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APPROXIMATION PROPERTIES OF PARTIAL SUMS OF FOURIER SERIES

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Abstract. In this paper we find class of functions for which the Lebesgue estimate can be improved.

Let $C([0,2\pi])$ denote the space of continuous function f with period 2π. If $f \in C([0,2\pi])$ then the function

$$\omega_p(\delta, f) = \sup_x \sup_{|h| \le \delta} |\Delta_p(x; h, f)|, \omega_1(\delta, f) = \omega(\delta, f)$$

is called the modulus of continuity of the function f, where

$$\Delta_1(x; h, f) = f(x+h) - f(x),$$
$$\Delta_{p+1}(x; h, f) = \Delta_p(x+h; h, f) - \Delta_p(x; h, f).$$

Denote by $Lip\alpha$ the class of function $f \in C([0,2\pi])$ for which $\omega(\delta, f) \le c(f)\,\delta^\alpha$ and let $S_n(f, x)$ be the n-th partial sum of the trigonometric Fourier series of the function f.

The estimation of Lebesgue (see [Dz, p. 116], or [Ba, Ch. 1])is well known

$$\|f - S_n(f)\|_C \le c\omega\left(\frac{1}{n}, f\right)\log(n+2).$$

Generalization of this estimation were studied by Chanturia [Ch], Oskolkov [Os], Karchava [Ka]. The questions devoted to estimation the uniform deviation of f from its partial Fourier sums with respect to the Walsh, Vilenkin (bounded and unbounded case) systems were discussed by Fine [Fi], Onnewer [On] Tevzadze [Te], Gát [Gá].

In this paper we consider the following characteristic of function f

$$\varphi_p(n, \delta; f) = \sup_x \sup_{|h| \le \delta} \sum_{i=1}^{n} |\Delta_p(x_i; h, f)|,$$

where $x_i = x + (i-1)h$.

There arises a question: for what subclasses of classes of $C([0,2\pi])$ the Lebesgue estimate can be improved?

We prove that the following are true

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Theorem 1. *Let $f \in C([0, 2\pi])$. Then*

$$\|f - S_n(f)\|_C \leq c(p) \sum_{k=1}^{n} \frac{\varphi_p(k; \pi/n, f)}{k^2}.$$

Corollary 1. *Let the function f has a finite number of intervals of monotonicity on $[0, 2\pi]$ and $f \in \operatorname{Lip}\alpha, 0 < \alpha < 1$. Then*

$$\|f - S_n(f)\|_C \leq \frac{c(f, \alpha)}{n^\alpha}.$$

Corollary 2. *Let the function f has a finite number of intervals of convexity or concavity on $[0, 2\pi]$, then*

$$\|f - S_n(f)\|_C \leq c(f)\, \omega\left(\frac{1}{n}; f\right).$$

Corollary 3. *Let the function f has a finite number of intervals of monotonicity on $[0, 2\pi]$. Then*

$$\|f - S_n(f)\|_C \leq c(f) \frac{1}{n} \int_{1/n}^{\pi} \frac{\omega(t, f)}{t^2} dt.$$

Proof of Theorem 1. Let $T_n(x)$ be Vale-Poisson polynomial which provides best approximation of function f in the space $C([0, 2\pi])$ (see [Dz]), in particular

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_n(t)\, dt, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)|\, dt \leq 1.$$

Denote

$$g(t) := f(x+t) + f(x-t) - 2f(x) - T_n(x+t) - T_n(x-t) + 2T_n(x).$$

It evident that

$$\varphi_p(k, \delta, T_n) \leq c\varphi_p(k, \delta, f)$$

and

$$\sum_{i=k+1}^{n} \frac{(-1)^i}{i} = \frac{(-1)^{k+1}}{2k} + P_k + Q_n \text{ where } P_k \leq \frac{c}{k^2}, Q_n \leq \frac{c}{n}. \tag{1}$$

We write

$$\int_0^\pi g(t) D_n(t)\, dt = \int_0^\pi g(t) \left(D_n(t) - \frac{\sin nt}{t}\right) dt + \int_0^\pi g(t) \frac{\sin nt}{t} dt. \tag{2}$$

Since

$$D_n(t) - \frac{\sin nt}{t}$$

is bounded function and

$$|g(t)| \leq 4E_n(f),$$

we have

$$\left|\int_0^\pi g(t) \left(D_n(t) - \frac{\sin nt}{t}\right) dt\right| \leq cE_n(f). \tag{3}$$

where $E_n(f)$ is best approximation of function f.

On the other hand

$$\left|\int\limits_0^{\pi/n} g(t)\,\frac{\sin nt}{t}dt\right| \le \|g\|_C \int\limits_0^{\pi/n} \frac{\sin nt}{t}dt \le cE_n(f)\,. \tag{4}$$

Combining (2)–(4) we have

$$\int\limits_0^{\pi} g(t)\,D_n(t)\,dt = \int\limits_0^{\pi} g(t)\,\frac{\sin nt}{t}dt + \gamma, \tag{5}$$

where

$$\gamma \le cE_n(f)\,.$$

It is evident that

$$\begin{aligned}
\left|\int\limits_{\pi/n}^{\pi} g(t)\,\frac{\sin nt}{t}dt\right| &= \left|\sum_{k=1}^{n-1}\int\limits_{\pi k/n}^{\pi(k+1)/n} g(t)\,\frac{\sin nt}{t}dt\right| \\
&= \left|\sum_{k=1}^{n-1}\int\limits_0^{\pi/n} g\left(t+\frac{\pi k}{n}\right)\frac{(-1)^k\sin nt}{t+\frac{\pi k}{n}}dt\right| \\
&= \left|\sum_{k=1}^{n-1}\int\limits_0^{\pi} g\left(\frac{u+\pi k}{n}\right)\frac{(-1)^k\sin u}{u+\pi k}du\right| \\
&\le \pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{u+\pi k}\right| \\
&\le \pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\left(\frac{(-1)^k}{u+\pi k}-\frac{(-1)^k}{\pi k}\right)\right| \\
&\quad+\pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{\pi k}\right| \\
&= I+II,
\end{aligned} \tag{6}$$

where

$$\sup_u\left|\sum_{k=1}^{n-1} g\left(\frac{u+\pi k}{n}\right)\frac{(-1)^k}{u+\pi k}\right| = \left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{u_0+\pi k}\right|.$$

It is clear that

$$I \le \|g\|_C \sum_{k=1}^{n-1}\frac{1}{k^2} = O\left(E_n(f)\right). \tag{7}$$

Denote

$$u_k := \frac{u_0+\pi k}{n}.$$

Using Abel transformation and (1), for II we obtain

$$\begin{aligned}
II &= \left|\sum_{k=1}^{n-1} g\left(u_k\right)\frac{(-1)^k}{k}\right| \\
&\le \left|\sum_{k=1}^{n-2}\left(g\left(u_{k+1}\right)-g\left(u_k\right)\right)\sum_{i=k+1}^{n-1}\frac{(-1)^i}{i}\right| \\
&\quad + \frac{1}{\pi}\left|g\left(u_1\right)\sum_{i=1}^{n-1}\frac{(-1)^i}{i}\right| \\
&\le \left|\sum_{k=1}^{n-2}\left(g\left(u_{k+1}\right)-g\left(u_k\right)\right)\left(\frac{(-1)^{k+1}}{2k}+P_k+Q_n\right)\right| \\
&\quad + \left|g\left(u_1\right)\sum_{i=1}^{n-1}\frac{(-1)^i}{i}\right| \\
&\le \left|\sum_{k=1}^{n-2}\left(g\left(u_{k+1}\right)-g\left(u_k\right)\right)\frac{(-1)^{k+1}}{2k}\right| + O\left(E_n\left(f\right)\right) \\
&= \left|\sum_{k=1}^{n-2}\Delta_1\left(u_k;\frac{\pi}{n},g\right)\frac{(-1)^{k+1}}{2k}\right| + O\left(E_n\left(f\right)\right).
\end{aligned}$$

Iterating this inequality we obtain

$$\begin{aligned}
II &\le c\left(p\right)\left|\sum_{k=1}^{n-2}\Delta_p\left(u_k;\frac{\pi}{n},g\right)\frac{(-1)^{k+1}}{2k}\right| + O\left(E_n\left(f\right)\right) \\
&\le c\left(p\right)\sum_{k=1}^{n-2}\frac{\left|\Delta_p\left(u_k;\frac{\pi}{n},g\right)\right|}{k} + O\left(E_n\left(f\right)\right).
\end{aligned}$$

Using Abel transformation we obtain

$$\begin{aligned}
\sum_{k=1}^{n-2}\frac{\left|\Delta_p\left(u_k;\frac{\pi}{n},g\right)\right|}{k} &\le \sum_{k=1}^{n-3}\frac{1}{k^2}\sum_{i=1}^{k}\left|\Delta_p\left(u_i;\frac{\pi}{n},g\right)\right| \\
&\quad + \frac{1}{n-2}\sum_{k=1}^{n-2}\left|\Delta_p\left(u_k;\frac{\pi}{n},g\right)\right|,
\end{aligned}$$

consequently,

$$\begin{aligned}
II &= O\left(\sum_{k=1}^{n}\frac{1}{k^2}\varphi_p\left(k,\frac{\pi}{n},g\right) + O\left(E_n\left(f\right)\right)\right) \\
&= O\left(\sum_{k=1}^{n}\frac{1}{k^2}\varphi_p\left(k,\frac{\pi}{n};f\right)\right).
\end{aligned} \tag{8}$$

Combining (5)–(8) we complete the proof of the Theorem 1. □

Proof of Corollary 1. Since

$$\varphi_1\left(k,\frac{\pi}{n};f\right) \le c\left(f\right)\omega\left(\frac{\pi k}{n};f\right) \le c\left(f\right)\left(\frac{\pi k}{n}\right)^{\alpha},$$

we have

$$\sum_{k=1}^{n}\frac{1}{k^2}\varphi_1\left(k,\frac{\pi}{n};f\right)\leq c\left(f\right)\sum_{k=1}^{n}\frac{\omega\left(\frac{\pi k}{n};f\right)}{k^2}$$
$$\leq\frac{c\left(f\right)}{n^{\alpha}}\sum_{k=1}^{n}\frac{1}{k^{2-\alpha}}\leq\frac{c\left(f,\alpha\right)}{n^{\alpha}}.$$

□

Proof of Corollary 2. Since the function f has a finite number of intervals of convexity or concavity on $[0,2\pi]$ we have

$$\varphi_2\left(k,\delta;f\right)=c\left(f\right)\omega\left(\delta,f\right),$$

consequently from Theorem 1 we get,

$$\left\|f-S_n\left(f\right)\right\|_C\leq c\sum_{k=1}^{n}\frac{1}{k^2}\varphi_2\left(k,\frac{\pi}{n};f\right)\leq c\left(f\right)\omega\left(\frac{1}{n},f\right).$$

□

Proof of Corollary 3. From the fact that the function f has a finite number of intervals of monotonicity on $[0,2\pi]$ we write

$$\varphi_1\left(k,\frac{\pi}{n};f\right)\leq c\left(f\right)\omega\left(\frac{\pi k}{n};f\right),$$

then using Theorem 1 we complete the proof of Corollary 3 in the following way

$$\left\|f-S_n\left(f\right)\right\|_C\leq c\left(f\right)\sum_{k=1}^{n}\frac{1}{k^2}\varphi_1\left(k,\frac{\pi}{n};f\right)\leq c\left(f\right)\sum_{k=1}^{n}\frac{\omega\left(\frac{\pi k}{n};f\right)}{k^2}$$
$$\leq c\left(f\right)\frac{1}{n}\sum_{k=1}^{n}\frac{\omega\left(\frac{k}{n};f\right)}{\left(k/n\right)^2}\frac{1}{n}\leq c\left(f\right)\frac{1}{n}\int_{1/n}^{\pi}\frac{\omega\left(t,f\right)}{t^2}dt.$$

□

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