}(x_2) + \cdots + \alpha_{n0}(x_n))$$

holds for all $x_1 \in X_1, x_k \in X_0^\circ$, $k = 2, \ldots, n$. The extension of α_{k0} from X_k° to X_k, $k = 2, \ldots, n$ can be done similarly such that

$$Q(x_1, x_2, \ldots, x_n) = \varphi_0(\alpha_1(x_1) + \cdots + \alpha_n(x_n))$$

should hold for all $x_k \in X_k$, $k = 1, \ldots, n$. Finally, let

$$\xi^* \in \alpha_1(X_1) + \cdots + \alpha_n(X_n)$$

be boundary point. Then ξ^* is the maximum or the minimum of the function $(x_1, \ldots, x_n) \mapsto \alpha_1(x_1) + \cdots + \alpha_n(x_n)$, $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. Therefore there is a unique point $(x_1^*, \ldots, x_n^*) \in (X_1 \setminus X_1^\circ) \times \cdots \times (X_n \setminus X_n^\circ)$ such that $\xi^* = \alpha_1(x_1^*) + \cdots + \alpha_n(x_n^*)$. Let $\varphi(\xi^*) = Q(x_1^*, \ldots, x_n^*)$ and $\varphi(\xi) = \varphi_0(\xi)$ if $\xi \in (\alpha_1(X_1) + \cdots + \alpha_n(X_n))^\circ$. Thus, by Lemma 3,

$$\begin{aligned} (\alpha_1(X_1) + \cdots + \alpha_n(X_n))^\circ &= \alpha_1(X_1^\circ) + \cdots + \alpha_n(X_n^\circ) \\ &= \alpha_{10}(X_1^\circ) + \cdots + \alpha_{n0}(X_n^\circ). \end{aligned}$$

Therefore $\varphi_0 = \varphi | \alpha_{10}(X_1^\circ) + \cdots + \alpha_{n0}(X_n^\circ)$. On the other hand

$$\varphi(\xi^*) = \inf\{\alpha_{10}(X_1^\circ) + \cdots + \alpha_{n0}(X_n^\circ)\} \ \text{ or } \ \varphi(\xi^*) = \sup\{\alpha_{10}(X_1^\circ) + \cdots + \alpha_{n0}(X_n^\circ)\}$$

hence φ is CM function and $(\varphi, \alpha_1, \dots, \alpha_n)$ is a generator of Q on

$$X_1 \times \cdots \times X_n.$$

□

3. Main result

In this section we prove that local quasi-sums are also quasi-sums. To do this we need a fitting result on quasi-sums. This says that, if a function is quasi-sum on finitely many n-dimensional intervals fitting each other in a particular way, then it is quasi-sum on the union of these intervals, too. Our basic tool is the following lemma that is an easy consequence of Corollary 3 in Radó-Baker [5].

Lemma 4. *Let $1 < N$ be a fixed integer, $X_k \subset \mathbb{R}$, $k = 1, \dots, N$ be intervals, $\gamma_k \colon X_k \to \mathbb{R}$, $k = 1, \dots, N$ and $\kappa \colon X_1 \times \cdots \times X_N$ be CM functions. Then the functional equation*

$$\kappa(x_1 + \cdots + x_N) = \gamma_1(x_1) + \cdots + \gamma_N(x_N) \qquad ((x_1, \dots, x_N) \in X_1 \times \cdots \times X_N) \tag{3.1}$$

holds if, and only if, there exist real numbers $a \neq 0, b_1, \dots, b_N$ such that

$$\gamma_k(x) = ax + b_k \quad (x \in X_k,\ k = 1, \dots, N)$$

and

$$\kappa(x) = ax + b_1 + \cdots + b_N \quad (x \in X_1 + \cdots + X_N).$$

Now we are ready to prove the following

Lemma 5. *Let $R \subset \mathbb{R}^n$ be an n-dimensional interval and $Q \colon R \to \mathbb{R}$ be quasi-sum on the n-dimensional interval $X_1 \times \cdots \times X_n \subset R$. Then, for each $\xi \in X_1, \eta_k \in X_k$, $k = 1, \dots, n$, $\xi \neq \eta_1$ and $p, q_k \in \mathbb{R}$, $k = 1, \dots, n$, $p \neq q_1$, there exists a unique generator $(\varphi, \alpha_1, \dots, \alpha_n)$ of Q on $X_1 \times \cdots \times X_n$ such that*

$$\alpha_1(\xi) = p \quad \textit{and} \quad \alpha_k(\eta_k) = q_k \quad (k = 1, \dots, n). \tag{3.2}$$

Proof. By definition, Q has a generator $(\psi, \beta_1, \dots, \beta_n)$ on $X_1 \times \cdots \times X_n$. Define the $(n+1)$-tuple $(\varphi, \alpha_1, \dots, \alpha_n)$ by

$$\alpha_k(x) = \frac{p - q_1}{\beta_1(\xi) - \beta_1(\eta_1)}(\beta_k(x) - \beta_k(\eta_k)) + q_k \quad (x \in X_k,\ k = 1, \dots, n)$$

$$\varphi(x) = \psi\left(\frac{\beta_1(\xi) - \beta_1(\eta_1)}{p - q_1}\left(x - \sum_{k=1}^{n}\left(q_k - \frac{p - q_1}{\beta_1(\xi) - \beta_1(\eta_1)}\beta_k(\eta_k)\right)\right)\right)$$

for $x \in \alpha_1(X_1) \times \cdots \times \alpha_n(X_n)$. A simple calculation shows that $(\varphi, \alpha_1, \dots, \alpha_n)$ is a generator of Q on $X_1 \times \cdots \times X_n$ having property (3.2).

To prove the uniqueness suppose that $(\varphi, \alpha_1, \dots, \alpha_n)$ and $(\chi, \delta_1 \dots, \delta_n)$ are two generators of Q on $X_1 \times \cdots \times X_n$ so that the equalities

$$\alpha_1(\xi) = \delta_1(\xi) = p \quad \text{and} \quad \alpha_k(\eta_k) = \delta_k(\eta_k) = q_k \quad (k = 1, \dots, n) \tag{3.3}$$

hold. Then

$$\varphi(\alpha_1(x_1)+\cdots+\alpha_n(x_n))=\chi(\delta_1(x_1)+\cdots+\delta_n(x_n))\quad (x_k\in X_k, k=1,\dots,n)$$

whence

$$\chi^{-1}\circ\varphi(\omega_1+\cdots+\omega_n)=\delta_1\circ\alpha^{-1}(\omega_1)+\cdots+\delta_n\circ\alpha^{-1}(\omega_n)$$

follows for all $\omega_k\in\alpha_k(X_k)$, $(k=1,\dots,n)$. With the notations $n=N$, $\kappa=\chi^{-1}\circ\varphi, \gamma_k=\delta_k\circ\alpha_k^{-1}$, $(k=1,\dots,n)$, this implies equation (3.1). Thus Lemma 4 can be applied and we have the following connections between the elements of the two generators:

$$\begin{aligned}\delta_k(x)&=a\alpha_k(x)+b_k\quad (x\in X_k,\ k=1,\dots,n),\\ \varphi(x)&=\chi(ax+b_1+\cdots+b_n)\quad (x\in\alpha_1(X_1)+\cdots+\alpha_n(X_n))\end{aligned}\tag{3.4}$$

with some real numbers $a\neq 0, b_1,\dots,b_n$. However, by (3.3), these numbers can easily be determined and finally we get that $a=1, b_1=\cdots=b_n=0$. Thus the proof is complete. □

An easy calculation shows that, if $(\varphi,\alpha_1,\dots,\alpha_n)$ is a generator of the quasi-sum Q on $X_1\times\cdots\times X_n$ and the $(n+1)$-tuple $(\chi,\delta_1\dots,\delta_n)$ is defined by (3.4) with arbitrary real numbers $0\neq a, b_1,\dots,b_n$, then it is also a generator of Q on $X_1\times\cdots\times X_n$. Thus the generators can be "re-defined" if necessary. The following lemma is an immediate consequence of the previous one.

Lemma 6. *Let $R\subset\mathbb{R}^n$ be an n-dimensional interval and $Q\colon R\to\mathbb{R}$ be quasi-sum on the n-dimensional interval $X_1\times\cdots\times X_n\subset R$. Suppose that $(\varphi,\alpha_1,\dots,\alpha_n)$ and $(\psi,\beta_1,\dots,\beta_n)$ are two generators of Q on $X_1\times\cdots\times X_n$ so that the equalities*

$$\alpha_1(\xi)=\beta_1(\xi)\quad\textit{and}\quad \alpha_k(\eta_k)=\beta_k(\eta_k)\quad (k=1,\dots,n)$$

hold for some $\xi\in X_1$ and $\eta_k\in X_k$, $(k=1,\dots,n)$, $\xi\neq\eta_1$. Then the two generators coincide, that is, $\alpha_k=\beta_k$ on X_k, $(k=1,\dots,n)$ and $\varphi=\psi$ on $\alpha_1(X_1)+\cdots+\alpha_n(X_n)$.

In the following lemma we show how quasi-sums can be fitted.

Lemma 7. *Let $1\leqq k\leqq n$ be fixed integer, $R\subseteq\mathbb{R}^n$ be n-dimensional interval and $Q\colon R\to\mathbb{R}$ be quasi-sum on the n-dimensional intervals*

$$X_1\times\cdots\times X_k\times\cdots\times X_n\subseteq\mathbb{R}^n$$

and also on $X_1\times\cdots\times X_k^*\times\cdots\times X_n\subseteq\mathbb{R}^n$. Further, suppose that $X_k\cap X_k^*$ has inner point. Then Q is quasi-sum on $X_1\times\cdots\times(X_k\cup X_k^*)\times\cdots\times X_n$, as well.

Proof. If $X_k\subset X_k^*$ or $X_k^*\subset X_k$ then the statement is obvious. Suppose that $X_k\not\subset X_k^*$ and $X_k^*\not\subset X_k$. Let $(\varphi,\alpha_1,\dots,\alpha_n)$ be a generator of Q on $X_1\times\cdots\times X_n$ and $\xi\in X_1, \eta_i\in X_i$, $i\in\{1,\dots,n\}\setminus\{k\}$, $\eta_k\in X_k\cap X_k^*$, $\xi\neq\eta_1$. Since Q is quasi-sum also on $X_1\times\cdots\times X_k^*\times\cdots\times X_n$, therefore, by Lemma 5, it has a generator $(\psi,\beta_1,\dots,\beta_n)$ on $X_1\times\cdots\times X_k^*\times\cdots\times X_n$ so that $\beta_1(\xi)=$

$\alpha_1(\xi)$ and $\beta_k(\eta_k) = \alpha_k(\eta_k)$, $(k = 1,\dots,n)$. Obviously, Q is quasi-sum also on $X_1 \times \dots \times (X_k \cap X_k^*) \times \dots \times X_n$. Thus, by Lemma 6, we obtain that

$$\begin{aligned}
&\beta_i(x) = \alpha_i(x)\ (x \in X_i,\ i \in \{1,\dots,n\} \setminus \{k\}),\\
&\beta_k(x) = \alpha_k(x)\ (x \in X_k \cap X_k^*) \text{ and}\\
&\varphi(x) = \psi(x)\ (x \in \alpha_1(X_1) + \dots + \alpha_k(X_k \cap X_k^*) + \dots + \alpha_n(X_n)).
\end{aligned} \tag{3.5}$$

Define the functions $\gamma_i\colon X_i \to \mathbb{R}$ $(i \in \{1,\dots,n\} \setminus \{k\})$, $\gamma_k\colon X_k \cup X_k^* \to \mathbb{R}$ and $\Gamma\colon \alpha_1(X_1) + \dots + (\alpha_k(X_k) \cup \alpha_k(X_k^*)) + \dots + \alpha_n(X_n) \to \mathbb{R}$ by

$$\gamma_i(x) = \alpha_i(x) \qquad (x \in X_i,\ (i \in \{1,\dots,n\} \setminus \{k\})$$

$$\gamma_k(x) = \begin{cases} \alpha_k(x) & \text{if } x \in X_k \\ \beta_k(x) & \text{if } x \in X_k^* \end{cases}$$

and

$$\Gamma(x) = \begin{cases} \varphi(x) & \text{if } x \in \alpha_1(X_1) + \dots + \alpha_k(X_k) + \dots + \alpha_n(X_n) \\ \psi(x) & \text{if } x \in \alpha_1(X_1) + \dots + \alpha_k(X_k^*) + \dots + \alpha_n(X_n). \end{cases}$$

Then it is obvious that γ_i is CM function for all $i \in \{1,\dots,n\} \setminus \{k\}$. Since $X_k \cap X_k^*$ is an interval of positive length, thus, by (3.5), α_k and β_k are strictly monotonic in the same sense. Hence γ_k is CM function, too. If

$$x \in (\alpha_1(X_1)+\dots+\alpha_k(X_k)+\dots+\alpha_n(X_n)) \cap (\alpha_1(X_1)+\dots+\alpha_k(X_k^*)+\dots+\alpha_n(X_n))$$

then Lemma 1 and (3.5) imply that the definition of Γ is correct and Γ is CM function. Finally, it is obvious that $(\Gamma, \gamma_1, \dots, \gamma_n)$ is a generator of Q on $X_1 \times \dots \times (X_k \cup X_k^*) \times \dots \times X_n$. □

We note that the lemma above can also be used repeatedly and for different values of $1 \leqq k \leqq n$. The following result makes possible to restrict our considerations to compact n-dimensional intervals.

Lemma 8. *Let $R \subset \mathbb{R}$ be n-dimensional interval, $X_{ij} \subset \mathbb{R}$ be interval for all $i = 1,\dots,n$ and for every positive integer j, $R_j = X_{1j} \times \dots \times X_{nj} \subset R$, $R_j \subset R_{j+1}$ for every positive integer j and $R_0 = \bigcup_{j=1}^{\infty} R_j$. Suppose that $Q\colon R \to \mathbb{R}$ is quasi-sum on R_j for every positive integer j. Then Q is quasi-sum also on R_0.*

Proof. Let $(\varphi_1, \alpha_{11}, \dots, \alpha_{n1})$ be a generator of Q on R_1 and

$$\xi \in X_{11}, \eta_k \in X_{k1},\ (k = 1,\dots,n),\ \xi \neq \eta_1$$

and if we have chosen a generator $(\varphi_j, \alpha_{1j}, \dots, \alpha_{nj})$ of Q on R_j for the positive integer j then choose the generator $(\varphi_{j+1}, \alpha_{1j+1}, \dots, \alpha_{nj+1})$ of Q on R_{j+1} so that

$$\alpha_{1j+1}(\xi) = \alpha_{1j}(\xi) \quad \text{and} \quad \alpha_{ij+1}(\eta_i) = \alpha_{ij}(\eta_i) \quad (i = 1,\dots,n)$$

be fulfilled. This is possible by Lemma 5, and from Lemma 6 we get that

$$\alpha_{ij+1}(x_i) = \alpha_{ij}(x_i)\ \ (x_i \in X_{ij},\ (i = 1,\dots,n))$$

and

$$\varphi_{j+1}(x) = \varphi_j(x) \ \ (x \in \alpha_{1j}(X_{1j}) + \cdots + \alpha_{nj}(X_{nj}))$$

for every positive integer j. This shows that the functions

$$\alpha_i = \bigcup_{j=1}^{\infty} \alpha_{ij} \ \ (i = 1, \ldots, n)$$

and $\varphi = \bigcup\limits_{j=1}^{\infty} \varphi_j$ are well-defined, they are CM functions, and $(\varphi, \alpha_1, \ldots, \alpha_n)$ is a generator of Q on R_0. □

Now we prove the main result of our paper.

Theorem 4. *Let $X_1 \times \cdots \times X_n$ be an n-dimensional interval and suppose that $Q\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ is local quasi-sum on it. Then Q is quasi-sum on $X_1 \times \cdots \times X_n$.*

Proof. By Lemma 8, it is enough to prove that Q is quasi-sum on any compact n-dimensional subinterval $C = [a_1, b_1] \times \cdots \times [a_n, b_n]$ of $X_1 \times \cdots \times X_n$. For this, let $\xi \in [a_n, b_n]$ be fixed and

$$C_\xi = \{(\eta_1, \ldots, \eta_{n-1}, \xi) : (\eta_1, \ldots, \eta_{n-1}) \in \mathop{\times}_{i=1}^{n-1} [a_i, b_i]\}.$$

Then $C_\xi \subset C$ is compact. Since Q is local quasi-sum on C, for each point of C there exists an n-dimensional interval, open in C, containing the point, and on which Q is quasi-sum. On the other hand, the compactness of C_ξ implies that there are n-dimensional intervals $X_{1j}^\xi \times \cdots \times X_{nj}^\xi$, $(j = 1, \ldots, m)$ contained by C, such that they are open in C, Q is quasi-sum on each of them, and

$$C_\xi \subset \bigcup_{j=1}^{n} (X_{1j}^\xi \times \cdots \times X_{nj}^\xi).$$

Let

$$R_\xi = \left(\bigcap_{j=1}^{m} X_{1j}^\xi \right) \times \cdots \times \left(\bigcap_{j=1}^{m} X_{n-1j}^\xi \right) \times \left(\bigcup_{j=1}^{m} X_{nj}^\xi \right).$$

Then $C_\xi \subset R_\xi \subset C$ is n-dimensional interval and it is open in C. Applying Lemma 7 repeatedly we obtain that Q is quasi-sum on R_ξ. Hence, because of the compactness of C, there are numbers $\xi_1, \ldots, \xi_M \in [a_n, b_n]$ such that $C = \bigcup\limits_{j=1}^{M} R_{\xi_j}$. Applying Lemma 7 again we get that Q is quasi-sum on C. □

It is clear that the theorem above is a generalization of Theorem 1. Combining this result and Theorem 3 we have the generalization of Theorem 2, as well.

Theorem 5. *Let $X_1 \times \cdots \times X_n$ be an n-dimensional interval and suppose that the CM function $Q\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ is local quasi-sum on the interior of its domain. Then Q is quasi-sum on $X_1 \times \cdots \times X_n$.*

4. An application

Now we prove the following theorem as an application of our results on quasi-sums in several variables.

Theorem 6. *Let $1 < N$ and $1 < M$ be fixed integers, $U_{k\ell} \subset \mathbb{R}$ be intervals, and*

$$g\colon \sum_{\ell=1}^{N} U_{1\ell} \times \cdots \times \sum_{\ell=1}^{N} U_{M\ell} \to \mathbb{R}, \quad g_\ell\colon U_{1\ell} \times \cdots \times U_{M\ell} \to \mathbb{R},$$

and

$$f\colon V_1 \times \cdots \times V_N \to \mathbb{R},$$

where $V_\ell = g_\ell(U_{1\ell}, \dots, U_{M\ell})$, $(k = 1, \dots, M,\ \ell = 1, \dots, N)$, be CM functions. Suppose that

$$\begin{aligned}(4.1)\quad & g(u_{11} + \cdots + u_{1N}, \dots, u_{M1} + \cdots + u_{MN}) \\ &\qquad = f(g_1(u_{11}, \dots, u_{M1}), \dots, g_N(u_{1N}, \dots, u_{MN}))\end{aligned}$$

holds for all $u_{k\ell} \in U_{k\ell}$, $k = 1, \dots, M$, $\ell = 1, \dots, N$. Then there exist CM functions

$$\alpha_\ell\colon V_\ell \to \mathbb{R}\ (\ell = 1, \dots, N),\ \varphi\colon \sum_{\ell=1}^{N} \alpha_\ell(V_\ell) \to \mathbb{R},$$

$c \in \mathbb{R}^M$ with coordinates different from zero and $d_\ell \in \mathbb{R}$ $(\ell = 1, \dots, N)$ such that

$$(4.2)\quad \begin{aligned} & g(u) = \varphi(\langle c, u\rangle + d_1 + \cdots + d_N) \qquad \left(u \in \sum_{\ell}^{N} (U_{1\ell} \times \cdots \times U_{M\ell})\right) \\ & g_\ell(u_\ell) = \alpha_\ell^{-1}(\langle c, u_\ell\rangle + d_\ell),\ (u_\ell \in U_{1\ell} \times \cdots \times U_{M\ell},\ \ell = 1, \dots, N)\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^M$ and

$$f(v_1, \dots, v_N) = \varphi(\alpha_1(v_1) + \cdots + \alpha_N(v_N)) \quad ((v_1, \dots, v_N) \in V_1 \times, \dots, \times V_N).$$

Proof. First we show that f is an N-variable quasi-sum. Since f is CM function, by Theorem 5, it is enough to prove that f is local quasi-sum on $V_1^\circ \times \cdots \times V_N^\circ$. For this, let $(a_1, \dots, a_N) \in V_1^\circ \times \cdots \times V_N^\circ$. Then there exist $u^*_{k\ell} \in U^\circ_{k\ell}$, $(k = 1, \dots, M,\ \ell = 1, \dots, N)$ and $0 < \delta \in \mathbb{R}$ such that $a_\ell = g_\ell(u^*_{1\ell}, \dots, u^*_{M\ell})$ $(\ell = 1, \dots, N)$ and, with the notations

$$I_\ell =]u^*_{1\ell} - \delta, u^*_{1\ell} + \delta[,\ (\ell = 1, \dots, N)$$

and

$$S = g_1(I_1, u^*_{21}, \dots, u^*_{M1}) \times \cdots \times g_N(I_N, u^*_{2N}, \dots, u^*_{MN}),$$

we have that $(a_1, \dots, a_N) \in S \subset V_1^\circ \times \cdots \times V_N^\circ$.

Now we prove that f is quasi-sum on S. Let $t_\ell \in I_\ell$ and $u_{1\ell} = t_\ell, u_{k\ell} = u^*_{k\ell}$, $(k = 2,\dots,M,\ \ell = 1,\dots,N)$. Then equation (4.1) implies that

$$\begin{aligned} &g(t_1 + \cdots + t_N, u^*_{21} + \cdots + u^*_{2N}, \dots, u^*_{M1} + \cdots + u^*_{MN}) \\ &= f(g_1(t_1, u^*_{21} \dots, u^*_{M1}), \dots, g_N(t_N, u^*_{2n} \dots, u^*_{MN})) \\ &= f(h_1(t_1), \dots, h_N(t_N)) \end{aligned}$$

where

$$h_\ell(t_\ell) = g_\ell(t_\ell, u^*_{2\ell} \dots, u^*_{M\ell}), \qquad t_\ell \in I_\ell, \quad (\ell = 1,\dots,N).$$

Thus

$$f(s_1,\dots,s_N) = g(h_1^{-1}(s_1) + \cdots + h_N^{-1}(s_N), u^*_{21} + \cdots + u^*_{2N}, \dots, u^*_{M1} + \cdots + u^*_{MN})$$

holds for all $(s_1,\dots,s_N) \in h_1(I_1) \times \cdots \times h_N(I_N) = S$.

Applying Theorem 5 we have that f is quasi-sum on its domain $V_1 \times \cdots \times V_N$, that is,

$$f(v_1,\dots,v_N) = \varphi(\alpha_1(v_1) + \cdots + \alpha_N(v_N)) \qquad ((v_1,\dots,v_N) \in V_1\times,\dots,\times V_N) \tag{4.3}$$

for some CM functions $\alpha_\ell \colon V_\ell \to \mathbb{R}$ $(\ell = 1\dots,N)$ and $\varphi\colon \sum_{\ell=1}^N \alpha_\ell(V_\ell) \to \mathbb{R}$. Therefore equation (4.1) can be written as

$$\begin{aligned} &g(u_{11} + \cdots + u_{1N}, \dots, u_{M1} + \cdots + u_{MN}) \\ &\qquad = \varphi(\alpha_1 \circ g_1(u_{11},\dots,u_{M1}) + \cdots + \alpha_N \circ g_N(u_{1N},\dots,u_{MN})) \end{aligned}$$

$(u_{k\ell} \in U_{k\ell},\ k = 1,\dots,M,\ \ell = 1,\dots,N)$ or shortly, with the notations

$$\gamma = \varphi^{-1} \circ g, \quad \beta_\ell = \alpha_\ell \circ g_\ell, \quad u_\ell = (u_{1\ell},\dots,u_{M\ell}) \quad (\ell = 1,\dots,N), \tag{4.4}$$

we have that

$$\gamma(u_1 + \cdots + u_N) = \beta_1(u_1) + \cdots + \beta_N(u_N)$$

holds for all $u_\ell \in U_{1\ell} \times \cdots \times U_{M\ell}$, $(\ell = 1,\dots,N)$. The CM solutions of this equation can easily be obtained from Corollary 3 of [5], and we have that

$$\beta_\ell(u) = \langle c, u\rangle + d_\ell \quad (u_\ell \in U_{1\ell} \times \cdots \times U_{M\ell},\ \ell = 1,\dots,N)$$

and

$$\gamma(u) = \langle c, u\rangle + d_1 + \cdots + d_N \quad \left(u \in \sum_{\ell=1}^N U_{1\ell} \times \cdots \times U_{M\ell}\right)$$

with some $c \in \mathbb{R}^M$ and $d_\ell \in \mathbb{R}$, $(\ell = 1,\dots,N)$. Taking into consideration (4.4), this and (4.3) imply (4.2). □

An easy calculation shows that the converse statement of this theorem is true, as well. That is, the functions g, g_ℓ and f defined by (4.2) with CM functions φ, α_ℓ and $d_\ell \in \mathbb{R}$, $(\ell = 1,\dots,N)$ are CM solutions of (4.1).

5. Final remark

As we shall see in this section the problem of finding all CM solutions of the general aggregation equation (1.1) leads to equations (1.2) and (4.1). In ([3]) we have solved (1.2) by using two variable quasi-sums while equation (4.1) can be solved by using several variable quasi-sums. In this section we give a relatively short proof for the following theorem (see also [3]).

Theorem 7. *Let $1 < n, 1 < m$ be fixed integers, $X_{ij} \subset \mathbb{R}$ be intervals, $G_i\colon X_{1i} \times \cdots \times X_{mi} \to \mathbb{R}$, $G_i(X_{1i}, \ldots, X_{mi}) = I_i$, $F_j\colon X_{j1} \times \cdots \times X_{jn} \to \mathbb{R}$, $F_j(X_{j1}, \ldots, X_{jn}) = J_j$, G_i, F_j be CM functions for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $G\colon J_1 \times \cdots \times J_m \to \mathbb{R}$, $F\colon I_1 \times \cdots \times I_n \to \mathbb{R}$ and G, F be CM functions, too. Suppose that equation (1.1)*

$$G(F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots, x_{mn})) = F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn}))$$

holds for all $x_{ji} \in X_{ji}$, $j = 1, \ldots, m$ and $i = 1, \ldots, n$. Then there exist an interval $I \subset \mathbb{R}$ and CM functions $\varphi\colon I \to \mathbb{R}$, $\alpha_i\colon I_i \to \mathbb{R}$, $\gamma_j\colon J_j \to \mathbb{R}$ and $\beta_{ji}\colon X_{ji} \to \mathbb{R}$, $j = 1, \ldots, m$, $i = 1, \ldots, n$ such that

$$F(z_1, \ldots, z_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \alpha_i(z_i)\right), \quad (z_1, \ldots, z_n) \in I_1 \times \cdots \times I_n, \tag{5.1}$$

$$G(y_1, \ldots, y_m) = \varphi^{-1}\left(\sum_{j=1}^{m} \gamma_j(y_j)\right), \quad (y_1, \ldots, y_m) \in J_1 \times \cdots \times J_m, \tag{5.2}$$

$$F_j(x_{j1}, \ldots, x_{jn}) = \gamma_j^{-1}\left(\sum_{i=1}^{n} \beta_{ji}(x_{ji})\right), \tag{5.3}$$

and

$$G_i(x_{1i}, \ldots, x_{mi}) = \alpha_i^{-1}\left(\sum_{j=1}^{m} \beta_{ji}(x_{ji})\right), \tag{5.4}$$

$$x_{ji} \in X_{ji}, \ j = 1, \ldots, m, \ i = 1, \ldots, n.$$

Proof. Part (A). First we prove the theorem for $m = 2$ by induction on n. In this case equation (1.1) has the form

$$G(F_1(x_{11}, \ldots, x_{1n}), F_2(x_{21}, \ldots, x_{2n})) = F(G_1(x_{11}, x_{21}), \ldots, G_n(x_{1n}, x_{2n})), \tag{5.5}$$

and the statement of our theorem is true for $n = 2$ (see Theorem 1 in [3]). Suppose that $n > 2$ and the statement is true for $n - 1$ instead of n. Fix the variables x_{1n}, x_{2n} in (5.5). Then, by the induction hypothesis, we obtain that (5.4) holds for $m = 2$ and for $n - 1$ instead of n with CM functions $\alpha_i, \beta_{1i}, \beta_{2i}, \ i = 1, \ldots, n - 1$. Next, fix the variables x_{11}, x_{21} in (5.5) and apply the induction hypothesis again. Thus we get (5.4) for $m = 2$ and also for $i = n$

with CM functions $\alpha_n, \beta_{1n}, \beta_{2n}$. Hence (5.4) holds for $m = 2$. Substitute this form of $G_i,\ i = 1, \dots, n$ into (5.5) we have that

$$(5.6)\quad G(F_1(x_{11}, \dots, x_{1n}), F_2(x_{21}, \dots, x_{2n})) = F(\alpha_1^{-1}(\beta_{11}(x_{11}) + \beta_{21}(x_{21})), \dots, \alpha_n^{-1}(\beta_{1n}(x_{1n}) + \beta_{2n}(x_{2n}))))$$

holds for all $x_{ji} \in X_{ji}\ j = 1, 2,\ i = 1, \dots, n$. Let $U_{ji} = \beta_{ji}(X_{ji}),\ j = 1, 2,\ i = 1, \dots, n$. Then U_{ji} is interval and for all $u_{ji} \in U_{ji}$ there exists $x_{ji} \in X_{ji}$ such that $u_{ji} = \beta_{ji}(x_{ji}),\ j = 1, 2,\ i = 1, \dots, n$. Thus (5.6) implies that

$$(5.7)\quad G(F_1(\beta_{11}^{-1}(u_{11}), \dots, \beta_{1n}^{-1}(u_{1n})), F_2(\beta_{21}^{-1}(u_{21}), \dots, \beta_{2n}^{-1}(u_{2n}))) = F(\alpha_1^{-1}(u_{11} + u_{21}), \dots, \alpha_n^{-1}(u_{1n} + u_{2n})).$$

With the definitions $N = 2, M = n$,

$$(5.8)\quad \begin{aligned} g(t_1, \dots, t_M) &= F(\alpha_1^{-1}(t_1), \dots, \alpha_M^{-1}(t_M)) \quad (t_i \in U_{1i} + U_{2i}), \\ g_j(u_{j1}, \dots, u_{jM}) &= F_j(\beta_{j1}^{-1}(u_{j1}), \dots, \beta_{jM}^{-1}(u_{jM})) \quad (u_{ji} \in U_{ji}), \\ f &= F \qquad\qquad (j = 1, 2,\ i = 1, \dots, M) \end{aligned}$$

equation (5.7) goes over into (4.1) and Theorem 6 can be applied. Thus we have (4.2) and, by definitions (5.8) and by re-defining the generators (see the remark after Lemma 4), we get (5.1)-(5.4) for $m = 2$.

Part (B). Now, for fixed $n > 1$ we continue the proof by induction on m. The statement of our theorem is true for $m = 2$, as we have shown in *Part (A)* of the proof. Suppose that $m > 2$ and the statement is true for $m - 1$ instead of m. First fix the variables $x_{m1}, \dots, x_{mn}$ in (1.1). Then, by the induction hypothesis, we have (5.3) for $j = 1, \dots, m - 1$ with CM functions $\gamma_j, \beta_{ji},\ j = 1, \dots, m - 1,\ i =, \dots, n$. Next, let $x_{11}, \dots, x_{1n}$ be fixed in (1.1) to obtain (5.3), by using the induction hypothesis again, also for $j = m$ with CM functions $\gamma_m, \beta_{mi},\ i = 1, \dots, n$. Thus we have proved (5.3). Substitute the known form of $F_j,\ j = 1, \dots, m$ into (1.1) to get

$$(5.9)\quad G(\gamma_1^{-1}(\beta_{11}(x_{11}) + \dots + \beta_{1n}(x_{1n})), \dots, \gamma_m^{-1}(\beta_{m1}(x_{m1}) + \dots + \beta_{mn}(x_{mn}))) = F(G_1(x_{11}, \dots, x_{m1}), \dots, G_n(x_{1n}, \dots, x_{mn}))$$

for all $x_{ji} \in X_{ji}\ j = 1, \dots, m,\ i = 1, \dots, n$. Let $U_{ji} = \beta_{ji}(X_{ji}),\ j = 1, \dots, m,$ $i = 1, \dots, n$. Then U_{ji} is interval and for all $u_{ji} \in U_{ji}$ there exists $x_{ji} \in X_{ji}$ such that $u_{ji} = \beta_{ji}(x_{ji}),\ j = 1, \dots, m\ i = 1, \dots, n$. Thus (5.9) implies that

$$(5.10)\quad G(\gamma_1^{-1}(u_{11} + \dots + u_{1n}), \dots, \gamma_m^{-1}(u_{m1} + \dots + u_{mn})) = F(G_1(\beta_{11}^{-1}(u_{11}), \dots, \beta_{m1}^{-1}(u_{m1})), \dots, G_n(\beta_{1n}^{-1}(u_{1n}), \dots, \beta_{mn}^{-1}(u_{mn}))).$$

With the definitions $N = n, M = m$,

$$(5.11)\quad \begin{aligned} g(t_1, \dots, t_M) &= G(\gamma_1^{-1}(t_1), \dots, \gamma_M^{-1}(t_M)) \quad (t_j \in U_{j1} + \dots + U_{jN}), \\ g_i(u_{1i}, \dots, u_{Mi}) &= G_i(\beta_{1i}^{-1}(u_{1i}), \dots, \beta_{Mi}^{-1}(u_{Mi})) \quad (u_{ji} \in U_{ji}), \\ f &= F \qquad (j = 1, \dots, M,\ i = 1, \dots, N) \end{aligned}$$

equation (5.10) goes over into (4.1) and Theorem 6 can be applied. Thus we have (4.2) and, by definitions (5.11) and by re-defining the generators (see the remark again after Lemma 4), we get (5.1)–(5.4). □

An easy calculation shows that the converse statement of this theorem is true, as well. That is, the functions F, G, F_j, G_i defined by (5.1)-(5.4) with CM functions $\varphi, \alpha_i, \gamma_j$ and β_{ji} $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ are CM solutions of (1.1).

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