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ON MONOTONIC BEHAVIOUR OF RELATIVE INCREMENTS OF UNIMODAL DISTRIBUTIONS

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Abstract. Sufficient conditions for monotonic behaviour of relative increment and hazard rate functions h of unimodal distributions of types U and J are being investigated, proved and then applied to some distributions. In addition, a general algorithm for checking monotonic properties of h is given, where we do not need the cumulative distribution function $F(x) = \int_{-\infty}^{x} f(t)dt$. Instead, we use the probability density function f and its first two derivatives only.

1. Introduction

We will need some concepts, definitions and results from [4]. By the *relative increment function* (briefly, RIF) of a probability distribution function F we mean the fraction

$$h(x) = \frac{F(x+c) - F(x)}{1 - F(x)},$$

where c is a positive constant, and $F(x) < 1$ for all x. The *hazard rate* (*failure rate*) is defined to be

$$\lim_{c \to 0} \frac{h(x)}{c} = \frac{f(x)}{1 - F(x)}.$$

Lemma 1. *Let F be a twice differentiable distribution function with*

$$F(x) < 1, \ F'(x) = f(x) > 0$$

for all x. We define the auxiliary function Ψ as follows:

$$\Psi(x) := \frac{(F(x) - 1) \cdot f'(x)}{f^2(x)}.$$

If $\Psi < 1$ ($\Psi > 1$), then the function h, the RIF of F strictly increases (strictly decreases). [4]

Remark 1. It is clear that $\Psi \lesseqgtr 1$ is equivalent to

$$\Phi(x) := f^2(x) + (1 - F(x)) \cdot f'(x) \gtreqless 0.$$

In some examples, it is more convenient to check Φ instead of Ψ.

Theorem 1. *Let f be a probability density function and F be the corresponding distribution function with the following properties.*

1. *$I = (r, s) \subseteq \mathbb{R}$ is the possible largest finite or infinite open interval in which $f > 0$ (i.e. I is the open support of f; r and s may belong to the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$);*
2. *there exists an $m \in I$ at which f' is continuous and $f'(m) = 0$;*
3. *$f' > 0$ in (r, m), and $f' < 0$ in (m, s);*
4. *f is twice differentiable in (m, s);*
5. *$(f/f')' = d/dx[f(x)/f'(x)] > 0$ in (m, s).*

Then the corresponding continuous RIF h is either strictly increasing in I, or strictly increasing in (r, y) and strictly decreasing in (y, s) for some $y \in I$.

Moreover, if $\Psi(s^-) = \lim_{x \to s^-} \Psi(x) \in \mathbb{R}^*$ exists, then

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– *h strictly increases in I, if $\Psi(s^-) \le 1$;*
– *h strictly increases in (r, y) and strictly decreases in (y, s) for some y in I, if $\Psi(s^-) > 1$.* [4]

Theorem 2. *Let f be a density function with* (1), (3-4), *$m = r$ and*

(6) *$(f/f')' < 0$ in (m, s).*

Then r is finite, and

(7) *if $\Psi(r^+) < 1$ or ($\Psi(r^+) = 1$ and $\Psi < 1$ in some right neighborhood of r), then $\Psi < 1$ in I, and the corresponding RIF strictly increases in I;*

(8) *if $\Psi(r^+) > 1$ then*

(8.1) *if $\Psi(s^-) \ge 1$, then $\Psi > 1$ and the RIF h strictly decreases in I;*

(8.2) *if $\Psi(s^-) < 1$, then $\Psi > 1$ in (r, y) and $\Psi < 1$ in (y, s) for some $y \in I$, thus the RIF strictly decreases first and, having reached its local minimum, it strictly increases.* [4]

The proof of Theorem 2 in [4] remains valid, if one replaces the relation sign $<$ in (12), page 109 of [4] by $\le$. Hence, the "main idea" in page 109, line 20 can be modified as follows: *in (m, s), if $\Psi \le 1$, then Ψ strictly decreases, provided $(f/f')' < 0$.*

There is an immediate connection to the theory of reliability. By the Mathematical Preliminaries of [1] (Sec. 1., p. 549), a distribution function F has increasing failure rate if $\ln(1 - F(x))$ is concave down i.e. if $\Psi(x) \le 1$. Similarly, F has decreasing failure rate if $\ln(1 - F(x))$ is concave up, i.e. $\Psi(x) \ge 1$. This is the reason why we always simultaneously investigate here the *hazard rates* and *relative increment functions* of (cumulative) distribution functions F (like we did in [4]).

Remark 2. Since $(f/f')' = -(\ln f)''/[(\ln f)']^2$, the condition (5) can be formulated as follows:

(5′) $(\ln f)'' < 0$ in (m, s).

Similarly, (6) can be written in the form

(6′) $(\ln f)'' > 0$ in (m, s). [4]

Remark 3. Theorem 1 is related to U-distributions, while Theorem 2 is related to J-distributions.

In this paper, we will try to extend our results in [4] to the case when $(f/f')'$ changes its sign in (m, s).

In (m, s), the so-called "*main ideas*" of the proofs of Theorems 1 and 2 in [4] apply: *once Ψ reaches a value more (less) than 1, it will strictly increase (decrease) and will remain more (less) than 1, provided $(f/f')' > 0$ ($(f/f')' < 0$).*

2. Main results

U-distributions.

Theorem 3. *Assume (1-4) are fulfilled.*

(9) *Suppose $(f/f')' > 0$ in (m, Y) and $(f/f')' < 0$ in (Y, s) for some $Y \in (m, s)$.*

Then both the corresponding RIF h and the hazard rate will either strictly increase; or strictly increase first and then strictly decrease; or first strictly increase, then strictly decrease and, finally, strictly increase in I.

The maximum or minimum (if exists) will be reached in $(m, Y]$ or (Y, s), respectively.

Proof. It follows that Ψ is continuous in $[m, s)$, and $\Psi < 1$ in $(r, m + p)$ for some $p > 0$. (See the proof of Theorem 1. in [4].) If $\Psi(Y) \le 1$, then $\Psi < 1$ in (m, Y).

If $\Psi(Y) < 1$ or ($\Psi(Y) = 1$ and $\Psi < 1$ in some right-neighborhood of Y) then, according to the "main ideas", Ψ will strictly decrease and it will remain below 1 in (Y, s). Thus $\Psi(s^-) < 1$ provided $\Psi(s^-)$ exists. In this case, $\Psi < 1$ in $I \setminus \{Y\}$ so, according to Lemma 1 in [4], the RIF h will strictly increase in I.

If $\Psi(Y) > 1$ or ($\Psi(Y) = 1$ and $\Psi > 1$ in some right-neighborhood of Y), then $\exists\ Y_0 \in (m, Y)$ such that $\Psi(Y_0) = 1$, $\Psi < 1$ in (m, Y_0) and $\Psi > 1$ in (Y_0, Y) because, according to the "main ideas", once $\Psi(x) \ge 1$, Ψ will strictly increase at x, since $(f(x)/f'(x))' > 0$. (Ψ is continuous, so Ψ will remain above 1 in $(Y_0, Y + \delta)$ for some $\delta > 0$.} We have two cases:

Case 1. Ψ remains above 1 in (Y,s). Then either $\Psi(s^-) > 1$, or $\Psi(s^-) = 1$ but $\Psi > 1$ in some left-neighborhood of s (provided $\exists\ \Psi(s^-)$). In this case, due to Lemma 1, the RIF h will first strictly increase, then strictly decrease, and its maximum will be reached in $(m, Y]$;

Case 2. $\Psi(Y_1) = 1$ for some $Y_1 \in (Y, s)$; then, according to the "main ideas", Ψ will remain below 1 in (Y_1, s). Then $\Psi(s^-) < 1$ (provided $\exists\ \Psi(s^-)$). In this case, the RIF h will have three monotonic phases: h will first strictly increase, then strictly decrease and, finally, strictly increase. The function h will reach its maximum in $(m, Y]$ and minimum in (Y, s).

According to the "main ideas" of the proofs of Theorems 1 and 2 in [4], there are no cases like:

– $\Psi(Y) = 1$ and $\Psi \geq 1$ in some left-neighborhood of Y,
– or $\Psi(Y) = 1$ and $\Psi \equiv 1$ in some right-neighborhood of Y,
– or $\Psi(s^-) = 1$ and $\Psi \leq 1$ in some left-neighborhood of s. □

Remark 4. If $\Psi(s^-) \geq 1$, then we do not have to check the value of $\Psi(Y)$ because, independently from the value of $\Psi(Y)$, the RIF h will have two monotonic phases: it will strictly increase first, and then strictly decrease.

Remark 5. $(f/f')' \equiv 0$ is impossible, since it would lead to a contradiction of the form $f(m) = f'(m) = 0$.

J-distributions.

Theorem 4. *Let f be a density function with (1), (3-5) and $m = r$. Then r is finite, and*

– *if $\Psi(m^+) > 1$ or ($\Psi(m^+) = 1$ and $\Psi > 1$ in some right-neighborhood of m), then $\Psi > 1$ in I, and the corresponding RIF h strictly decreases in I; in this case, $\Psi(s^-) > 1$, provided $\exists\ \Psi(s^-)$;*
– *if $\Psi(m^+) < 1$ or ($\Psi(m^+) = 1$ and $\Psi < 1$ in some right-neighborhood of m), then*
  - *if $\Psi(s^-) \leq 1$, then $\Psi < 1$ and h will strictly increase in I;*
  - *if $\Psi(s^-) > 1$, then $\Psi > 1$ in (m, y) and $\Psi < 1$ in (y, s) for some $y \in I$;*

thus, the RIF h will strictly increase first and, having reached its maximum, it will strictly decrease.

Proof. (2) and (4) imply that Ψ is continuous in I. If $\Psi(m^+) > 1$ or ($\Psi(m^+) = 1$ and $\Psi > 1$ in some right-neighborhood of m) then, according to the "main idea" of the proof of Theorem 1 in [4], Ψ will strictly increase in I, so $\Psi(s^-) > 1$ provided it exists. Thus, $\Psi > 1$ in I and, according to Lemma 1, the RIF h will strictly decrease in I.

If $\Psi(m^+) < 1$ or ($\Psi(m^+) = 1$ and $\Psi < 1$ in some right-neighborhood of m), then we have two cases.

Case 1. $\Psi < 1$ in I. Then h will strictly increase in I. Thus, $\Psi(s^-) \leq 1$ provided $\exists\ \Psi(s^-)$.

Case 2. $\Psi(x_0) \geq 1$ for some $x_0 \in (m, s)$. Then, according to the "main idea", Ψ will strictly increase in $[x_0, s)$. So, $\Psi > 1$ in (x_0, s), and $\Psi(s^-) > 1$ (provided $\exists\ \Psi(s^-)$). Hence, $\exists\ y \in I$ such that $\Psi(y) = 1$, $\Psi < 1$ in (m, y) and $\Psi > 1$ in (y, s). So, the RIF h will strictly increase in (m, y) and strictly decrease in (y, s). □

Theorem 5. *Assume $m = r$ and (1), (3-4) are fulfilled.*

(10) *Suppose the following relations hold: $(f/f')' < 0$ in (m, Y) and $(f/f')' > 0$ in (Y, s) for some $Y \in I = (m, s)$.*

Then both the corresponding RIF h and the hazard rate will, in I,

– *either strictly increase;*
– *or strictly decrease;*
– *or strictly increase first and then strictly decrease;*
– *or strictly decrease first and then strictly increase;*
– *or first strictly decrease, then strictly increase and, finally, strictly decrease.*

The maximum or minimum will, if exists, be reached in (Y, s) or in $(m, Y]$, respectively.

Proof. It follows that Ψ is continuous in I. If $\Psi(m^+) < 1$ or ($\Psi(m^+) = 1$ and $\Psi < 1$ in some right-neighborhood of m), then Ψ will strictly decrease, since $(f(x)/f'(x))' < 0$. Ψ is continuous, so Ψ will remain below 1 in $(m, Y + \delta)$ for some $\delta > 0$.

If $\Psi(x_0) \geq 1$ for some $x_0 \in [Y + \delta, s)$ then, according to the "main ideas", Ψ will strictly increase and it will remain above 1 in $(Y + \delta, s)$. In this case, we have $\Psi(s^-) > 1$ (provided $\exists\ \Psi(s^-)$). The RIF h will have two monotonic phases: it will first strictly increase, and then strictly decrease. Its maximum will be reached in (Y, s).

If there is no x_0 with this property, then $\Psi < 1$ in $(Y + \delta, s)$, and $\Psi(s^-) \leq 1$ (provided $\exists\ \Psi(s^-)$). In this case, the RIF h will strictly increase in I.

If $\Psi(m^+) > 1$ or ($\Psi(m^+) = 1$ and $\Psi > 1$ in some right-neighborhood of m) then, in (m, Y), we can follow the series of thoughts of the proof of Theorem 2. in [4]:

– if $\Psi(Y) \geq 1$, then $\Psi > 1$ in (m, Y);

– if $\Psi(Y) < 1$, then $\exists\ y \in (m, Y)$ such that $\Psi(y) = 1$, $\Psi > 1$ in (m, y) and Ψ will strictly decrease in $[y, Y)$, thus Ψ will remain below 1 in $(y, Y + \delta)$ for some $\delta > 0$.

In (Y, s), we have the following situation. If $\Psi(Y) > 1$ or ($\Psi(Y) = 1$ and $\Psi > 1$ in some right-neighborhood of Y) then, in (Y, s), Ψ will strictly increase and it will remain above 1. Thus, $\Psi > 1$ in $I \setminus \{Y\}$ and $\Psi(s^-) > 1$ (provided $\exists\ \Psi(s^-)$). In this case, the RIF h will strictly decrease in I.

If $\Psi(Y) < 1$ or ($\Psi(Y) = 1$ and $\Psi < 1$ in some right-neighborhood of Y), then either $\Psi < 1$ in (Y, s) (and then $\Psi(s^-) \leq 1$ provided $\exists\ \Psi(s^-)$); in this case, the RIF h will first strictly decrease, then strictly increase; h will reach its minimum in $(m, Y]$; or $\exists\ z \in (Y, s)$ such that $\Psi(z) = 1$; in this case, $\Psi < 1$ in (Y, z) and $\Psi > 1$ in (z, s) since, according to the "main ideas", once Ψ reaches the value 1, it will strictly increase and will remain more than 1. Thus, $\Psi(s^-) > 1$ (provided $\exists\ \Psi(s^-)$); in this case, due to Lemma 1, the RIF h will have three monotonic phases: it will first strictly decrease, then strictly increase and, finally, strictly decrease. The function h will reach its minimum in $(m, Y]$ and maximum in (Y, s).

According to the "main ideas" of the proofs of Theorems 1 and 2 in [4], there are no cases like:

– $\Psi(Y) = 1$ and $\Psi \leq 1$ in some left-neighborhood of Y,
– or $\Psi(Y) = 1$ and $\Psi \equiv 1$ in some right-neighborhood of Y,
– or $\Psi(s^-) = 1$ and $\Psi \geq 1$ in some left-neighborhood of s. □

We can formulate a symmetrical statement as follows.

Theorem 6. *Assume $m = r$ and (1), (3-4), (9) are fulfilled. Then both the corresponding RIF h and the hazard rate will, in I,*

– *either strictly increase;*
– *or strictly decrease;*
– *or strictly increase first and then strictly decrease;*
– *or strictly decrease first and then strictly increase;*
– *or first strictly increase, then strictly decrease and, finally, strictly increase.*

The maximum or minimum will, if exists, be reached in $(m, Y]$ or in (Y, s), respectively.

Proof. It follows that Ψ is continuous in I. If $\Psi(m^+) > 1$ or ($\Psi(m^+) = 1$ and $\Psi > 1$ in some right-neighborhood of m), then Ψ will strictly increase in (m, Y), since $(f(x)/f'(x))' > 0$. Ψ is continuous, so it will remain above 1 in $(m, Y + \delta)$ for some $\delta > 0$.

If $\Psi(x_0) \leq 1$ for some $x_0 \in [Y + \delta, s)$ then, according to the "main ideas", Ψ will strictly decrease and it will remain below 1 in $(Y + \delta, s)$. In this case, we have $\Psi(s^-) < 1$ (provided $\exists\ \Psi(s^-)$). Thus, the RIF h will have two monotonic phases: it will first strictly decrease, and then strictly increase. Its minimum will be reached in (Y, s).

If there is no x_0 with the above property, then $\Psi > 1$ in $(Y + \delta, s)$, and $\Psi(s^-) \geq 1$ (provided $\exists\ \Psi(s^-)$). In this case, the RIF h will strictly decrease in I.

If $\Psi(m^+) < 1$ or ($\Psi(m^+) = 1$ and $\Psi < 1$ in some right-neighborhood of m) then, in (m, Y), we have the following possibilities.

– If $\Psi(Y) \leq 1$, then $\Psi < 1$ in (m, Y);
– if $\Psi(Y) > 1$, then $\exists\ y \in (m, Y)$ such that $\Psi(y) = 1$, $\Psi < 1$ in (m, y) and Ψ will strictly increase in $[\,y, Y)$; thus, because of its continuity, Ψ will remain above 1 in $(\,y, Y + \delta)$ for some $\delta > 0$.

In (Y, s), we have the following situation. If $\Psi(Y) < 1$ or ($\Psi(Y) = 1$ and $\Psi < 1$ in some right-neighborhood of Y) then, in (Y, s), Ψ will strictly decrease and it will remain below 1. Thus, $\Psi < 1$ in $I \setminus \{Y\}$ and $\Psi(s^-) < 1$ (provided $\exists\ \Psi(s^-)$). In this case, according to Lemma 1, the RIF h will strictly increase in I.

If $\Psi(Y) > 1$ or ($\Psi(Y) = 1$ and $\Psi > 1$ in some right-neighborhood of Y), then

– either $\Psi > 1$ in (Y, s) (and then $\Psi(s^-) \geq 1$ provided $\exists\ \Psi(s^-)$);

in this case, h will first strictly increase, then strictly decrease; it will reach its maximum in $(m, Y]$;

– or $\exists\ z \in (Y, s)$ such that $\Psi(z) = 1$;

in this case, $\Psi > 1$ in (Y, z) and $\Psi < 1$ in (z, s) because, according to the "main ideas", once Ψ reaches the value of 1 in (Y, s), it will strictly decrease and will remain less than 1. Thus, $\Psi(s^-) < 1$ (provided $\exists\ \Psi(s^-)$); in this case, due to Lemma 1, the RIF h will have three monotonic phases: it will first strictly increase, then strictly decrease and, finally, strictly increase. The function h will reach its maximum in $(m, Y]$ and minimum in (Y, s).

According to the "main ideas", there are no cases like:

– $\Psi(Y) = 1$ and $\Psi \geq 1$ in some left-neighborhood of Y,
– or $\Psi(Y) = 1$ and $\Psi \equiv 1$ in some right-neighborhood of Y,
– or $\Psi(s^-) = 1$ and $\Psi \leq 1$ in some left-neighborhood of s. □

Due to the Remark 2.1 in [4], there is no distribution for which all the requirements (1-4) and (10) are fulfilled at the same time.

3. Algorithmic Investigation

If $\Psi(s^-)$ and $\Psi(m^+)$ exist, then the entire investigation of monotonic behaviour of both the RIF h and the hazard rate can briefly be summarized and described in an algorithmic way, in a flow-chart, in which the abbreviations Bs, Bm, BY (that can be considered to be Boolean variables) denote the following logical conditions:

$$Bs := \big(B \cdot (\Psi(s^-) - 1) > 0\big)$$

$$BY := \big(\Psi(Y) > 1 \text{ or } (\Psi(Y) = 1 \text{ and } \Psi > 1 \text{ in some right-neighborhood of } Y)\big)$$

$$Bm := \big(\Psi(m^+) < 1 \text{ or } (\Psi(m^+) = 1 \text{ and } \Psi < 1 \text{ in some right-neighborhood of } m)\big)$$

where

$$\Psi(s^-) = \lim_{x \to s-0} \Psi(x)$$

and

$$\Psi(m^+) = \lim_{x \to m+0} \Psi(x)$$

Actually, B is equal to the sign of $(f/f')'$ in a sufficiently small left-neighborhood of s.

If $(f/f')'$ changes sign in (m, s) only once, say at Y, then the locations of maxima/minima (if exist) obey the following rule:
the RIF h (and the corresponding hazard rate) reach the maximum (minimum) in $(m, Y]$ (or (Y, s)), respectively, provided $(f/f')' < 0$ in some left-neighborhood of s (see Theorems 3 and 6); the RIF (and the hazard rate) reach the maximum (minimum) in (Y, s) (or $(m, Y]$), respectively, provided $(f/f')' > 0$ in some left-neighborhood of s (see Theorem 5).

The algorithm for investigation of both the RIF h and the hazard rate of a specific distribution can be described by the flow-chart 1.

4. Applications

Our results apply to some distributions as follows.

Example 1. Inverse Gaussian distribution (p. 382 in [3]):

$$f(x) = (2\pi x^3/\lambda)^{-\frac{1}{2}} \cdot \exp\big(-\lambda \cdot (x-\mu)^2/(2\mu^2 \cdot x)\big)$$

where $\lambda, \mu > 0$ and $x \in (0, \infty) =: I$. We have $f/f' = 2x^2/L$ and $(f/f')' = 2x \cdot (2\lambda - 3x)/L^2$, where $L := \lambda - 3x - \lambda x^2/\mu^2$. The value of $(f/f')'$ is positive if $x < 2\lambda/3 =: Y$. The value of m is strictly positive, since $f'(x) = 0$ if $\lambda x^2 + 3\mu^2 x - \lambda\mu^2 = 0$, the only positive root of which is $m = \mu \cdot ((c^2+1)^{\frac{1}{2}} - c) \in I$, where $c := 3\mu/(2\lambda)$. On the other hand, $m < Y$, and f is of type U. So, $(f/f')' > 0$ in (m, Y) and $(f/f')' < 0$ in (Y, ∞).

According to our flow-chart, $B := -1$, and the logical expression Bs is equivalent to $\Psi(s^-) < 1$. By using Remark 1.4 in [1], one can obtain $\Psi(s^-) = \lim_{x\to\infty} (1 + (f/f')')^{-1} = 1$, since

$$f_\infty = \lim_{x\to\infty} f^2/f' = -\mu^2 \cdot \frac{2\lambda}{\pi} \cdot \lim_{x\to\infty} \frac{\exp(-\lambda \cdot (x-\mu)^2)/(2\mu^2 \cdot x)}{\lambda x \cdot \sqrt{x} + 3\mu^2 \cdot \sqrt{x} - \lambda\mu^2/\sqrt{x}} = 0.$$

So, $\Psi(s^-) \geq 1$ and, according to the flow-chart, both the corresponding RIF h and the hazard rate will, in I, first strictly increase and then strictly decrease.

Example 2. Lognormal distribution (p. 192, Table 5.7 in [3]):

$$f(x) = C \cdot \frac{\exp\big(-(\ln x)^2/(2\sigma^2)\big)}{x},$$

where $\sigma > 0$, $C = 1/(\sigma \cdot \sqrt{2\pi})$ and $x \in (0, \infty) =: I$. The equation $f' = 0$ gives the modus: $m = \exp(-\sigma^2) \in I$. We have $(f/f')' = -(x/(1+k \cdot \ln x))' = (1-\sigma^2-\ln x)/(\sigma+(\ln x)/\sigma)^2 > 0$ if $x < e \cdot m =: Y$. On the other hand, $m < Y$, and f is of type U. So, $(f/f')' > 0$ in (m, Y) and $(f/f')' < 0$ in (Y, ∞). The

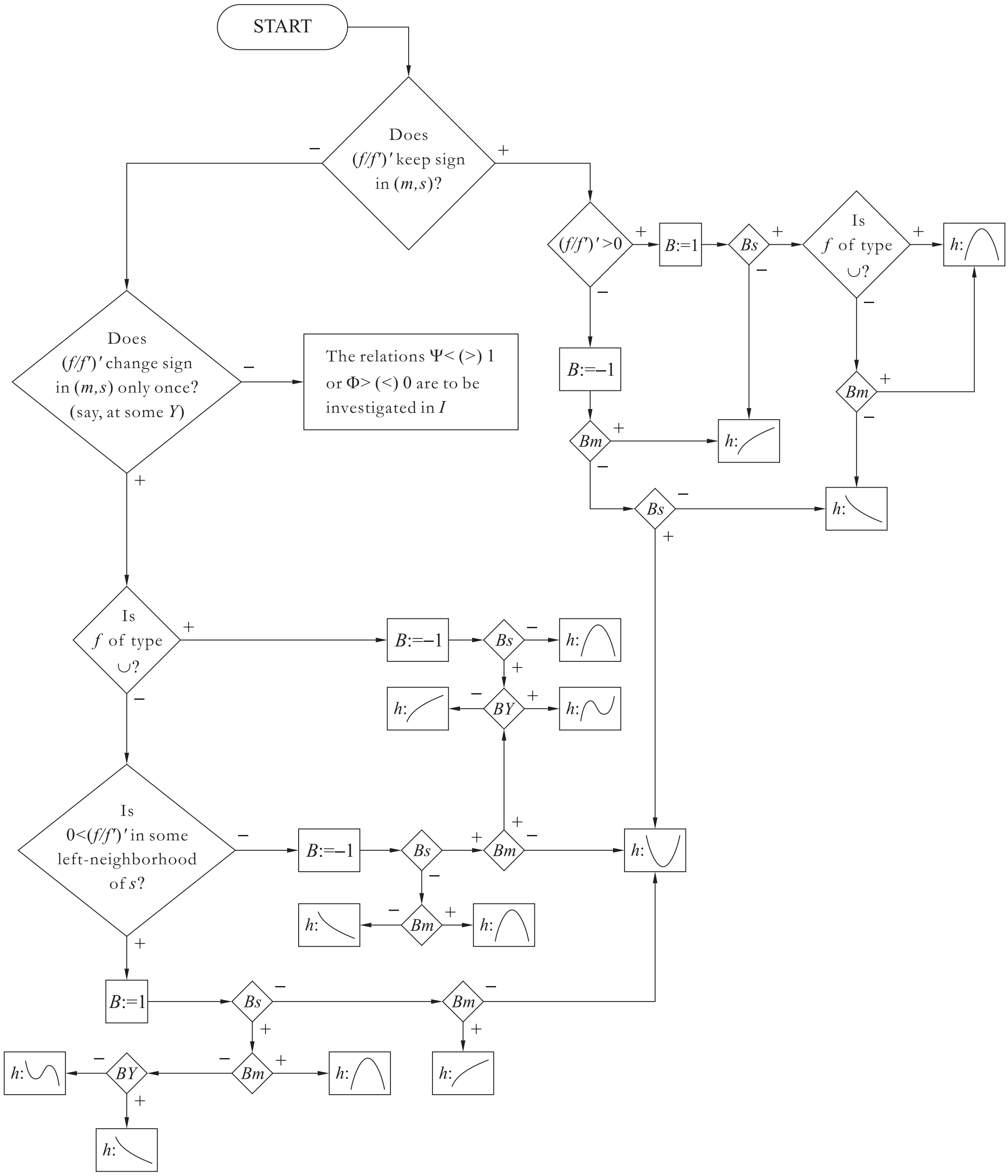

FIGURE 1

density function f is of type U because of uniqueness of m in I. Thus, $B = -1$, and Bs is equivalent to $\Psi(s^-) < 1$. By using Remark 1.4 in [1], one can obtain $\Psi(s^-)$ as follows:

$$f^2/f' = -C \cdot \sigma^2 \cdot \frac{\exp\bigl(-(\ln x)^2/(2\sigma^2)\bigr)}{\sigma^2 + \ln x}$$

tends to 0 as $x \to \infty$, so

$$\Psi(s^-) = \Psi(\infty) = \lim_{x\to\infty} \left(1 - \sigma^2 \cdot \frac{\sigma^2 + \ln x - 1}{(\sigma^2 + \ln x)^2}\right)^{-1} = 1$$

and, according to our algorithm, both the corresponding RIF h and the hazard rate will, in I, first strictly increase and then strictly decrease.

Example 3. $F(x) = (1-x^2)^{\frac{1}{2}}$, $x \in (-1,0) =: I$. We have $m = r = -1$, $s = 0$ and f has no local maximum in I. On the other hand, $(f/f')' = (x - x^3)' = 1 - 3x^2 < 0$ in $(-1,Y)$ and $(f/f')' > 0$ in $(Y,0)$, where $Y = -1/\sqrt{3}$. So, $B = 1$. The density function f is of type J. Bs is equivalent to $\Psi(s^-) < 1$. We have

$$\lim_{x\to 0} f^2/f' = \lim_{x\to 0} -x^2 \cdot (1-x^2)^{\frac{1}{2}} = 0 \ ,$$

$\Psi(0^-) = 1/2$, so the actual value of Bs is FALSE. Since $\Psi(-1^+) = +\infty$, Bm is FALSE. $\Psi(Y)$ is close to 0.67, so BY is FALSE and, according to our algorithm, both the corresponding RIF h and the hazard rate will, in I, first strictly decrease first, and then strictly increase. Our algorithm is working but, we have to admit, checking the relation $\Psi(x) < 1$ in this example is much easier.

Remark 6. The expression f/f' plays a central role in the entire investigation, throughout both [4] and the present paper. Sometimes, the actual form of f/f' is very simple, like in the case of Pearson distributions. This case is analyzed in [5] thoroughly.

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