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ABSOLUTE CONVERGENCE OF THE DOUBLE SERIES OF FOURIER – HAAR COEFFICIENTS

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Abstract. In this paper we study the absolute convergence of the double series of Fourier-Haar coefficients of the class PBV_p.

1. Introduction

The problems related to the behaviour of single series of Fourier-Haar are well studied [9]. Namely, P. Ulianov [14] and B. Golubov [8] received the results related to the problems of absolute convergence of the series of Fourier–Haar coefficients. Some generalization of these results related were received by Z. Chanturia [3], T. Akhobadze [1], U. Goginava [7] and by the author [2]. In the term of modulus of smoothness the problem of absolute convergence of the series of Fourier-Haar coefficients was studied by V. Krotov [10]. Multidimensional analogies corresponding to the results of V. Krotov were formulated in the works of V. Tsagareishvili [13] and G. Tabatadze [12].

The estimates of Fourier coefficients of functions of bounded fluctuation with respect to Walsh system were studied in [11] and with respect to Vilenkin system were studied by G. Gát and R. Toledo [4].

We consider the double Haar system $\{\chi_n(x) \times \chi_m(y) : n, m = 0, 1, 2, \ldots\}$ on the unit square $I^2 = [0,1] \times [0,1]$. As usual, $L_p\left(I^2\right)$ $(p \geqslant 1)$ denotes the set of all measurable functions defined on I^2, for which

$$\|f\|_p = \left(\int_0^1 \int_0^1 |f(x,y)|^p \, dxdy\right)^{\frac{1}{p}} < \infty$$

and $C\left(I^2\right)$ is the space of continuous functions on I^2 equipped with maximum norm

$$\|f\|_c = \max_{x,y \in I} |f(x,y)| .$$

If $f \in L\left(I^2\right)$, then

$$C_{n,m}(f) = \int_0^1 \int_0^1 f(x,y)\, \chi_n(x)\, \chi_m(y)\, dxdy$$

is the (n,m)th Fourier-Haar coefficient of f.

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We say that $f \in \operatorname{Lip}\alpha$ on $[0,1]^2$, if

$$\left\| f(\cdot + h, \cdot + \eta) - f(\cdot, \cdot) \right\|_c = O\left(\left(h^2 + \eta^2\right)^{\frac{\alpha}{2}} \right), \alpha \in (0,1].$$

We have the following theorem.

Theorem A ([12])**.** *a) Let $f \in \operatorname{Lip}\alpha$ on $[0,1]^2, \alpha \in (0,1]$. If $\beta > 0$ and $\gamma + 1 < \beta\frac{(\alpha+1)}{2}$, then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{\gamma} \left| C_{n,m}(f) \right|^{\beta} < \infty.$$

b) Let $\gamma + 1 = \beta\frac{(\alpha+1)}{2}$, for some $\alpha \in (0,1)$. Then there exists a function $f_\alpha \in \operatorname{Lip}\alpha$ for which

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{\gamma} \left| C_{n,m}(f_\alpha) \right|^{\beta} = \infty.$$

The case for $\gamma = 0$ was considered earlier by V. Tsagareishvili [13].

Let $f \in L_p\left(I^2\right)$. The partial integrated modulus of continuity are defined by

$$\omega_1(\delta_1, f)_p = \sup\left\{ \left\| f(x+u, y) - f(x,y) \right\|_p : |u| \leqslant \delta_1 \right\},$$
$$\omega_2(\delta_2, f)_p = \sup\left\{ \left\| f(x, y+v) - f(x,y) \right\|_p : |v| \leqslant \delta_2 \right\}.$$

We also use the notion of the mixed integrated modulus of continuity. It is defined as follows

$$\begin{aligned} &\omega_{1,2}(\delta_1, \delta_2, f)_p \\ &\quad = \sup\left\{ \left\| f(x+u, y+v) - f(x+u, y) - f(x, y+v) + f(x,y) \right\|_p : \right. \\ &\qquad\qquad \left. |u| \leqslant \delta_1, |v| \leqslant \delta_2 \right\}, f \in L_p\left(I^2\right). \end{aligned}$$

It is not difficult to show that

$$\omega_{1,2}(\delta_1, \delta_2, f)_p \leqslant 2\sqrt{\omega_1(\delta_1, f)_p}\sqrt{\omega_2(\delta_2, f)_p}. \tag{1}$$

We study the problem of absolute convergence of the series of Fourier-Haar coefficients for the classes of functions with bounded partial p-variations, which were first considered by U. Goginava (see [5] for $p = 1$ and [6] for $p > 1$).

Definition. A function $f\colon I^2 \to R$ is said to be of bounded partial p-variation $\left(f \in \mathrm{PBV}_p\left(I^2\right)\right)$if there exists a constant K such that for any partition

$$\Delta_1 : 0 \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant 1,$$
$$\Delta_2 : 0 \leqslant y_0 < y_1 < y_2 < \ldots < y_m \leqslant 1,$$

we have

$$\begin{aligned} V_1(f)_p &= \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} \left| f(x_i, y) - f(x_{i+1}, y) \right|^p \leqslant K, \\ V_2(f)_p &= \sup_x \sup_{\Delta_2} \sum_{j=0}^{m-1} \left| f(x, y_j) - f(x, y_{j+1}) \right|^p \leqslant K. \end{aligned}$$

Given a function $f(x,y)$, periodic in both variables with period 1. Denote by

$$\Delta_{h_1} f(x,y)_1 = f(x+h_1, y) - f(x,y),$$
$$\Delta_{h_2} f(x,y)_2 = f(x, y+h_2) - f(x,y),$$
$$\begin{aligned} \Delta_{h_1,h_2} f(x,y) &= \Delta_{h_1}\left(\Delta_{h_2} f(x,y)_2\right)_1 = \Delta_{h_2}\left(\Delta_{h_1} f(x,y)_1\right)_2 \\ &= f(x,y) - f(x+h_1, y) - f(x, y+h_2) + f(x+h_1, y+h_2). \end{aligned}$$

2. Main Results

The main results of this paper are presented in the following propositions.

Theorem 1. *Let* $f \in \mathrm{PBV}_p\left(I^2\right)$, $p \geqslant 1$ *and* $\beta > \frac{2p}{1+p}$. *Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left|C_{n,m}\left(f\right)\right|^{\beta} < \infty.$$

Theorem 2. *Let* $f \in \mathrm{PBV}_p\left(I^2\right)$, $p \geqslant 1$ *and* $\alpha < \frac{1}{2p} - \frac{1}{2}$. *Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\left(n+1\right)\left(m+1\right)\right]^{\alpha}\left|C_{n,m}\left(f\right)\right| < \infty.$$

Theorem 3. *Let* $f \in \mathrm{PBV}_p\left(I^2\right)$, $p \geqslant 1$ *and* $\beta > 0, \alpha + 1 < \beta\left(\frac{1}{2p} + \frac{1}{2}\right)$. *Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\left(n+1\right)\left(m+1\right)\right]^{\alpha}\left|C_{n,m}\left(f\right)\right|^{\beta} < \infty.$$

Since $\mathrm{Lip}\,\frac{1}{p} \subset \mathrm{PBV}_p$ in case $p > 1$ the sharpness of Theorems 1-3 follows from the works [12, 13].

3. Auxiliary results

Lemma 1. *Let* $f \in \mathrm{PBV}_p\left(I^2\right)$, $p \geqslant 1$. *Then*

$$\omega_i\left(\delta, f\right)_p \leqslant 3^{\frac{1}{p}}\delta^{\frac{1}{p}}V_i\left(f\right)_p \ \ (i=1,2), 0 < \delta < 1,$$

where $V_i\left(f\right)_p$ *is a partial p-variation of function.*

Using the method of [8], we can easily obtain the validity of Lemma 1.

4. Proof of main results

Proof of Theorem 1. We write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left|C_{n,m}\left(f\right)\right|^{\beta} = \sum_{n=0}^{\infty}\left|C_{n,0}\left(f\right)\right|^{\beta} + \sum_{m=1}^{\infty}\left|C_{0,m}\left(f\right)\right|^{\beta} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|C_{n,m}\left(f\right)\right|^{\beta}.$$

Let $n=2^{n_1}+i,\ m=2^{m_1}+j,\ n_1=0,1,\dots\ i=1,\dots,2^{n_1},\ m_1=0,1,\dots,\ j=1,\dots,2^{m_1}$. Then using Hölder inequality, from the Lemma 1 we get

$$\begin{aligned}
&\sum_{i=1}^{2^{n_1}}\left|C_{2^{n_1}+i,0}(f)\right|^p\\
&=2^{\frac{pn_1}{2}}\sum_{i=1}^{2^{n_1}}\left|\int_0^1\left(\int_{\frac{2i-2}{2^{n_1+1}}}^{\frac{2i-1}{2^{n_1+1}}}\left[f(x,y)-f\left(x+\frac{1}{2^{n_1+1}},y\right)\right]dx\right)dy\right|^p\\
&\leqslant 2^{\frac{pn_1}{2}}\sum_{i=1}^{2^{n_1}}\left[\int_0^1\left(\int_{\frac{2i-2}{2^{n_1+1}}}^{\frac{2i-1}{2^{n_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}}}f(x,y)_1\right|dx\right)dy\right]^p\\
&\leqslant 2^{\frac{pn_1}{2}}\sum_{i=1}^{2^{n_1}}\left[\left(\int_0^1\left(\int_{\frac{2i-2}{2^{n_1+1}}}^{\frac{2i-1}{2^{n_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}}}f(x,y)_1\right|^p dx\right)dy\right)^{\frac{1}{p}}\left(\int_0^1\int_{\frac{2i-2}{2^{n_1+1}}}^{\frac{2i-1}{2^{n_1+1}}}1dxdy\right)^{1-\frac{1}{p}}\right]^p\\
&\leqslant 2^{\frac{pn_1}{2}}\frac{1}{2^{n_1(p-1)}}\int_0^1\int_0^1\left|\Delta_{\frac{1}{2^{n_1+1}}}f(x,y)_1\right|^p dxdy\\
&\leqslant 2^{n_1\left(1-\frac{p}{2}\right)}\omega_1^p\left(\frac{1}{2^{n_1+1}},f\right)_p\leqslant 2^{n_1\left(1-\frac{p}{2}\right)}3\frac{1}{2^{n_1}}V_1^p(f)_p\leqslant c2^{-\frac{n_1p}{2}}V_1^p(f)_p.
\end{aligned}\tag{2}$$

Let $\frac{2p}{1+p}<\beta<p$. Using Hölder inequality, from (2) we get

$$\begin{aligned}
\sum_{i=1}^{2^{n_1}}\left|C_{n_1,0}^{(i)}(f)\right|^\beta&\leqslant\left(\sum_{i=1}^{2^{n_1}}\left|C_{n_1,0}^{(i)}(f)\right|^p\right)^{\frac{\beta}{p}}2^{n_1\left(1-\frac{\beta}{p}\right)}\\
&\leqslant 2^{n_1\left(1-\frac{\beta}{p}\right)}\left(c2^{-\frac{n_1p}{2}}V_1^p(f)_p\right)^{\frac{\beta}{p}}\\
&\leqslant c2^{n_1\left(1-\frac{\beta}{p}\right)}2^{-\frac{n_1\beta}{2}}\leqslant c2^{n_1\left[1-\frac{\beta}{p}-\frac{\beta}{2}\right]}.
\end{aligned}\tag{3}$$

By (3) and from the condition of the Theorem 1 we obtain

$$\sum_{n=2}^{\infty}\left|C_{n,0}(f)\right|^\beta=\sum_{n_1=0}^{\infty}\sum_{i=1}^{2^{n_1}}\left|C_{2^{n_1}+i,0}(f)\right|^\beta\leqslant\sum_{n_1=0}^{\infty}2^{n_1\left[1-\frac{\beta}{p}-\frac{\beta}{2}\right]}<\infty.$$

Analogously, we obtain that

$$\sum_{m=1}^{\infty}\left|C_{0,m}(f)\right|^\beta<\infty,\ \text{for }\beta>\frac{2p}{1+p}.$$

Using Hölder inequality, by (1) and from Lemma 1 we get

$$\sum_{i=0}^{2^{n_1}-1}\sum_{j=0}^{2^{m_1}-1}\left|\int_{\frac{2i}{2^{n_1}}}^{\frac{i+1}{2^{n_1}}}\int_{\frac{2j}{2^{m_1}}}^{\frac{j+1}{2^{m_1}}} f(x,y)\,\chi_{2^{n_1}+i}(x)\,\chi_{2^{m_1}+j}(y)\,dxdy\right|^p$$

$$\leqslant 2^{p\frac{n_1+m_1}{2}}\sum_{i=0}^{2^{n_1}-1}\sum_{j=0}^{2^{m_1}-1}\left[\int_{\frac{2i}{2^{n_1+1}}}^{\frac{2i+1}{2^{n_1+1}}}\int_{\frac{2j}{2^{m_1+1}}}^{\frac{2j+1}{2^{m_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}},\frac{1}{2^{m_1+1}}} f(x,y)\right|dxdy\right]^p$$

$$\leqslant 2^{p\frac{n_1+m_1}{2}}\sum_{i=0}^{2^{n_1}-1}\sum_{j=0}^{2^{m_1}-1}\left[\left(\int_{\frac{2i}{2^{n_1+1}}}^{\frac{2i+1}{2^{n_1+1}}}\int_{\frac{2j}{2^{m_1+1}}}^{\frac{2j+1}{2^{m_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}},\frac{1}{2^{m_1+1}}} f(x,y)\right|^p dxdy\right)^{\frac{1}{p}}\right.$$

$$(4)\qquad \left.\times\left(\int_{\frac{2i}{2^{n_1+1}}}^{\frac{2i+1}{2^{n_1+1}}}\int_{\frac{2j}{2^{m_1+1}}}^{\frac{2j+1}{2^{m_1+1}}} 1dxdy\right)^{1-\frac{1}{p}}\right]^p$$

$$\leqslant 2^{p\frac{n_1+m_1}{2}}\frac{1}{2^{(n_1+m_1)(p-1)}}\int_0^1\int_0^1\left|\Delta_{\frac{1}{2^{n_1+1}},\frac{1}{2^{m_1+1}}} f(x,y)\right|^p dxdy$$

$$\leqslant 2^{(n_1+m_1)\left(1-\frac{p}{2}\right)}\omega_{1,2}^p\left(\frac{1}{2^{n_1+1}},\frac{1}{2^{m_1+1}},f\right)_p$$

$$\leqslant 2^{(n_1+m_1)\left(1-\frac{p}{2}\right)}2^p\omega_1^{\frac{p}{2}}\left(\frac{1}{2^{n_1+1}},f\right)_p\omega_2^{\frac{p}{2}}\left(\frac{1}{2^{m_1+1}},f\right)_p$$

$$\leqslant c2^{(n_1+m_1)\left(1-\frac{p}{2}\right)}\frac{2^p}{2^{\frac{n_1+m_1}{2}}}\leqslant c2^{(n_1+m_1)\left(\frac{1}{2}-\frac{p}{2}\right)}.$$

Let $\frac{2p}{1+p}<\beta<p$. Using Hölder inequality, by (4) we write

$$(5)\qquad \begin{aligned}\sum_{i=0}^{2^{n_1}-1}\sum_{j=0}^{2^{m_1}-1}\left|C_{2^{n_1}+i,2^{m_1}+j}(f)\right|^\beta &\leqslant c2^{(n_1+m_1)\left(1-\frac{p}{2}\right)\frac{\beta}{p}}2^{(n_1+m_1)\left(1-\frac{\beta}{p}\right)}\\ &= c2^{(n_1+m_1)\left[\frac{\beta}{2p}-\frac{\beta}{2}+1-\frac{\beta}{p}\right]}\\ &= c2^{n_1\left[1-\frac{\beta}{2}-\frac{\beta}{2p}\right]}2^{m_1\left[1-\frac{\beta}{2}-\frac{\beta}{2p}\right]}.\end{aligned}$$

By (5) and from the condition of the Theorem 1 we get

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|C_{n,m}(f)\right|^\beta &= \sum_{n_1=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{i=0}^{2^{n_1}-1}\sum_{j=0}^{2^{m_1}-1}\left|C_{2^{n_1}+i,2^{m_1}+j}(f)\right|^\beta\\ &\leqslant c\sum_{n_1=0}^{\infty}2^{n_1\left[1-\frac{\beta}{2}-\frac{\beta}{2p}\right]}\sum_{m_1=0}^{\infty}2^{m_1\left[1-\frac{\beta}{2}-\frac{\beta}{2p}\right]}<\infty.\end{aligned}$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. We write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[(n+1)(m+1)\right]^{\alpha}\left|C_{n,m}(f)\right| = \sum_{n=0}^{\infty}(n+1)^{\alpha}\left|C_{n,0}(f)\right| + \sum_{m=1}^{\infty}(m+1)^{\alpha}\left|C_{0,m}(f)\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[(n+1)(m+1)\right]^{\alpha}\left|C_{n,m}(f)\right|.$$

Let $\beta = 1$. Then from (3) we get

$$\sum_{i=1}^{2^{n_1}}\left(2^{n_1}+i+1\right)^{\alpha}\left|C_{2^{n_1}+i,0}(f)\right| \leqslant c2^{n_1\alpha}\sum_{i=1}^{2^{n_1}}\left|C_{2^{n_1}+i,0}(f)\right| \leqslant c2^{n_1\alpha}2^{n_1\left(\frac{1}{2}-\frac{1}{p}\right)} = c2^{n_1\left(\alpha+\frac{1}{2}-\frac{1}{p}\right)}. \tag{6}$$

By (6) and from the condition of the Theorem 2 we obtain

$$\sum_{n=0}^{\infty}(n+1)^{\alpha}\left|C_{n,0}(f)\right| = \sum_{n_1=0}^{\infty}\sum_{i=1}^{2^{n_1}}\left(2^{n_1}+i+1\right)^{\alpha}\left|C_{2^{n_1}+i,0}(f)\right| \leqslant c\sum_{n_1=0}^{\infty}2^{n_1\alpha}\sum_{i=1}^{2^{n_1}}\left|C_{2^{n_1}+i,0}(f)\right| \leqslant c\sum_{n_1=0}^{\infty}2^{n_1\left(\alpha+\frac{1}{2}-\frac{1}{p}\right)} < \infty.$$

Analogously, we obtain that

$$\sum_{m=1}^{\infty}(m+1)^{\alpha}\left|C_{0,m}(f)\right| < \infty, \text{ for } \alpha < \frac{1}{2p}-\frac{1}{2}.$$

Let $\beta = 1$. Then by (5) and from the condition of the Theorem 2 we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[(n+1)(m+1)\right]^{\alpha}\left|C_{n,m}(f)\right| \leqslant \sum_{n_1=0}^{\infty}\sum_{m_1=0}^{\infty}2^{(n_1+m_1)\alpha}\sum_{i=1}^{2^{n_1}}\sum_{j=1}^{2^{m_1}}\left|C_{2^{n_1}+i,2^{m_1}+j}(f)\right| \leqslant c\sum_{n_1=0}^{\infty}2^{n_1\left(\alpha+\frac{1}{2}-\frac{1}{2p}\right)}\sum_{m_1=0}^{\infty}2^{m_1\left(\alpha+\frac{1}{2}-\frac{1}{2p}\right)} < \infty.$$

The proof of Theorem 2 is complete. □

Combining the methods of Theorems 1-2 we can prove validity of Theorem 3. Observe that the result of this paper can be proved in the same way for dimension more than 2.

References

[1] T. Akhobadze. Generalized BV($P(n) \uparrow \infty, \phi$) class of bounded variation. *Bull. Georgian Acad. Sci.*, 163(3):426–428, 2001.

[2] A. Aplakov. On the absolute convergence of the series of Fourier-Haar coefficients. *Bull. Georgian Acad. Sci.*, 164(2):238–241, 2001.

[3] Z. A. Chanturia. On the absolute convergence of the series of Fourier-Haar coefficients. *Comment. Math. Special Issue*, 2:25–35, 1979.

[4] G. Gát and R. Toledo. Fourier coefficients and absolute convergence on compact totally disconnected groups. *Math. Pannon.*, 10(2):223–233, 1999.

[5] U. Goginava. On the uniform summability of multiple Walsh-Fourier series. *Anal. Math.*, 26(3):209–226, 2000.

[6] U. Goginava. Uniform convergence of N-dimensional trigonometric Fourier series. *Georgian Math. J.*, 7(4):665–676, 2000.

[7] U. Goginava. On the absolute convergence of the series of Fourier-Haar coefficients. *Bull. Georgian Acad. Sci.*, 164(1):21–23, 2001.
[8] B. I. Golubov. On Fourier series of continuous functions with respect to a Haar system. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:1271–1296, 1964. Russian.
[9] B. I. Golubov. Series in the Haar system. In *Mathematical analysis 1970*, pages 109–146. VINITI, Moscow, 1971. Russian.
[10] V. G. Krotov. Fourier coefficients with respect to a certain orthonormal system that forms a basis in the space of continuous functions. *Izv. VUZ. Matematika*, 10(161):33–46, 1975. Russian.
[11] F. Schipp, W. R. Wade, and P. Simon. *Walsh series. An introduction to dyadic harmonic analysis.* Adam Hilger Ltd., Bristol, 1990.
[12] G. Z. Tabatadze. On absolute convergence of Fourier-Haar series. *Bull. Acad. Sci. Georgian SSR*, 103(3):541–543, 1981. Russian.
[13] V. Tsagareishvili. Fourier-Haar coefficients. *Bull. Acad. Sci. Georgian SSR*, 81(1):29–31, 1976. Russian.
[14] P. L. Ulijanov. On Haar series. *Mat. Sb. (N.S.)*, 63 (105):356–391, 1964. Russian.

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