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AUTOREGRESSIVE TYPE MARTINGALE FIELDS

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Abstract. In this paper a generalization of d-parameter martingales is studied. A d-parameter process is called an autoregressive martingale field if it satisfies certain autoregressive type stochastic difference equations. An almost sure convergence theorem is proved for autoregressive martingale fields.

1. Introduction

There are several extensions of the notion of a martingale. The so called linear martingales were studied in [Mac73], [Hey80], [Faz87]. The notion of a linear martingale was extended to the two index case in [Faz88]. We mention that a lot of papers are devoted to the study of multiindex martingales (e.g. [Cai70], [Faz83]). It is well-known that the almost sure (a.s.) convergence of a multiindex sequence (in particular a martingale) requires stronger conditions than that of a single index sequence. The a.s. convergence of multiindex martingales is described in [Cai70].

In this paper we extend the notion of a linear martingale to the multiindex case. Then we obtain an a.s. convergence result for it (Theorem 6.1). This theorem contains previous results of [Faz87] and [Faz88] as special cases. In order to prove our result we have to use a new martingale convergence theorem (Theorem 3.1). Theorem 3.1 is a uniform a.s. convergence result for Banach space valued multiindex martingales.

2. Notation and preliminary remarks

In the following $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively. Let d be a fixed positive integer. Throughout the paper $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}$ denote elements of $\mathbb{N}_0^d$ (in particular, elements of $\mathbb{N}^d$). $\mathbf{n}$ always means the vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d$, a d-parameter sequence of random variables, and let $\mathcal{F}_{\mathbf{n}} \subseteq \mathcal{F}$ be σ-algebras for all $\mathbf{n} \in \mathbb{N}^d$.

We shall use $\mathbf{1} := (1, \ldots, 1) \in \mathbb{N}^d$ and $\mathbf{0} := (0, \ldots, 0) \in \mathbb{N}_0^d$. In $\mathbb{N}_0^d$ we consider the coordinate-wise partial ordering: $\mathbf{m} \leq \mathbf{n}$ means $m_i \leq n_i,\ i = 1, \ldots, d$ ($\mathbf{m} < \mathbf{n}$ means $\mathbf{m} \leq \mathbf{n}$ and $\mathbf{m} \neq \mathbf{n}$). Relations $\leq$, max, min, $\to$ are interpreted coordinate-wise. E.g. $\mathbf{n} \to \infty$ is interpreted as $n_i \to \infty$ for every $i = 1, \ldots, d$. Let $|\log\ \mathbf{n}| := \prod_{i=1}^d \log^+ n_i$, where $\log^+ x = \log x$, if $x \geq e$ and $\log^+ x = 1$, if $x < e$.

Let $\overline{\mathbf{n}}$ denote certain coordinates of $\mathbf{n}$ and let $\underline{\mathbf{n}}$ denote the rest of the coordinates of $\mathbf{n}$. Denote by $(\underline{\mathbf{n}}, \infty)$ a length d sequence that consists of those coordinates of $\mathbf{n}$ which belong to $\underline{\mathbf{n}}$ while the remaining coordinates of $\mathbf{n}$ are substituted by ∞. For example when $\underline{\mathbf{n}}$ consists of the second and third coordinates of $\mathbf{n}$ then $(\underline{\mathbf{n}}, \infty) = (\infty, n_2, n_3, \infty, \ldots, \infty)$ and $(\overline{\mathbf{n}}, \infty) = (n_1, \infty, \infty, n_4, \ldots, n_d)$.

Let $\mathcal{F}_{(\underline{\mathbf{n}},\infty)}$ denote the σ-algebra generated by the σ-algebras $\mathcal{F}_{\mathbf{k}}$ where $\mathbf{k} \leq (\underline{\mathbf{n}}, \infty), \mathbf{k} \in \mathbb{N}^d$. For example in the above case $\mathcal{F}_{(\underline{\mathbf{n}},\infty)} = \sigma\{\mathcal{F}_{\mathbf{k}} : k_2 \leq n_2, k_3 \leq n_3\}$, and $\mathcal{F}_{(\overline{\mathbf{n}},\infty)} = \sigma\{\mathcal{F}_{\mathbf{k}} : k_1 \leq n_1, k_4 \leq n_4, \ldots, k_d \leq n_d\}$.

Let $(B, \|.\|)$ be a real separable Banach space. Let $c(B)$ denote the set of all convergent sequences in B. If $X = (x_1, x_2, \ldots) \in c(B)$, let $\|X\|_c = \sup_i \|x_i\|$. $c^d(B)$ (and the norm in this space) is defined by induction, i.e. $c^d = c(c^{d-1}(B))$. Let $c_0(B)$ denote the set of sequences converging to 0 (the 0 element of B).

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Recall the notion of a martingale. Suppose that $\mathcal{F}_{\mathbf{m}} \subseteq \mathcal{F}_{\mathbf{n}}$ for every $\mathbf{m} \leq \mathbf{n}$. Assume that $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and integrable for every $\mathbf{n} \in \mathbb{N}^d$. We say that $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$ is a martingale if $E(X_{\mathbf{n}+\mathbf{k}} \,|\mathcal{F}_{\mathbf{n}}) = X_{\mathbf{n}}$ a.s., for all $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{k} \in \mathbb{N}_0^d$.

Throughout the paper we shall assume that

$$E\left(E(X_{\mathbf{l}}|\mathcal{F}_{\mathbf{m}})|\mathcal{F}_{\mathbf{n}}\right) = E\left(X_{\mathbf{l}}|\mathcal{F}_{\min\{\mathbf{m},\mathbf{n}\}}\right) \tag{1}$$

holds for every $\mathbf{l}, \mathbf{n}, \mathbf{m} \in \mathbb{N}^d$. This property is widely used in the theory of multiindex martingales (see e.g. [Faz83]).

We shall use the following condition. For any η with finite expectation

$$E\left(\eta|\mathcal{F}_{\mathbf{n}}\right) = E\left(\dots E\left(\eta|\mathcal{F}_{\mathbf{n}}^{(i_1)}\right)\dots|\mathcal{F}_{\mathbf{n}}^{(i_d)}\right) \tag{2}$$

for any permutation $(i_1, \dots, i_d)$ of $(1, \dots, d)$, where $\mathcal{F}_{\mathbf{n}}^{(i)} = \sigma\{\mathcal{F}_{\mathbf{l}} : l_i = n_i\}$ for any fixed $\mathbf{n}$ and i (in the notation $\mathcal{F}_{\mathbf{n}}^{(i)}$ (i) shows the appropriate coordinate). Actually $\mathcal{F}_{\mathbf{n}}^{(i)}$ is a particular case of $\mathcal{F}_{(\underline{\mathbf{n}},\infty)}$. That is $\mathcal{F}_{\mathbf{n}}^{(i)} = \mathcal{F}_{(\underline{\mathbf{n}},\infty)}$, if $(\underline{\mathbf{n}}, \infty) = (\infty, \dots, \infty, n_i, \infty, \dots, \infty)$.

It is easy to see that (2) implies the following property

$$E\left\{\eta|\mathcal{F}_{\mathbf{n}}\right\} = E\{E\{\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)}\}|\mathcal{F}_{(\overline{\mathbf{n}},\infty)}\}, \tag{3}$$

for every η with finite expectation. To prove it let $\underline{\mathbf{n}}$ denote the i_1st,..., i_lth coordinates of $\mathbf{n}$. Applying (2) to $E\{\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)}\}$ we obtain

$$\begin{aligned} E\{\eta|\mathcal{F}_{\mathbf{n}}\} &= E\{E(\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)})|\mathcal{F}_{\mathbf{n}}\} = \\ &= E\{\dots E[E\dots[E(\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)})|\mathcal{F}_{\mathbf{n}}^{(i_1)}]\dots|\mathcal{F}_{\mathbf{n}}^{(i_l)}]\dots|\mathcal{F}_{\mathbf{n}}^{(i_d)}\} = \\ &= E\{\dots E[\dots E(\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)})|\mathcal{F}_{\mathbf{n}}^{(i_{l+1})}]\dots|\mathcal{F}_{\mathbf{n}}^{(i_d)}\} \end{aligned} \tag{4}$$

because $\mathcal{F}_{\mathbf{n}}^{(i_1)}, \dots, \mathcal{F}_{\mathbf{n}}^{(i_l)}$ contain $\mathcal{F}_{(\underline{\mathbf{n}},\infty)}$.

$$\begin{aligned} E\left\{\eta|\mathcal{F}_{\mathbf{n}}\right\} &= E\{E(\eta|\mathcal{F}_{\mathbf{n}})|\mathcal{F}_{(\overline{\mathbf{n}},\infty)}\} = \\ &= E\{E[E\dots[E(\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)})|\mathcal{F}_{\mathbf{n}}^{(i_{l+1})}]\dots|\mathcal{F}_{\mathbf{n}}^{(i_d)}]|\mathcal{F}_{(\overline{\mathbf{n}},\infty)}\} = \\ &= E\{E\{\eta|\mathcal{F}_{(\underline{\mathbf{n}},\infty)}\}|\mathcal{F}_{(\overline{\mathbf{n}},\infty)}\}, \end{aligned}$$

where we applied (4) in the second step.

It is easy to see that (3) implies

$$E(E(\eta|\mathcal{F}_{\mathbf{m}})|\mathcal{F}_{\mathbf{n}}) = E(\eta|\mathcal{F}_{\min\{\mathbf{m},\mathbf{n}\}}) \tag{5}$$

for any η having finite expectation. In particular, (3) implies (1). It is known (see [Kho02], p. 36) that (5) implies (2). Therefore (2), (3) and (5) are equivalent and they imply (1).

Proposition 2.1. *Let* $\varepsilon_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d$, *be independent random variables,* $\mathcal{F}_{\mathbf{n}} = \sigma\{\varepsilon_{\mathbf{k}} : \mathbf{k} \leq \mathbf{n}\}$. *Then* $\mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d$, *satisfies* (2).

Proof. We consider only the case $d = 2$. Let ξ_{12}, ξ_1 and ξ_2 be the following random elements: $\xi_{12} = (\varepsilon_{ij} : i \leq n_1, j \leq n_2), \xi_1 = (\varepsilon_{ij} : i \leq n_1, j > n_2)$ and $\xi_2 = (\varepsilon_{ij} : j \leq n_2, i > n_1)$. Then $E(\eta|\xi_{12}, \xi_1) = f(\xi_{12}, \xi_1)$, where f is measurable.
Then

$$E\{E(\eta|\mathcal{F}_{\mathbf{n}}^{(1)})|\mathcal{F}_{\mathbf{n}}^{(2)}\} = E\{E(\eta|\xi_{12}, \xi_1)|\xi_{12}, \xi_2\} = E\{f(\xi_{12}, \xi_1)|\xi_{12}, \xi_2\} = g(\xi_{12}). \tag{6}$$

To see it, first we observe that the independence of ξ_{12}, ξ_1 and ξ_2 implies

$$E\{f(\xi_{12}, \xi_1)|\xi_{12} = x_{12}, \xi_2 = x_2\} = E(f(x_{12}, \xi_1)) = g(x_{12}),$$

where g is a measurable function. Now we substitute ξ_{12} and ξ_2 in this equation.

Using the fact $\mathcal{F}_{\mathbf{n}} \subseteq \mathcal{F}_{\mathbf{n}}^{(1)}, \mathcal{F}_{\mathbf{n}}^{(2)}$ and (6), we obtain

$$E\{\eta|\mathcal{F}_{\mathbf{n}}\} = E\{E[E(\eta|\mathcal{F}_{\mathbf{n}}^{(1)})|\mathcal{F}_{\mathbf{n}}^{(2)}]|\mathcal{F}_{\mathbf{n}}\} = E\{g(\xi_{12})|\mathcal{F}_{\mathbf{n}}\} = g(\xi_{12}) = E[E(\eta|\mathcal{F}_{\mathbf{n}}^{(1)})|\mathcal{F}_{\mathbf{n}}^{(2)}].$$

For $d > 2$ the proof is similar. $\square$

Our first result is the uniform convergence of B-valued multiindex martingales. Actually the following theorem is a version of Theorem 4.4 in [Faz83] where the uniform convergence was not studied.

Recall the notion of Radon-Nikodym property (see [Cha68]). The Banach space B has the Radon-Nikodym property with respect to $(\Omega, \mathcal{F}, \mathbf{P})$ if every B-valued σ-additive set-function μ of bounded variation (that is, $V_\mu(\Omega) < \infty$) which is absolutely continuous with respect to P (that is, $P(A) = 0 \Rightarrow \mu(A) = 0$ or equivalently, $V_\mu \ll P$) has an integral representation, that is, there exists an $f \in L^1(\mathcal{F}, B)$ such that $\mu(A) = \int_A f(s)P(ds)$ for all $A \in \mathcal{F}$.

The space B will be said to have Radon-Nikodym property if it has the Radon-Nikodym property with respect to Lebesgue measure on the Borel sets of the unit interval.

Theorem 3.1. *Let B be a real separable Banach space. Let $(X_\mathbf{n}, \mathcal{F}_\mathbf{n}), \mathbf{n} \in \mathbb{N}^d$, be a B-valued martingale. Assume that the σ-algebras $\mathcal{F}_\mathbf{n}$ satisfy* (3)*. Let B have Radon-Nikodym property or let $X_\mathbf{n}$ be of the form $X_\mathbf{n} = E(X|\mathcal{F}_\mathbf{n}), \mathbf{n} \in \mathbb{N}^d$, for an $X \in L^1(\mathcal{F}, B)$. Assume that $\sup_\mathbf{n} E\|X_\mathbf{n}\|(\log^+ \|X_\mathbf{n}\|^{d-1}) < \infty$. Then there exists an event A with $P(A) = 1$ such that for $\omega \in A$ we have: if arbitrary coordinates of $\mathbf{n}$ converge to ∞ while the remaining coordinates remain fixed, then $X_\mathbf{n}(\omega)$ converges uniformly. (The limit is a random variable depending on the coordinates remaining fixed.)*

For a two index martingale the convergence in our theorem means the following. Let $\varepsilon > 0$. Then for any n_2 we have $\|X_{n_1,n_2}(\omega) - X_{\infty,n_2}(\omega)\| < \varepsilon$ if $n_1 > n_{1\varepsilon}$, for any n_1 we have $\|X_{n_1,n_2}(\omega) - X_{n_1,\infty}(\omega)\| < \varepsilon$ if $n_2 > n_{2\varepsilon}$, moreover $\|X_{n_1,n_2}(\omega) - X_{\infty,\infty}(\omega)\| < \varepsilon$ if $n_1, n_2 > n_\varepsilon$.

Proof of Theorem 3.1. We use induction. For $d = 1$ the result is known (see [Cha68]).

Suppose that the result is valid for dimension not exceeding $d - 1$. Now we prove for $d, d \geq 2$. (We shall fix the last coordinate of $\mathbf{n}$.)

We see that $(X_\mathbf{n}, \mathcal{F}_\mathbf{n}), \mathbf{n} \in \mathbb{N}^d$, satisfies the conditions of Theorem 4.4 of [Faz83]. In [Faz83] it is proved that $X_\mathbf{n} = E(X|\mathcal{F}_\mathbf{n}),\ \mathbf{n} \in \mathbb{Z}^d$, and $X_\mathbf{n} \to X$ $(\mathbf{n} \to \infty)$ in L^1, where X is $\mathcal{F}_\infty$ -measurable.

We show, that $X_\mathbf{n}$ converges uniformly with probability 1, when some coordinates of $\mathbf{n}$ tend to infinity.

Let $Z^{(k)}_\mathbf{m} = X_{(\mathbf{m},k)}$, where $k \in \mathbb{N}$ is fixed and $\mathbf{m} \in \mathbb{N}^{d-1}$ is running. This is a $(d-1)$ index martingale.

We see that $Z^{(k)}_\mathbf{m} = E(X|\mathcal{F}_{(\mathbf{m},k)})$. From here

$$\begin{aligned} Z^{(k)}_\mathbf{m} &= X_{(\mathbf{m},k)} = E(X_{(\mathbf{m},k)}|\mathcal{F}_{(\infty,k)}) = E[E(X|\mathcal{F}_{(\mathbf{m},k)})|\mathcal{F}_{(\infty,k)}] \\ &= E[E(X|\mathcal{F}_{(\infty,k)})|\mathcal{F}_{(\mathbf{m},k)}] = E(Z^{(k)}_\infty|\mathcal{F}_{(\mathbf{m},k)}), \end{aligned}$$

where $Z^{(k)}_\infty = E(X|\mathcal{F}_{(\infty,k)})$.

Now we explain the main ideas of the proof. We use that $Z^{(k)}_\mathbf{m}$ (for each fixed k) is a $(d-1)$ index martingale. Because of the induction hypothesis $Z^{(k)}_\mathbf{m}$ converges uniformly if any subset of coordinates of $\mathbf{m}$ tends to infinity. The structure of $Z^{(k)}_\mathbf{m}$ is the following. If only the last coordinate m_{d-1} of $\mathbf{m}$ goes to ∞, then $Z^{(k)}_\mathbf{m}$ is a convergent sequence (with probability 1), i.e. that sequence is an element of $c(B)$. Now the coordinate m_{d-2} is running. Then the previous elements of $c(B)$ are convergent. So it is an element of $c(c(B))$. Finally $Z^{(k)}_\mathbf{m} \in \underbrace{c(\ldots c}_{d-1}(B)) = c^{d-1}(B)$. Actually we shall show by induction that $X_\mathbf{n} \in c^d(B)$.

We need to show that $Z^{(k)}_\mathbf{m}$ converges with probability 1. Now we create its limit.

We can suppose that X is $\mathcal{F}_\infty$ -measurable therefore $X_\mathbf{n} = E(X|\mathcal{F}_\mathbf{n})$ implies $X_\mathbf{n} \to X$ in L_1.

Let $Z^{(\infty)}_\mathbf{m} = E(X|\mathcal{F}_{(\mathbf{m},\infty)})$. Therefore $(Z^{(\infty)}_\mathbf{m}, \mathcal{F}_{(\mathbf{m},\infty)}),\ \mathbf{m} \in \mathbb{N}^{d-1}$, is a martingale. From the submartingale convergence theorem we get

$$E\|X\|(\log^+ \|X\|)^{d-1} \leq \sup_{\mathbf{n} \in \mathbb{N}^d} E\|X_\mathbf{n}\|(\log^+ \|X_\mathbf{n}\|)^{d-1} \leq K < \infty.$$

From the Jensen inequality we obtain

$$E\|Z^{(\infty)}_\mathbf{m}\|(\log^+ \|Z^{(\infty)}_\mathbf{m}\|)^{d-1} \leq E\|X\|(\log^+ \|X\|)^{d-1} \leq K < \infty. \tag{7}$$

That is the martingale $(Z^{(\infty)}_\mathbf{m}, \mathcal{F}_{(\mathbf{m},\infty)}), \mathbf{m} \in \mathbb{N}^{d-1}$, satisfies the conditions of the theorem, therefore we can consider this martingale as a random element of $c^{d-1}(B)$.

We need to prove that the equation

$$E(Z^{(\infty)}|\mathcal{F}_{(\infty,k)}) = Z^{(k)} \tag{8}$$

holds for every k. Here $Z^{(\infty)} = \{Z^{(\infty)}_{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^{d-1}\}$ and $Z^{(k)} = \{Z^{(k)}_{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^{d-1}\}$. We prove this equation for each fixed $\mathbf{m}$.

$$E(Z^{(\infty)}_{\mathbf{m}}|\mathcal{F}_{(\infty,k)}) = E[E(X|\mathcal{F}_{(\mathbf{m},\infty)})|\mathcal{F}_{(\infty,k)}] = E(X|\mathcal{F}_{(\mathbf{m},k)}) = Z^{(k)}_{\mathbf{m}}.$$

If $E\|Z^{(\infty)}\|_{c^{d-1}(B)} < \infty$ is satisfied, then Lemma 2.4 of [Faz83] implies that it is enough to prove relation (8) coordinate-wise. (In the expression $\|Z^{(\infty)}\|_{c^{d-1}(B)}$ we use the norm of the space $c^{d-1}(B)$.)

By equation (7), we have $\sup_{\mathbf{m}} E\|Z^{(\infty)}_{\mathbf{m}}\|(\log^+\|Z^{(\infty)}_{\mathbf{m}}\|)^{d-1} < \infty$. Applying the proof of Theorem 4.4 of [Faz83], the Cairoli inequality (see [Faz83], p.158) and induction we obtain $E\|Z^{(\infty)}\|_{c^{d-1}(B)} < \infty$. So equation (8) is valid. Therefore $Z^{(k)} \to Z^{(\infty)}$ a.s.

Now we prove, if arbitrarily many coordinates of $\mathbf{n}$ tend to infinity then $X_{\mathbf{n}}$ converges uniformly with probability 1. Divide the $\mathbf{n}$ into parts: $\mathbf{n} = (\mathbf{m}, \mathbf{l}, k)$, where $k \in \mathbb{N}$. By the martingale convergence theorem in $c^{d-1}(B)$, $\lim_{k\to\infty} X_{(\mathbf{m},\mathbf{l},k)} = X_{(\mathbf{m},\mathbf{l},\infty)}$ in the space $c^{d-1}(B)$ a.s. That is for $\varepsilon > 0$ there exists k_ε so that in the case $k > k_\varepsilon$ we have $\|X_{(\mathbf{m},\mathbf{l},k)} - X_{(\mathbf{m},\mathbf{l},\infty)}\| < \varepsilon$ for all $\mathbf{l}, \mathbf{m}$. But $X_{(\mathbf{m},\mathbf{l},\infty)} \in c^{d-1}(B)$ is satisfied, so $\|X_{(\mathbf{m},\mathbf{l},\infty)} - X_{(\mathbf{m},\infty,\infty)}\| < \varepsilon$ when $\mathbf{l}$ is sufficiently large. But then $\|X_{(\mathbf{m},\mathbf{l},k)} - X_{(\mathbf{m},\infty,\infty)}\| < 2\varepsilon$ when k and $\mathbf{l}$ are sufficiently large.

Therefore the proof is complete. □

We shall use the d-index version of the Burkholder inequality.

Lemma 3.2 (Noszály, Tómács [NT00], Fazekas [Faz05])**.** *Let* $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, *be a martingale with values in* $\mathbb{R}^m$. *Assume that* (1) *is satisfied. Let* $p > 1$. *There exist finite and positive constants* C *and* D *depending only on* m, p *and* d *such that*

$$CE\left(\sum_{\mathbf{m}\le\mathbf{n}} \|\Delta_{\mathbf{m}}\|^2\right)^{p/2} \le E\|X_{\mathbf{n}}\|^p \le DE\left(\sum_{\mathbf{m}\le\mathbf{n}} \|\Delta_{\mathbf{m}}\|^2\right)^{p/2}$$

for every $\mathbf{n} \in \mathbb{N}^d$, *where* $\Delta_{\mathbf{k}}$ *is the martingale difference, i.e.* $X_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} \Delta_{\mathbf{k}}$.

Above and in what follows $\|.\|$ denotes the Euclidean norm.

4. The definition of an autoregressive martingale field

To describe the structure of the random field $\xi_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d$, we shall use the Kronecker product (denoted by $\otimes$) and the vec operation (see, e.g. [MN88]).

Let A be an $m \times n$ type matrix and let $\mathbf{a}_j$ be its jth column. Then $\operatorname{vec} A$ is the $mn \times 1$ type vector

$$\operatorname{vec} A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}.$$

Thus the vec operator transforms a matrix into a vector by stacking the columns of the matrix one underneath the other (for further properties see [MN88]).

The vec operator transforms a d-dimensional array into a vector. At first the first index is running, then second one, and so on. E.g. for the 3-index array $A = (a_{ijk})_{i=1}^{l}{}_{j=1}^{m}{}_{k=1}^{n}$

$$\operatorname{vec} A = (a_{111}, a_{211}, \ldots, a_{l11}, a_{121}, \ldots, a_{l21}, \ldots, a_{1mn}, \ldots, a_{lmn})^\top.$$

Definition 4.1. The process $\{\xi_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}\}$, $\mathbf{n} \in \mathbb{N}^d$, is called an *autoregressive martingale field* if $\xi_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and integrable for every $\mathbf{n} \in \mathbb{N}^d$,

$$E\left(\xi_{\mathbf{n}}|\mathcal{F}^{(j)}_{\mathbf{n}-\mathbf{e}_j}\right) = a^{(j)}_1(n_j)\xi_{\mathbf{n}-\mathbf{e}_j} + a^{(j)}_2(n_j)\xi_{\mathbf{n}-2\mathbf{e}_j} + \cdots + a^{(j)}_m(n_j)\xi_{\mathbf{n}-m\mathbf{e}_j} \tag{9}$$

for every $\mathbf{n}$ and j, with $n_j > m$, $j = 1, \ldots, d$, where m is a fixed positive integer, $\mathbf{e}_j = (0, \ldots, \underbrace{1}_{jth}, \ldots 0) \in \mathbb{N}^d_0$ is the jth unit vector, $j = 1, \ldots, d$, and $a^{(j)}_i(n_j)$ are non-negative non-random coefficients with $\sum_{i=1}^m a^{(j)}_i(n_j) = 1$ for every $n_j = m+1, m+2, \ldots,$ $j = 1, \ldots, d$.

If the coefficients $a^{(j)}_k(l)$ do not depend on l, then $\xi_{\mathbf{n}}$ is called a *homogeneous autoregressive martingale field*.

Let $\xi_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, be a d-index random field. Using $\xi_{\mathbf{n}}$, we shall construct another random field $X_{\mathbf{n}}$. The values of this new field are d-index arrays. For any fixed $m \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^d$ (with $n_i \geq m, i = 1, \dots, d$) $X_{\mathbf{n}}$ denotes the elements of the random field $\xi_{\mathbf{k}}$ with indices being in a hypercube of size $\underbrace{m \times m \times \cdots \times m}_{d}$.

More precisely let the $\mathbf{k}$th element of the array $X_{\mathbf{n}}$ be $X_{\mathbf{n}}^{\mathbf{k}} = \xi_{\mathbf{n}-\mathbf{m}+\mathbf{k}}$, where $\mathbf{m} = (m, \dots, m) \in \mathbb{N}^d$, $\mathbf{k} = (k_1, \dots, k_d)$, $k_i = 1, \dots, m$, $i = 1, \dots, d$. If we consider the index as time and $\mathbf{n}$ is the present, then $X_{\mathbf{n}}$ contains the present value $\xi_{\mathbf{n}}$ and $m^d - 1$ past values of the underlying field $\xi_{\mathbf{n}}$.

Proposition 4.2. *Let $(\xi_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$ be the autoregressive martingale field introduced in Definition 4.1. Let $X_{\mathbf{n}}$ be the array valued random field corresponding to $\xi_{\mathbf{n}}$. Then*

$$\operatorname{vec}\left[E(X_{\mathbf{n}}|\mathcal{F}^{(j)}_{\mathbf{n}-\mathbf{e}_j})\right] = \Big(\underbrace{I \otimes \cdots \otimes I}_{d-j} \otimes A_j^{(n_j)} \otimes \underbrace{I \otimes \cdots \otimes I}_{j-1}\Big) \operatorname{vec}(X_{\mathbf{n}-\mathbf{e}_j}) \tag{10}$$

for every $\mathbf{n}$ with $n_j > m$, $j = 1, \dots, d$, where $A_j^{(l)}$ denotes the following $m \times m$ matrix

$$A_j^{(l)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_m^{(j)}(l) & a_{m-1}^{(j)}(l) & \cdots & \cdots & a_1^{(j)}(l) \end{pmatrix},$$

for every $j = 1, \dots, d$ and $l = m+1, m+2, \dots$.

We see that $A_j^{(l)}$ is the transition matrix of a Markov chain. We assume that $a_m^{(j)}(l) > 0$. Then the chain is irreducible. If this chain is aperiodic then it is ergodic. If $a_k^{(j)}(l) > 0$, then k is a return time of the last state of the chain. Therefore if the greatest common divisor of $\{k : a_k^{(j)}(l) > 0\}$ is equal to 1 then the chain is aperiodic.

Proposition 4.3. *Let $X_{\mathbf{n}}$ be an array-valued random field satisfying* (10). *Assume that* (2) *is valid. Then for the process $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$ the equation*

$$\operatorname{vec}\left[E(X_{\mathbf{n}+\mathbf{t}}|\mathcal{F}_{\mathbf{n}})\right] = \left[A_d(n_d + t_d, n_d) \otimes \cdots \otimes A_2(n_2 + t_2, n_2) \otimes A_1(n_1 + t_1, n_1)\right] \operatorname{vec}(X_{\mathbf{n}}) \tag{11}$$

holds, where

$$A_j(n_j + t_j, n_j) = A_j^{(n_j+t_j)} A_j^{(n_j+t_j-1)} \cdots A_j^{(n_j+1)} \tag{12}$$

for every $n_j > m,\ j = 1, \dots, d$ and $\mathbf{n} \in \mathbb{N}^d$, $\mathbf{t} \in \mathbb{N}_0^d$.

Above and in the following $A_j(n_j, n_j) = I$ (the unit matrix).

Generalizing property (11), we get the following notion.

Definition 4.4. An array-valued process $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}),\ \mathbf{n} \in \mathbb{N}^{\mathbf{d}}$, is called an *A-martingale field* if

1) $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and integrable for every $\mathbf{n} \in \mathbb{N}^d$,
2) equation (11) is satisfied for every $\mathbf{n}, \mathbf{t} \in \mathbb{N}^d$, where the matrices $A_j(n_j + t_j, n_j)$ are given by (12). (All matrices $A_j^{(l_j)}$ considered are nonrandom and of type $m \times m$.)

For the A-martingale field $X_{\mathbf{n}}$ let $\Delta_{\mathbf{n}}$ denote the martingale difference type field.

$$\Delta_{\mathbf{n}} = \sum (-1)^{\sum_{k=1}^d \varepsilon_k} \left(E(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{c}})\right), \tag{13}$$

where $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d, \mathbf{c} = (c_1, \dots, c_d)$ and $c_k = \varepsilon_k(n_k - 1) + (1 - \varepsilon_k)n_k$ for every $k = 1, \dots, d$, $\mathbf{n} \geq \mathbf{1}$. We sum for all values of $\varepsilon_k = 0$ or $\varepsilon_k = 1, k = 1, \dots, d$. In (13) we consider $E\left(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{c}}\right)$ and $(X_{\mathbf{c}})$ being equal to 0 if $\mathbf{c} \in \mathbb{N}_0^d \setminus \mathbb{N}^d$.

If (1) is true, then $\Delta_{\mathbf{n}}$ is a martingale difference, i.e. $\Delta_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and $E(\Delta_{\mathbf{n}}|\mathcal{F}_{\mathbf{m}}) = 0$ if $\mathbf{m} \leq \mathbf{n},\ \mathbf{m} \neq \mathbf{n}$.

If (2) is true, then by (11), we have

$$\operatorname{vec}(\Delta_{\mathbf{n}}) = \sum (-1)^{\sum_{k=1}^d \varepsilon_k} \left[\left[\varepsilon_d A_d^{(n_d)} + (1 - \varepsilon_d)I\right] \otimes \cdots \otimes \left[\varepsilon_1 A_1^{(n_1)} + (1 - \varepsilon_1)I\right]\right] \operatorname{vec}(X_{\mathbf{c}}). \tag{14}$$

We remark that if $\xi_{\mathbf{n}}$ is an autoregressive martingale field and $X_{\mathbf{n}}$ is the corresponding A-martingale field, then

$$\operatorname{vec}(\Delta_{\mathbf{n}}) = \begin{pmatrix} \mathit{0} \\ \delta_{\mathbf{n}} \end{pmatrix},$$

where $\delta_{\mathbf{n}} = \sum (-1)^{\sum_{k=1}^{n} \varepsilon_k} E(\xi_{\mathbf{n}}|\mathcal{F}_{\mathbf{c}})$ and $0 \in \mathbb{N}_0^{m^d-1}$.

Proposition 4.5. *Assume* (2). *For the A-martingale field $X_{\mathbf{n}}$, we have the representation:*

$$\operatorname{vec}(X_{\mathbf{n}}) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} [A_d(n_d, k_d) \otimes \cdots \otimes A_1(n_1, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}}), \tag{15}$$

where $A_j(k_j, k_j) = I, \ j = 1, \dots, d$.

Proof. For fixed $\mathbf{n}$ consider $Z_{\underline{\mathbf{k}}} = [A_d(n_d, k_d) \otimes \cdots \otimes A_1(n_1, k_1)] \operatorname{vec}(X_{\mathbf{k}}), \ \mathbf{k} \leq \mathbf{n}$. Then $Z_{\mathbf{n}} = X_{\mathbf{n}}$. Moreover, (15) contains the summation of the difference sequence of the sequence $Z_{\mathbf{k}}, \ \mathbf{k} \leq \mathbf{n}$. $\square$

In the special cases $d = 1, 2$ we get the models studied in [Faz87] and [Faz88]. If $d = 1$, then

$$X_n = \begin{pmatrix} \xi_{n-m+1} \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{pmatrix}.$$

In this case, by (9),

$$E\left(X_n|\mathcal{F}_{n-1}\right) = \begin{pmatrix} \xi_{n-m+1} \\ \vdots \\ \xi_{n-1} \\ a_1\xi_{n-1} + \cdots + a_m\xi_{n-m} \end{pmatrix} = AX_{n-1}.$$

This model was considered in [Faz87].

Consider $d = 2$. Let $\mathbf{n} = (n_1, n_2), \mathbf{t} = (t_1, t_2)$. Then

$$X_{\mathbf{n}} = \begin{pmatrix} \xi_{n_1-m+1,n_2-m+1} & \cdots & \xi_{n_1-m+1,n_2} \\ \xi_{n_1-m+2,n_2-m+1} & \cdots & \xi_{n_1-m+2,n_2} \\ \vdots & \ddots & \vdots \\ \xi_{n_1,n_2-m+1} & \cdots & \xi_{n_1,n_2} \end{pmatrix}.$$

Using $[A_2 \otimes A_1] \operatorname{vec}(X_{\mathbf{n}}) = \operatorname{vec}(A_1 X_{\mathbf{n}} A_2^\top)$, and

$$\operatorname{vec}[E(X_{\mathbf{n+t}}|\mathcal{F}_{\mathbf{n}})] = [A_2(n_2 + t_2, n_2) \otimes A_1(n_1 + t_1, n_1)] \operatorname{vec}(X_{\mathbf{n}}),$$

we get

$$E(X_{\mathbf{n+t}}|\mathcal{F}_{\mathbf{n}}) = A_1(n_1 + t_1, n_1) X_{\mathbf{n}} A_2^\top(n_2 + t_2, n_2).$$

Therefore we obtain the model studied in [Faz88].

5. Convergence of A-martingale fields

In this section, we prove convergence theorems for A-martingale fields under the following conditions. Suppose that

$$A_j(i_j + t_j, i_j) \to A_j(\infty, i_j), \text{ as } t_j \to \infty, \text{ for every } i_j, j \in \mathbb{N} \tag{16}$$

and that the convergence is "fast" in the following sense:

$$\|A_j(\infty, i_j) - A_j(i_j + t_j, i_j)\| \leq c_{t_j}^{(j)}, \qquad \forall i_j, j \in \mathbb{N}, \tag{17}$$

where $\sum_{t_j=1}^{\infty} c_{t_j}^{(j)} < \infty$ for every j.

Let the norm of the matrix $A = (a_{ij})$ be $\|A\| = \sqrt{\sum_i \sum_j a_{ij}^2}$.

We shall use the following properties of this norm.

1) $\|A \cdot B\|^2 = \sum_i \sum_k \left(\sum_j a_{ij} \cdot b_{jk} \right)^2 \leq \sum_i \sum_k \left(\sum_j a_{ij}^2 \cdot \sum_j b_{jk}^2 \right) = \|A\|^2 \cdot \|B\|^2$. In particular, $\|A \cdot \mathbf{v}\| \leq \|A\| \cdot \|\mathbf{v}\|$, for every $\mathbf{v} \in \mathbb{R}^n$.
2) $\|A\| \geq 0$, $\|A\| = 0 \iff A \equiv 0$,
3) $\|\lambda A\| = |\lambda| \cdot \|A\|$,
4) $\|A + B\| \leq \|A\| + \|B\|$.

It is easy to see that $\|A\|^2 = \mathrm{tr}(A^\top \cdot A)$.

The norm of the Kronecker product of the matrixes A and B is the following

$$\|A \otimes B\|^2 = \mathrm{tr}\left[(A \otimes B)^\top (A \otimes B)\right] = \mathrm{tr}\left[(A^\top \otimes B^\top)(A \otimes B)\right] =$$

$$= \mathrm{tr}(A^\top A)\mathrm{tr}(B^\top B) = \|A\|^2 \|B\|^2.$$

For the limit matrices $A_j(\infty, k_j) = \lim_{t_j \to \infty} A_j(k_j + t_j, k_j)$, we assume that there exists a positive number C such that

$$\|[A_d(\infty, k_d) \otimes \cdots \otimes A_1(\infty, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}})\| \geq C \|\operatorname{vec}(\Delta_{\mathbf{k}})\|, \tag{18}$$

for every $\mathbf{k} = (k_1, \dots, k_d)$.

For $S \subseteq \{1, \dots, d\}$ denote by $\mathbf{k}_S$ the coordinates of $\mathbf{k} \in \mathbb{N}^d$ with indices in S and with $\mathbf{k}_{\overline{S}}$ the complement of coordinates. We shall use the following condition. For arbitrary $S \subseteq \{1, \dots, d\}$ and arbitrary $\mathbf{n}$

$$\begin{aligned} &\left\| \sum_{\mathbf{k}_S \leq \mathbf{n}_S} [A_d(\infty, k_d) \otimes A_{d-1}(\infty, k_{d-1}) \otimes \cdots \otimes A_1(\infty, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}}) \right\| \geq \\ &\geq C \left\| \sum_{\mathbf{k}_S \leq \mathbf{n}_S} [D_{k_d} \otimes D_{k_{d-1}} \otimes \cdots \otimes D_{k_1}] \operatorname{vec}(\Delta_{\mathbf{k}}) \right\|, \end{aligned} \tag{19}$$

where the matrix $D_{k_l} = A_l(\infty, k_l)$ if $l \in S$ and $D_{k_l} = I$ if $l \notin S$.

The behaviour of $X_{\mathbf{n}}$ is closely related to a certain martingale $Y_{\mathbf{n}}$. This martingale is called the accompanying martingale of $X_{\mathbf{n}}$, and it is defined by the equation

$$\operatorname{vec}(Y_{\mathbf{n}}) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} [A_d(\infty, k_d) \otimes \cdots \otimes A_2(\infty, k_2) \otimes A_1(\infty, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}})$$

for every $\mathbf{n} \in \mathbb{N}^d$. We know, that under condition (1), $\Delta_{\mathbf{n}}$ is a martingale difference. Therefore, if (1) is satisfied, then $Y_{\mathbf{n}}$ is a martingale.

Lemma 5.1. *Assume that* (2) *and* (16) *are satisfied. If* $\sup_{\mathbf{n}} E\varphi(\|\operatorname{vec}(X_{\mathbf{n}})\|) < C < \infty$*, where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a convex non-decreasing function, then for the accompanying martingale* $\sup_{\mathbf{n}} E\varphi(\|\operatorname{vec}(Y_{\mathbf{n}})\|) < C$ *holds as well.*

Proof. Let

$$\operatorname{vec}(Y_{\mathbf{i}}^{\mathbf{t}}) = \sum_{k_1=1}^{i_1} \sum_{k_2=1}^{i_2} \cdots \sum_{k_d=1}^{i_d} [A_d(t_d, k_d) \otimes \cdots \otimes A_2(t_2, k_2) \otimes A_1(t_1, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}}),$$

where $\mathbf{t} = (t_1, \dots, t_d)$ is fixed, $\mathbf{t} \geq \mathbf{k}$, while $\mathbf{i} = (i_1, \dots, i_d)$. As $\Delta_{\mathbf{k}}$ is a martingale difference, it is easy to see that

$$(Y_{\mathbf{i}}^{\mathbf{t}}, \mathcal{F}_{\mathbf{i}}), \quad \mathbf{1} \leq \mathbf{i} \leq \mathbf{t},$$

is a martingale. Since $\varphi(\|\operatorname{vec}(Y_{\mathbf{i}}^{\mathbf{t}})\|)$ is a real submartingale, we have, by Proposition 4.5,

$$E(\varphi(\|\operatorname{vec}(Y_{\mathbf{i}}^{\mathbf{t}})\|) \leq E(\varphi(\|\operatorname{vec}(Y_{\mathbf{t}}^{\mathbf{t}})\|) = E\varphi(\|\operatorname{vec}(X_{\mathbf{t}})\|) < C,$$

for every $\mathbf{i} \leq \mathbf{t}$. On the other hand, $Y_{\mathbf{i}}^{\mathbf{t}} \to Y_{\mathbf{i}}$, as $\mathbf{t} \to \infty$. Thus, by Fatou's lemma, $E\varphi(\|\operatorname{vec}(Y_{\mathbf{i}})\|) < C$ for every $\mathbf{i} \in \mathbb{N}^d$. □

Theorem 5.2. *Assume that the A-martingale field* $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$*, satisfies* (2)*,* (17)*,* (18) *and* (19)*. If*

$$\sup_{\mathbf{k} \in \mathbb{N}^d} E\|\operatorname{vec}(X_{\mathbf{k}})\| \left[\log^+(\|\operatorname{vec}(X_{\mathbf{k}})\|)\right]^{d-1} < \infty, \tag{20}$$

then $X_{\mathbf{n}}$ *converges a.s. as* $n_j \to \infty$ *for all* j*. If, moreover,* $d \geq 2$ *then* $X_{\mathbf{n}}$ *converges in* $\mathcal{L}_1$*, as* $n_j \to \infty$ *for all* j*.*

Proof. Let $Y_{\mathbf{n}}$ be the accompanying martingale of $X_{\mathbf{n}}$.

$$\operatorname{vec}(Y_{\mathbf{n}}) = \sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} \underbrace{[A_d(\infty, k_d) \otimes \cdots \otimes A_1(\infty, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}})}_{\Delta_{\mathbf{k}}^{(Y)}}$$
$$= \sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} \Delta_{\mathbf{k}}^{(Y)},$$

where $\Delta_{\mathbf{k}}^{(Y)}$ is the difference of the martingale $Y_{\mathbf{n}}$. By Lemma 5.1, condition (20) is satisfied for $Y_{\mathbf{n}}$, namely $\sup_{\mathbf{n}} E\|\operatorname{vec}(Y_{\mathbf{n}})\| \left[\log^+(\|\operatorname{vec}(Y_{\mathbf{n}})\|)\right]^{d-1} < \infty$. First, we prove the a.s. convergence. By Theorem 3.1 $Y_{\mathbf{n}} \to Y$ a.s., as $\mathbf{n} \to \infty$. We show that $X_{\mathbf{n}} \to Y$ a.s.. We mention that $Y_{\mathbf{n}}$ converges uniformly if at least one of the coordinates of $\mathbf{n}$ tends to infinity. From here $\|\Delta_{\mathbf{n}}^{(Y)}\| < \varepsilon$, if at least one coordinate of $\mathbf{n}$ is greater than n_ε. Therefore $\|\Delta_{\mathbf{n}}^{(Y)}\|$ is bounded. Therefore, by (18), $\Delta_{\mathbf{k}} \to 0$ a.s. and $\{\Delta_{\mathbf{k}};\ \mathbf{k} \in \mathbb{N}^d\}$ is bounded. Proposition 4.5 and the definition of $Y_{\mathbf{n}}$ imply that

$$\|\operatorname{vec}(X_{\mathbf{n}}) - \operatorname{vec}(Y_{\mathbf{n}})\| = \Bigg\| \sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} [A_d(n_d, k_d) \otimes \cdots \otimes A_1(n_1, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}})$$
$$- \sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} [A_d(\infty, k_d) \otimes \cdots \otimes A_1(\infty, k_1)] \operatorname{vec}(\Delta_{\mathbf{k}}) \Bigg\|$$
$$= \left\| \sum_{G_1,\dots,G_d} \left(\sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} [G_d \otimes \cdots \otimes G_1] \operatorname{vec}(\Delta_{\mathbf{k}}) \right) \right\|, \tag{21}$$

where $G_i = A_i(\infty, k_i)$ or $G_i = A_i(n_i, k_i) - A_i(\infty, k_i)$ and at least one G_i is equal to the difference. (So the sum $\sum_{G_1,\dots,G_d}$ contains $2^d - 1$ terms.)

Consider a particular term of the above sum with only one difference. By condition (17), we obtain

$$\left\| \sum_{k_d=1}^{n_d} \cdots \sum_{k_1=1}^{n_1} \Big[\big(A_d(n_d, k_d) - A_d(\infty, k_d)\big) \otimes A_{d-1}(\infty, k_{d-1}) \otimes \cdots \otimes A_1(\infty, k_1) \Big] \operatorname{vec}(\Delta_{\mathbf{k}}) \right\|$$
$$\leq c \sum_{k_d=1}^{n_d} c^{(d)}_{\underbrace{n_d - k_d}_{l_d}} \left\| \sum_{k_{d-1}=1}^{n_{d-1}} \cdots \sum_{k_1=1}^{n_1} [I \otimes A_{d-1} \otimes \cdots \otimes A_1] \operatorname{vec}(\Delta_{\mathbf{k}}) \right\|$$
$$= c \sum_{l_d=0}^{n_d-1} c^{(d)}_{l_d} \left\| \sum_{k_{d-1}=1}^{n_{d-1}} \cdots \sum_{k_1=1}^{n_1} [I \otimes A_{d-1} \otimes \cdots \otimes A_1] \operatorname{vec}(\Delta_{k_1,\dots,k_{d-1},(n_d - l_d)}) \right\|$$
$$= c \sum_{l_d=0}^{v} c^{(d)}_{l_d} \left\| \sum_{k_{d-1}=1}^{n_{d-1}} \cdots \sum_{k_1=1}^{n_1} [I \otimes A_{d-1} \otimes \cdots \otimes A_1] \operatorname{vec}(\Delta_{k_1,\dots,k_{d-1},(n_d - l_d)}) \right\| \tag{22}$$
$$+ c \sum_{l_d=v}^{n_d-1} c^{(d)}_{l_d} \left\| \sum_{k_{d-1}=1}^{n_{d-1}} \cdots \sum_{k_1=1}^{n_1} [I \otimes A_{d-1} \otimes \cdots \otimes A_1] \operatorname{vec}(\Delta_{k_1,\dots,k_{d-1},(n_d - l_d)}) \right\|, \tag{23}$$

where v is an appropriately chosen fixed integer.

Now we consider the limiting behaviour of these expressions when $n_d \to \infty$. To this end we shall use that

$$\sum_{k_{d-1}=1}^{n_{d-1}} \cdots \sum_{k_1=1}^{n_1} [I \otimes A_{d-1} \otimes \cdots \otimes A_1] \operatorname{vec}(\Delta_{k_1,\dots,k_{d-1},s}) \to 0 \tag{24}$$

uniformly, if $s \to \infty$.

Let $\varepsilon > 0$ be arbitrary. If v is sufficiently large, then by (17), $\displaystyle\sum_{l=v}^{n_d-1} c_l^{(d)} < \varepsilon$. By (24), the term $\|\dots\|$ of (23) is bounded. So the expression in (23) is less than $c\varepsilon$, where $c < \infty$. Also by (24), the expression in (22) converges to 0, as $n_d \to \infty$. Now we prove (24). By (19), the left hand side of (24) is less than

$$\frac{1}{C}\left\|\sum_{k_{d-1}=1}^{n_{d-1}}\cdots\sum_{k_1=1}^{n_1}[A_d(\infty,k_d)\otimes\cdots\otimes A_1(\infty,k_1)]\operatorname{vec}(\Delta_{k_1,\ldots,k_{d-1},s})\right\|.$$

It is a difference of $Y_{\mathbf{n}}$ according to the last coordinate of $\mathbf{n}$. So it converges to 0, as $s \to \infty$.

Now, consider another term from (21) that contains two differences:

$$\begin{aligned}
&\left\|\sum_{k_d=1}^{n_d}\cdots\sum_{k_1=1}^{n_1}\{A_d(\infty,k_d)\otimes\cdots\otimes A_3(\infty,k_3)\otimes\right.\\
&\left.\otimes[A_2(n_2,k_2)-A_2(\infty,k_1)]\otimes[A_1(n_1,k_1)-A_1(\infty,k_1)]\}\operatorname{vec}(\Delta_{\mathbf{k}})\right\|\\
&\le c\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}c_{\underbrace{n_1-k_1}_{l_1}}c_{\underbrace{n_2-k_2}_{l_2}}\left\|\sum_{k_3=1}^{n_3}\cdots\sum_{k_d=1}^{n_d}[A_d\otimes\cdots\otimes A_3\otimes I\otimes I]\operatorname{vec}(\Delta_{\mathbf{k}})\right\|\\
&= c\sum_{l_1=0}^{n_1-1}\sum_{l_2=0}^{n_2-1}c_{l_1}c_{l_2}\underbrace{\left\|\sum_{k_3=1}^{n_3}\cdots\sum_{k_d=1}^{n_d}[A_d\otimes\cdots\otimes A_3\otimes I\otimes I]\operatorname{vec}(\Delta_{(n_1-l_1),(n_2-l_2),k_3,\ldots,k_d})\right\|}_{f_{n_1-l_1,n_2-l_2,n_3,\ldots,n_d}}\\
&\le c\sum_{l_1=0}^{v_1}\sum_{l_2=0}^{v_2}\underbrace{c_{l_1}c_{l_2}}_{\text{bounded}}\underbrace{f_{n_1-l_1,n_2-l_2,n_3,\ldots,n_d}}_{\to 0\ \textit{if}\ n_1,n_2\to\infty}+c\underbrace{\sum_{l_1=v_1}^{n_1-1}c_{l_1}}_{\le\varepsilon_1}\underbrace{\sum_{l_2=0}^{n_2-1}c_{l_2}}_{\text{bounded}}\underbrace{f_{n_1-l_1,n_2-l_2,n_3,\ldots,n_d}}_{\text{bounded}}\\
&\quad+c\underbrace{\sum_{l_2=v_2}^{n_2-1}c_{l_2}}_{\le\varepsilon_2}\underbrace{\sum_{l_1=0}^{n_1-1}c_{l_1}}_{\text{bounded}}\underbrace{f_{n_1-l_1,n_2-l_2,n_3,\ldots,n_d}}_{\text{bounded}}\to 0,\ \text{if } n_1,n_2\to\infty.
\end{aligned}$$

Above $\varepsilon_1 > 0, \varepsilon_2 > 0$ are arbitrary and v_1, v_2 are chosen to be large enough. Similarly for more differences.

Finally, if $d \ge 2$, the assumptions of the theorem imply the uniform integrability of $X_{\mathbf{n}}$, so $X_{\mathbf{n}}$ converges in $\mathcal{L}_1$, too. $\square$

Theorem 5.3. *Suppose that for the A-martingale field* $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d$, *condition* (2), (17) *and* (19) *hold, and*

$$\|A_j(i_j,u_j)\| < K < \infty \tag{25}$$

if $i_j > u_j,\ j = 1,\ldots,d$. *If* $\sup_{\mathbf{n}} E\|\operatorname{vec}(X_{\mathbf{n}})\|^{\alpha} < \infty$, *where* $\alpha > 1$, *then* $X_{\mathbf{n}}$ *converges in* $\mathcal{L}_\alpha$, *as* $\mathbf{n} \to \infty$.

Proof. By Lemma 5.1, $\sup_{\mathbf{n}} E\|\operatorname{vec}(Y_{\mathbf{n}})\|^{\alpha} < \infty$. Therefore, according to Theorem 4.6 of [Faz83], $Y_{\mathbf{n}}$ converges in $\mathcal{L}_\alpha$ if one of the coordinates of $\mathbf{n}$ goes to infinity. The main step of our proof is the following sequence of inequalities:

$$\begin{aligned}
W_{\mathbf{k}}^{\mathbf{i}} &= E\left\|\sum_{u_1=k_1}^{i_1}\sum_{u_2=k_2}^{i_2}\cdots\sum_{u_d=k_d}^{i_d}[A_d(i_d,u_d)\otimes A_{d-1}(i_{d-1},u_{d-1})\otimes\cdots\otimes A_1(i_1,u_1)]\operatorname{vec}(\Delta_{\mathbf{u}})\right\|^{\alpha}\\
&\le C_1 E\left(\sum_{u_1=k_1}^{i_1}\sum_{u_2=k_2}^{i_2}\cdots\sum_{u_d=k_d}^{i_d}\|\operatorname{vec}(\Delta_{\mathbf{u}})\|^2\right)^{\frac{\alpha}{2}}\\
&\le C_2 E\left\|\sum_{u_1=k_1}^{i_1}\sum_{u_2=k_2}^{i_2}\cdots\sum_{u_d=k_d}^{i_d}\operatorname{vec}(\Delta_{\mathbf{u}})\right\|^{\alpha}\\
&\le C_3 E\left\|\sum_{u_1=k_1}^{i_1}\sum_{u_2=k_2}^{i_2}\cdots\sum_{u_d=k_d}^{i_d}[A_d(\infty,u_d)\otimes A_{d-1}(\infty,u_{d-1})\otimes\cdots\otimes A_1(\infty,u_1)]\operatorname{vec}(\Delta_{\mathbf{u}})\right\|^{\alpha}\\
&= C_3\left\|\operatorname{vec}\left[\sum(-1)^{\sum_{z=1}^{d}\varepsilon_z}Y_{\mathbf{c}}\right]\right\|^{\alpha},
\end{aligned}\tag{26}$$

for $i_j > k_j,\ j = 1, \dots, d$, where $\mathbf{c} = \varepsilon_z k_z + (1-\varepsilon_z) i_z,\ z = 1, \dots, d$. We sum for all values of $\varepsilon_z = 0$ or $1,\ z = 1, \dots, d$. These inequalities are consequences of Burkholder's inequalities (Lemma 3.2). In the first inequality, we applied also condition (25) and in the third one condition (19).

Consequently, $W_{\mathbf{k}}^{\mathbf{i}} \to 0$ if at least one of the coordinates of $\mathbf{k}$ and $\mathbf{i}$ tends to infinity.

Let $Y_\infty = \lim_{\mathbf{k}\to\infty} Y_{\mathbf{k}}$. We show that $X_{\mathbf{n}} \to Y_\infty$ in $\mathcal{L}_\alpha$. For $\mathbf{i} \geq \mathbf{k}$

$$\|X_{\mathbf{i}} - Y_\infty\|_{\mathcal{L}_\alpha} \leq \|Y_{\mathbf{k}} - Y_\infty\|_{\mathcal{L}_\alpha} +$$

$$+ \left\| \sum_{u_1=1}^{k_1} \sum_{u_2=1}^{k_2} \cdots \sum_{u_d=1}^{k_d} [A_d(i_d, u_d) \otimes A_{d-1}(i_{d-1}, u_{d-1}) \otimes \cdots \otimes A_1(i_1, u_1)] \operatorname{vec}(\Delta_{\mathbf{u}}) - Y_{\mathbf{k}} \right\|_{\mathcal{L}_\alpha} +$$

$$+ \left\| \sum_{\mathbf{u}\leq\mathbf{i}, \mathbf{u}\not\leq\mathbf{k}} [A_d(i_d, u_d) \otimes A_{d-1}(i_{d-1}, u_{d-1}) \otimes \cdots \otimes A_1(i_1, u_1)] \operatorname{vec}(\Delta_{\mathbf{u}}) \right\|_{\mathcal{L}_\alpha}.$$

Let $\varepsilon > 0$ be arbitrary. As $Y_{\mathbf{k}} \to Y_\infty$, and by (26), one can fix $\mathbf{k}$ such that the first and the third terms in the above expression are less than ε. If $\mathbf{k}$ is fixed, then the second term tends to zero, as $\mathbf{i} \to \infty$. □

6. Convergence of autoregressive martingale fields

Theorem 6.1. *Let $(\xi_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d$, be a homogeneous autoregressive martingale field and suppose that (2) is satisfied. Assume, for each $j = 1, \dots, d$, $a_m^{(j)} > 0$, and the greatest common divisor of $\{k : 1 \leq k \leq m, a_k^{(j)} > 0\}$ is equal to 1.*

a) *If $\sup_{\mathbf{n}} E|\xi_{\mathbf{n}}| \left[\log^+ |\xi_{\mathbf{n}}|\right]^{d-1} < \infty$, then $\xi_{\mathbf{n}}$ converges a.s, if moreover, $d \geq 2$, then ξ_n converges in $\mathcal{L}_1$, as $\mathbf{n} \to \infty$.*

b) *Let be $\alpha > 1$. If $\sup_{\mathbf{n}} E|\xi_{\mathbf{n}}|^\alpha < \infty$, then $\xi_{\mathbf{n}}$ converges in $\mathcal{L}_\alpha$ (and a.s.), as $\mathbf{n} \to \infty$.*

Proof. In Section 4, we have constructed the A-martingale $X_{\mathbf{n}}$, the matrices $A_z^{(i_z)}$, and the martingale difference $\Delta_{\mathbf{n}}$ corresponding to $\xi_{\mathbf{n}}$. Because of the conditions of our theorem, $A_z = A_z^{(i_z)}$ is the transition matrix of a non-decomposable acyclic Markov-chain ($z = 1, \dots, d$). The elements of the matrices $A_z(i_z + t_z, i_z) = (A_z)^{t_z}$ converge exponentially fast to the elements of the matrix $A_z(\infty) = A_z(\infty, i_z) = (a_{kj})_{k,j=1}^m$, as $t_z \to \infty$, where $a_{kj} = b_j$ $(k, j = 1, \dots, m)$ is the stationary distribution of the chain ([Sen81]). The system of equation of stationarity is the following: $\mathbf{b}^\top = \mathbf{b}^\top A$ with $A = A_z$, for any $z = 1, \dots, d$, namely

$$(b_1, b_2, \dots, b_m) = (b_1, b_2, \dots, b_m) \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_m & a_{m-1} & \cdots & \cdots & a_1 \end{pmatrix}.$$

Therefore $b_1 = a_m b_m,\ b_2 = b_1 + a_{m-1} b_m\ , \dots,\ b_m = b_{m-1} + a_1 b_m$. We obtain $b_1 = a_m b_m,\ b_2 = (a_m + a_{m-1}) b_m,\ , \cdots,\ b_{m-1} = (a_m + \cdots + a_2) b_m,\ b_m = (a_m + \cdots + a_1) b_m$.

So $1 = b_1 + \cdots + b_m = (m a_m + (m-1) a_{m-1} + \cdots + a_1) b_m$. Hence

$$b_m = \frac{1}{\sum_{i=1}^m i a_i}$$

and

$$b_j = \left(\sum_{l=0}^{j-1} a_{m-l} \right) b_m = \frac{\sum_{i=m-j+1}^m a_i}{\sum_{i=1}^m i a_i}, \quad j = 1, \dots, m-1,$$

is the unique stationary distribution of the Markov chain. Therefore condition (17), (19) and (25) hold. Now, it is easy to see that Theorem 5.2 and Theorem 5.3 imply the desired result. □

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