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# ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

ZHIGANG WANG, CHUNYI GAO AND SHAOMOU YUAN

ABSTRACT. In the present paper, the authors introduce a new subclass $\mathcal{K}_s(\alpha, \beta)$ of close-to-convex functions. Several coefficient inequalities, growth, distortion and covering theorem for this class are provided.

## 1. INTRODUCTION

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclasses of $\mathcal{S}$ whose members are close-to-convex and starlike in $\mathcal{U}$, respectively. Also let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$.

Sakaguchi [5] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

In a later paper, Gao and Zhou [1] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions, that is the subclass of $f(z) \in \mathcal{S}$ satisfying the following inequality

$$\Re\left\{\frac{z^2 f'(z)}{g(z)g(-z)}\right\} < 0 \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

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Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [3]).

In the present paper, we introduce the following class of analytic functions, and obtain some interesting results.

**Definition 1.** Let $\mathcal{K}_s(\alpha, \beta)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left|\frac{z^2 f'(z)}{g(z)g(-z)} + 1\right| < \beta \left|\frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1\right| \quad (z \in \mathcal{U}), \tag{1.1}$$

where $0 \leq \alpha \leq 1,\ 0 < \beta \leq 1$ and $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

It is easy to know that $\mathcal{K}_s(1, 1) = \mathcal{K}_s$, so $\mathcal{K}_s(\alpha, \beta)$ is a generalization of $\mathcal{K}_s$.

In the present paper, we shall provide several coefficient inequalities, growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$.

## 2. Coefficient estimate

First we give a meaningful conclusion about the class $\mathcal{K}_s(\alpha, \beta)$.

**Theorem 1.** *A function $f(z) \in \mathcal{K}_s(\alpha, \beta)$ if and only if*

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + \beta z}{1 - \alpha\beta z} \quad (z \in \mathcal{U}). \tag{2.1}$$

*Proof.* Let $f(z) \in \mathcal{K}_s(\alpha, \beta)$, then from (1.1) we have

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)} - 1\right|^2 < \beta^2 \left|\frac{\alpha z^2 f'(z)}{-g(z)g(-z)} + 1\right|^2.$$

Expanding it we get

$$(1 - \alpha^2\beta^2)\left|\frac{z^2 f'(z)}{-g(z)g(-z)}\right|^2 - 2(1 + \alpha\beta^2)\Re\left\{\frac{z^2 f'(z)}{-g(z)g(-z)}\right\} < \beta^2 - 1.$$

If $\alpha \neq 1$ or $\beta \neq 1$, we have

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)}\right|^2 - \frac{2(1 + \alpha\beta^2)}{1 - \alpha^2\beta^2}\Re\left\{\frac{z^2 f'(z)}{-g(z)g(-z)}\right\} + \left(\frac{1 + \alpha\beta^2}{1 - \alpha^2\beta^2}\right)^2 < \frac{\beta^2 - 1}{1 - \alpha^2\beta^2} + \left(\frac{1 + \alpha\beta^2}{1 - \alpha^2\beta^2}\right)^2,$$

that is,

$$\left|\frac{z^2 f'(z)}{-g(z)g(-z)} - \frac{1 + \alpha\beta^2}{1 - \alpha^2\beta^2}\right|^2 < \frac{\beta^2(1 + \alpha)^2}{(1 - \alpha^2\beta^2)^2},$$

or equivalently,

$$\left|\frac{z^2f'(z)}{-g(z)g(-z)}-\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|<\frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

This tells us that the value region of $G(z)=(z^2f'(z))/(-g(z)g(-z))$ is contained in the disk whose center is $(1+\alpha\beta^2)/(1-\alpha^2\beta^2)$ and radius is $[\beta(1+\alpha)]/(1-\alpha^2\beta^2)$. And we know that the function $\omega=q(z)=(1+\beta z)/(1-\alpha\beta z)$ maps the unit disk to the disk:

$$\left|\omega-\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|<\frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

Notice that $G(0)=q(0)$, $G(\mathcal{U})\subset q(\mathcal{U})$, and $q(z)$ is univalent in $\mathcal{U}$, we obtain the following conclusion

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec q(z)=\frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, let

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

then

$$-\frac{z^2f'(z)}{g(z)g(-z)}=\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}, \tag{2.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$, and $\omega(0)=0$, $|\omega(z)|<1$. By calculation we can easily obtain from (2.2) that

$$\left|\frac{z^2f'(z)}{g(z)g(-z)}+1\right|<\beta\left|\frac{\alpha z^2f'(z)}{g(z)g(-z)}-1\right|,$$

that is $f(z)\in\mathcal{K}_s(\alpha,\beta)$.

If $\alpha=\beta=1$, inequality (1.1) becomes

$$\left|\frac{z^2f'(z)}{-g(z)g(-z)}-1\right|<\left|\frac{z^2f'(z)}{-g(z)g(-z)}+1\right|.$$

It is obvious that

$$-\frac{z^2f'(z)}{g(z)g(-z)}\prec\frac{1+z}{1-z}.$$

This completes the proof of Theorem 1. □

*Remark* 1. From Theorem 1 we know that

$$\Re\left\{\frac{zf'(z)}{(-g(z)g(-z))/z}\right\}>0\quad(z\in\mathcal{U}), \tag{2.3}$$

because of

$$\Re\left\{\frac{1+\beta z}{1-\alpha\beta z}\right\}>0\quad(z\in\mathcal{U}).$$

In order to give the coefficient estimate of functions belonging to the class $\mathcal{K}_s(\alpha,\beta)$, we shall require the following two lemmas.

**Lemma 1** ([1]). *Let* $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*(\frac{1}{2})$, *then*

$$\frac{-g(z)g(-z)}{z}=z+\sum_{n=2}^{\infty}B_{2n-1}z^{2n-1}\in\mathcal{S}^*,$$

*where*

(2.4)

$$B_{2n-1}=2b_{2n-1}-2b_2b_{2n-2}+\cdots+(-1)^n2b_{n-1}b_{n+1}+(-1)^{n+1}b_n^2\quad(n=2,3,\ldots).$$

*Remark* 2. From Lemma 1 and inequality (2.3), we know that if $f(z)\in\mathcal{K}_s(\alpha,\beta)$, then $f(z)$ is a close-to-convex function. So $\mathcal{K}_s(\alpha,\beta)$ is a subclass of the class of close-to-convex functions.

**Lemma 2** ([6]). *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}$, *and satisfy the inequality*

$$\left|\frac{zf'(z)}{g(z)}-1\right|<\beta\left|\frac{\alpha zf'(z)}{g(z)}+1\right|\quad(z\in\mathcal{U}),$$

*where* $0\le\alpha\le1$ *and* $0<\beta\le1$, *then for* $n\ge2$, *we have*

$$|na_n-b_n|^2\le2(1+\alpha\beta^2)\sum_{k=1}^{n-1}k\,|a_k|\,|b_k|\quad(|a_1|=|b_1|=1).\tag{2.5}$$

Now we give the following theorem.

**Theorem 2.** *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}$, *and satisfy the inequality (1.1), then for* $n\ge2$, *we have*

$$|na_n-B_{2n-1}|^2\le2(1+\alpha\beta^2)\sum_{k=1}^{n-1}k\,|a_k|\,|B_{2k-1}|\quad(|a_1|=|B_1|=1),\tag{2.6}$$

*where* $B_{2n-1}$ *is given by (2.4).*

*Proof.* It is easy to know that inequality (1.1) can be written as

$$\left|\frac{zf'(z)}{(-g(z)g(-z))/z}-1\right|<\beta\left|\frac{\alpha zf'(z)}{(-g(z)g(-z))/z}+1\right|.\tag{2.7}$$

By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z}=z+\sum_{n=2}^{\infty}B_{2n-1}z^{2n-1}\in\mathcal{S}^*\subset\mathcal{S}.$$

Now, suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}$ and satisfy (2.7), so $f(z)$ and $(-g(z)g(-z))/z$ satisfy the condition of Lemma 2, thus, from (2.5), we can get (2.6). $\square$

## 3. Sufficient condition

In this section, we give the sufficient condition for functions belonging to the class $\mathcal{K}_s(\alpha,\beta)$.

**Theorem 3.** *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be analytic in $\mathcal{U}$, if for $0\le\alpha\le 1$ and $0<\beta\le 1$, we have*

$$\sum_{n=2}^{\infty}n(1+\alpha\beta)\left|a_n\right|+\sum_{n=2}^{\infty}(1+\beta)\left|B_{2n-1}\right|\le(1+\alpha)\beta, \tag{3.1}$$

*where $B_{2n-1}$ is given by (2.4), then $f(z)\in\mathcal{K}_s(\alpha,\beta)$.*

*Proof.* By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z}=z+\sum_{n=2}^{\infty}B_{2n-1}z^{2n-1}\in\mathcal{S}^*.$$

Suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then for $z\in\mathcal{U}$, we have

$$\begin{aligned}M&=\left|zf'(z)-\frac{-g(z)g(-z)}{z}\right|-\beta\left|\alpha zf'(z)+\frac{-g(z)g(-z)}{z}\right|\\&=\left|z+\sum_{n=2}^{\infty}na_nz^n-z-\sum_{n=2}^{\infty}B_{2n-1}z^{2n-1}\right|\\&\quad-\beta\left|\alpha z+\sum_{n=2}^{\infty}n\alpha a_nz^n+z+\sum_{n=2}^{\infty}B_{2n-1}z^{2n-1}\right|.\end{aligned}$$

Now, for $|z|=r<1$, we have

$$\begin{aligned}M&\le\sum_{n=2}^{\infty}n\left|a_n\right|r^n+\sum_{n=2}^{\infty}\left|B_{2n-1}\right|r^{2n-1}\\&\quad-\beta\left[(1+\alpha)r-\sum_{n=2}^{\infty}n\alpha\left|a_n\right|r^n-\sum_{n=2}^{\infty}\left|B_{2n-1}\right|r^{2n-1}\right]\\&<\left[-(1+\alpha)\beta+\sum_{n=2}^{\infty}n(1+\alpha\beta)\left|a_n\right|+\sum_{n=2}^{\infty}(1+\beta)\left|B_{2n-1}\right|\right]r.\end{aligned}$$

From (3.1) we know that $M<0$, thus we have

$$\left|\frac{z^2f'(z)}{g(z)g(-z)}+1\right|<\beta\left|\frac{\alpha z^2f'(z)}{g(z)g(-z)}-1\right|\quad(z\in\mathcal{U}),$$

that is $f(z)\in\mathcal{K}_s(\alpha,\beta)$, and the proof is complete. □

## 4. Growth, distortion and covering theorem

Finally, we provide the growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$. For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\mathcal{U}$, $\phi(\mathcal{U})$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ (z \in \mathcal{U}),$$

which satisfy the condition $\Re\{p(z)\} > 0$. A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$. The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [2]. And the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$. The functions $k_{\phi n}(z)$ $(n = 2, 3, \ldots)$ defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}(z)$ satisfying $h_{\phi n}(z) = zk'_{\phi n}(z)$ are examples of functions in $\mathcal{S}^*(\phi)$. Write $k_{\phi 2}(z)$ simply as $k_\phi(z)$ and $h_{\phi 2}(z)$ simply as $h_\phi(z)$.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{K}_s(\phi)$ if

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $\phi(z) \in \mathcal{P}$.

In order to prove our next theorem, we shall require the following lemma. The proof of Lemma 3 below is much akin to that of Theorem 7 in [4], here we omit the details.

**Lemma 3.** *Let* $\min_{|z|=r} |\phi(z)| = \phi(-r),\ \max_{|z|=r} |\phi(z)| = \phi(r),\ |z| = r < 1.$ *If* $f(z) \in \mathcal{K}_s(\phi) \subset \mathcal{K}$*, then we have*

$$h'_\phi(-r) \leq |f'(z)| \leq h'_\phi(r),\ -h_\phi(-r) \leq |f(z)| \leq h_\phi(r),$$

*and*

$$f(\mathcal{U}) \supset \{\omega : |\omega| \leq -h(-1)\}.$$

*These results are sharp.*

**Theorem 4.** *Let* $\min_{|z|=r} \left|\frac{1+\beta z}{1-\alpha\beta z}\right| = \psi(-r),\ \max_{|z|=r} \left|\frac{1+\beta z}{1-\alpha\beta z}\right| = \psi(r),\ |z| = r < 1.$ *If* $f(z) \in \mathcal{K}_s(\alpha, \beta)$*, then we have*

$$h'_\psi(-r) \leq |f'(z)| \leq h'_\psi(r),\ -h_\psi(-r) \leq |f(z)| \leq h_\psi(r),$$

*and*

$$f(\mathcal{U}) \supset \{\omega : |\omega| \leq -h(-1)\}.$$

*These results are sharp.*

*Proof.* Setting $\phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ in Lemma 3, we can get the assertion of Theorem 4 easily. $\square$

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College of Mathematics and Computing Science,
Changsha University of Science and Technology,
Changsha, Hunan 410076,
People's Republic of China
*E-mail address*: zhigwang@163.com