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CONTINUED FRACTIONS AND THE GAUSS MAP

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Abstract. We discover properties of the Gauss Map and its iterates using continued fractions. In particular, we find all fixed points and show that the graph of an iterate over $\left[0,\frac{1}{2}\right]$ is symmetric to the graph of the next higher iterate over $\left[\frac{1}{2},1\right]$.

1. Introduction and Preliminaries

Following on from Corless [1], we consider the Gauss Map defined on the unit interval by

Definition 1.

$$G(x)=\begin{cases}0, & \text{if } x=0\\ \frac{1}{x}\bmod 1=\operatorname{frac}\left(\frac{1}{x}\right), & \text{if } 0<x\leqslant 1.\end{cases}$$

The notation $\operatorname{frac}\left(\frac{1}{x}\right)$ in Definition 1 is shorthand for the fractional part of $\frac{1}{x}$. This map is shown at Figure 1a.

If $x\in(0,1]$, we can represent it by its continued fraction $\{0;a_1,a_2,\ldots\}$ where each a_i is a positive integer and i is bounded for rational numbers and unbounded for irrational numbers. Hence for $\frac{1}{1+a_1}<x\leqslant\frac{1}{a_1}$,

$$\begin{aligned}G(x)&=\frac{1}{x}-a_1\\&=\{0;a_2,a_3,\ldots\}\end{aligned}\tag{1}$$

and therefore $0\leqslant G(x)<1$. It follows that G is continuous for $\frac{1}{1+a_1}<x\leqslant\frac{1}{a_1}$. Now

$$\lim_{x\to\frac{1}{1+a_1}^-}G(x)=0$$

whereas

$$\lim_{x\to\frac{1}{1+a_1}^+}G(x)=1.$$

Thus G is discontinuous at each of the points $x=\frac{1}{i}$ for $i=1,2,3,\ldots$. It is useful to picture the map as being made up of disjoint truncated parts of the hyperbola, $y=\frac{1}{x}$, displaced vertically downwards by the integer parts, $\left\lfloor\frac{1}{x}\right\rfloor$ (See Figure 1a).

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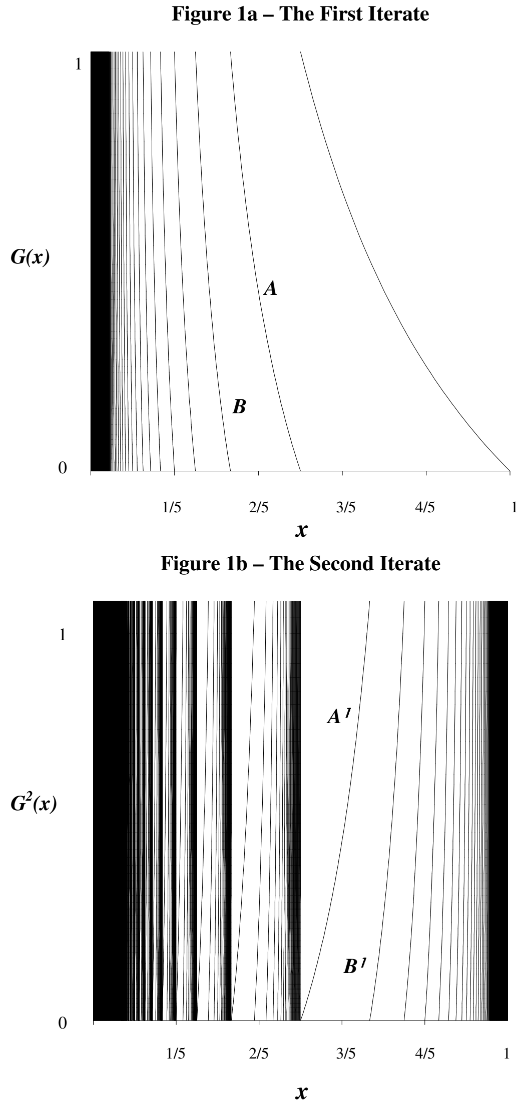
Figure 1: Symmetry Partners between the First and Second Iterates of the Gauss Map



Definition 2. For $x = \{a_0; a_{1,}a_2, \ldots\}$ the n^{th} *total convergent*, C_n is defined as

$$\begin{aligned} C_n &= \frac{p_n}{q_n} \\ &= \{a_0; a_1, a_2, \ldots, a_n\} \end{aligned}$$

where p_n and q_n are the numerator and denominator respectively of an irreducible fraction.

For $n \geq 0$, it can be readily shown (for example Khintchine [2]) that

$$q_n = q_{n-2} + a_n q_{n-1} \tag{2}$$

and

$$p_n = p_{n-2} + a_n p_{n-1} \tag{3}$$

where $p_{-1} = q_{-2} = 1$, $p_{-2} = q_{-1} = 0$. It can also be shown that, for $n \geq -1$,

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n . \tag{4}$$

Using Definition 1, we can readily build up expressions for the n^{th} *iterate*, $G^n(x)$. In particular,

$$\begin{aligned} G^n(0) &= G^{n-1}(G(0)) = G^{n-1}(0) = \ldots = G(0) = 0, \text{ and} \\ G^n(1) &= G^{n-1}(G(1)) = G^{n-1}(0) = \ldots = G(0) = 0. \end{aligned} \tag{5}$$

That is, $G^n(0) = G^n(1) = 0$.

The graph of G^2 is shown at Figure 1b.

Theorem 3. *For* $x = \{0; a_1, a_2, \ldots\}$,

$$G^n(x) = \{0; a_{n+1}, a_{n+2}, \ldots\}.$$

Proof. By repeated use of (1) we have

$$\begin{aligned} G(x) &= \{0; a_2, a_3, \ldots\}, \\ G^2(x) &= \{0; a_3, a_4, \ldots\}, \\ &\vdots \\ G^n(x) &= \{0; a_{n+1}, a_{n+2}, \ldots\}. \end{aligned}$$

□

Remark 4. Theorem 3 suggests an interesting representation for the continued fraction expansion of any non-negative number. Since for $x \in [0,1]$, $\left[\frac{1}{G^n(x)}\right] = a_{n+1}$, we have

$$x = \left\{0; \left[\frac{1}{x}\right], \left[\frac{1}{G(x)}\right], \left[\frac{1}{G^2(x)}\right], \ldots\right\}.$$

More generally, for $x \geq 0$,

$$x = \left\{[x]; \left[\frac{1}{\operatorname{frac}(x)}\right], \left[\frac{1}{G(\operatorname{frac}(x))}\right], \left[\frac{1}{G^2(\operatorname{frac}(x))}\right], \ldots\right\}. \tag{6}$$

If x is rational, that is, $x = \{a_0; a_1, a_2, \ldots, a_n\}$ for some n, then the above continued fraction terminates at the $(n-1)^{th}$ iterate of G.

2. Parts within the Gauss Map and Its Iterates

In this section we introduce the concept of *parts* within the Gauss map and its iterates. Parts are useful for two reasons: they help identify symmetry among the iterates of the map, and; they show the linkages that exist between the Gauss map and the Stern-Brocot tree. This paper only deals with the first situation, symmetry within the Gauss map.

Definition 5. The $(j_1, j_2, \ldots, j_n)^{th}$*part*, designated as $G^n_{j_1,\ldots,j_n}$, is that part of G^n whose domain is composed of the points

$$x \in [\{0; j_1, \ldots, j_n\}, \{0; j_1, \ldots, j_n, 1\})$$

if n is even, and the points

$$x \in (\{0; j_1, \ldots, j_n, 1\}, \{0; j_1, \ldots, j_n\}]$$

if n is odd.

We designate the half-open interval above (for n even or odd) by $I_{j_1,\ldots,j_n}$.

Note that in Definition 5, $I_{j_1,\ldots,j_n}$ contains all points

$$x = \{0; j_1, \ldots, j_n, a_{n+1}, a_{n+2}, \ldots\}.$$

We have seen that each part in $G(x)$ is discontinuous only at its endpoints. We generalise this result in Theorem 12 to show that each part in $G^n(x)$ is discontinuous only at its endpoints.

Example 6.

$$I_{j_1} = \left(\frac{1}{j_1+1}, \frac{1}{j_1}\right]$$

is the domain of G_{j_1}, that is, the domain of the j_1^{th} part.

Definition 7. $L_{j_1,\ldots,j_n}$ and $U_{j_1,\ldots,j_n}$ are the *greatest lower* and *least upper bounds* respectively of $I_{j_1,\ldots,j_n}$, the domain of the $(j_1, \ldots, j_n)^{th}$ part of the n^{th} iterate, $G^n(x)$.

Combining Definitions 5 and 7 we have for n odd,

$$\begin{aligned} L_{j_1,\ldots,j_n} &= \{0; j_1, j_2, \ldots, j_n, 1\} \\ U_{j_1,\ldots,j_n} &= \{0; j_1, j_2, \ldots, j_n\} \end{aligned} \tag{7}$$

and for n even,

$$\begin{aligned} L_{j_1,\ldots,j_n} &= \{0; j_1, j_2, \ldots, j_n\} \\ U_{j_1,\ldots,j_n} &= \{0; j_1, j_2, \ldots, j_n, 1\}. \end{aligned} \tag{8}$$

Definition 8. We define the *width* of $I_{j_1,\ldots,j_n}$ as

$$W_{j_1,\ldots,j_n} = U_{j_1,\ldots,j_n} - L_{j_1,\ldots,j_n}$$

Those parts of the Gauss map and its iterates whose domains have the same width are styled *width partners.*

By (2) and (3) with $x = \{0; j_1, \ldots, j_n, a_{n+1}, a_{n+2}, \ldots\}$,

$$\begin{aligned} \frac{p_n + p_{n-1}}{q_n + q_{n-1}} &= \frac{p_{n-2} + (j_n+1)p_{n-1}}{q_{n-2} + (j_n+1)q_{n-1}} \\ &= \{0; j_1, j_2, \ldots, j_{n-1}, j_n + 1\} \\ &= \{0; j_1, j_2, \ldots, j_n, 1\} \\ &= \begin{cases} U_{j_1,\ldots,j_n} & \text{for } n \text{ even} \\ L_{j_1,\ldots,j_n} & \text{for } n \text{ odd} \end{cases} \end{aligned} \tag{9}$$

and

$$\frac{p_n}{q_n} = \{0; j_1, j_2, \ldots, j_n\}$$

$$= \begin{cases} U_{j_1,\ldots,j_n} & \text{for } n \text{ odd} \\ L_{j_1,\ldots,j_n} & \text{for } n \text{ even.} \end{cases} \tag{10}$$

Hence for all n, and utilising (4)

$$\begin{aligned} W_{j_1,\ldots,j_n} &= \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n} \right| \\ &= \frac{1}{q_n (q_n + q_{n-1})} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{11}$$

Example 9. $G^2_{59,1}$ and G_{84} are width partners by (11).

3. General Equation of $G^n(x)$

Theorem 10. *Let* $x = \{0; a_1, a_2, \ldots\}$*. Then for* $n \geq 0$,

$$G^n(x) = \frac{q_n x - p_n}{p_{n-1} - q_{n-1} x}$$

where $\frac{p_n}{q_n} = \{0; a_1, a_2, \ldots, a_n\}$.

Proof. We prove this by induction. Since $q_0 = p_{-1} = 1$ and $q_{-1} = p_0 = 0$, the theorem holds for $n = 0$. Now by our induction hypothesis,

$$\begin{aligned} G^{k+1}(x) &= \frac{1}{G^k(x)} - a_{k+1} \\ &= \frac{p_{k-1} - q_{k-1} x}{q_k x - p_k} - a_{k+1} \\ &= \frac{-(p_{k-1} + a_{k+1} p_k - x(q_{k-1} + a_{k+1} q_k))}{p_k - q_k x} \\ &= \frac{q_{k+1} x - p_{k+1}}{p_k - q_k x} \text{ by (2) and (3)} \end{aligned}$$

and so our theorem is proved. □

Corollary 11. *Let* $x = \{0; a_1, a_2, \ldots\}$*. Then for* $n \geq 0$,

$$\{0; a_{n+1}, a_{n+2}, \ldots\} = \frac{q_n x - p_n}{p_{n-1} - q_{n-1} x}$$

where $\frac{p_n}{q_n} = \{0; a_1, a_2, \ldots, a_n\}$.

Proof. This follows immediately by Theorems 3 and 10. □

Theorem 12. $G^n(x)$ *is discontinuous only at the endpoints of its parts.*

Proof. For any $x \in (L_{j_1,\ldots,j_n}, U_{j_1,\ldots,j_n})$, Theorem 10 holds and $G^n_{j_1,\ldots,j_n}(x)$ is continuous. For n odd, from (7)

$$U_{j_1,\ldots,j_n} = \{0; j_1, j_2, \ldots, j_n\} \text{ and } G^n_{j_1,\ldots,j_n}(U_{j_1,\ldots,j_n}) = 0,$$
$$L_{j_1,\ldots,j_n} = \{0; j_1, j_2, \ldots, j_n, 1\} \text{ and } \lim_{x \to L_{j_1,\ldots,j_n}} G^n_{j_1,\ldots,j_n}(x) = 1.$$

Now since $\{0; j_1, j_2, \ldots, j_n, 1\} = \{0; j_1, j_2, \ldots, j_n + 1\}$, $L_{j_1,\ldots,j_n} = U_{j_1,\ldots,j_{n-1},j_n+1}$. But $G^n_{j_1,\ldots,j_{n-1},j_n+1}(U_{j_1,\ldots,j_{n-1},j_n+1}) = 0$. Hence at $x = \{0; j_1, j_2, \ldots, j_n, 1\}$, $G^n(x)$ is discontinuous. Similarly, at

$$x = \{0; j_1, j_2, \ldots, j_n - 1, 1\} = \{0; j_1, j_2, \ldots, j_n\}$$

$G^n(x)$ is discontinuous.

A similar argument holds for n even. So $G^n(x)$ is continuous except at the endpoints of its parts. □

4. Fixed Points of $G^n(x)$

We now show that every part in every iterate possesses a unique fixed point. Recall that a fixed point of a function, f, is a real number x such that $f(x) = x$.

Theorem 13. *For all n, $G^n_{j_1,j_2,\dots,j_n}$ has a unique fixed point given by*

$$\alpha_{j_1,j_2,\dots,j_n} = \left\{0; \overline{j_1, j_2, \dots, j_n}\right\}.$$

Proof. Let $x = \{0; j_1, j_2, \dots, j_n, a_{n+1}, a_{n+2}, \dots\}$ be any point in the domain of $G^n_{j_1,j_2,\dots,j_n}$. By Theorem 3, $G^n_{j_1,j_2,\dots,j_n}(x) = \{0; a_{n+1}, a_{n+2}, \dots\} = x$ if and only if $a_{mn+i} = j_i$ for $1 \le i \le n$ and $m \ge 1$. Thus $x = \left\{0; \overline{j_1, j_2, \dots, j_n}\right\}$ is the unique fixed point of $G^n_{j_1,j_2,\dots,j_n}$. Note that G^n has an infinite number of fixed points, one for each of the infinite number of parts in G^n. □

The following theorem gives us an exact expression for the fixed point of each part of the Gauss Map and its iterates based on convergents.

Theorem 14 (Fixed Point theorem)**.** *The fixed point of $G^n_{j_1,j_2,\dots,j_n}$, designated as $\alpha_{j_1,j_2,\dots,j_n}$, is*

$$\begin{aligned}\alpha_{j_1,j_2,\dots,j_n} &= \frac{p_{n-1} - q_n + \sqrt{q_n^2 + p_{n-1}^2 + 2q_{n-1}p_n + 2(-1)^{n-1}}}{2q_{n-1}} \\ &= \left\{0; \overline{j_1, j_2, \dots, j_n}\right\}\end{aligned}$$

where $\frac{p_{n-1}}{q_{n-1}} = \{0; j_1, j_2, \dots, j_{n-1}\}$ and $\frac{p_n}{q_n} = \{0; j_1, j_2, \dots, j_n\}$.

Proof. Let x be the fixed point of Corollary 11 and Theorem 13, that is,

$$x = \frac{q_n x - p_n}{p_{n-1} - q_{n-1}x}$$

Solving for x, and ensuring that x is non-negative, we have

$$x = \frac{p_{n-1} - q_n + \sqrt{(q_n - p_{n-1})^2 + 4q_{n-1}p_n}}{2q_{n-1}} \tag{12}$$

The result now follows by (4). □

Example 15. $\alpha_{1,2,3} = \left\{0; \overline{1,2,3}\right\}$ possesses the convergents

$$C_2 = \frac{p_2}{q_2} = \frac{2}{3}$$

and

$$C_3 = \frac{p_3}{q_3} = \frac{7}{10}.$$

Therefore by Theorem 14,

$$\begin{aligned}\alpha_{1,2,3} &= \left\{0; \overline{1,2,3}\right\} \\ &= \frac{p_2 - q_3 + \sqrt{q_3^2 + p_2^2 + 2q_2p_3 + 2}}{2q_2}\end{aligned}$$

$$= \frac{\sqrt{37}-4}{3}$$
$$\approx .69$$

Note that for $j_1 = 1$, by Theorem 14,

$$\alpha_1 = \frac{\sqrt{5}-1}{2} = \{0; 1, 1, 1, \ldots\}.$$

That is, the fixed point of the first part is the Golden Section, τ.

Corollary 16. *For any $k > 0$, if $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ are the convergents of $\alpha_{j_1,j_2,\ldots,j_n}$ then*

$$\alpha_{j_1,j_2,\ldots,j_n} = \frac{p_{kn-1} - q_{kn} + \sqrt{q_{kn}^2 + p_{kn-1}^2 + 2q_{kn-1}p_{kn} + 2(-1)^{kn-1}}}{2q_{kn-1}}.$$

Proof. By Corollary 11, for $x = \{0; a_1, a_2, \ldots\}$,

$$\{0; a_{kn+1}, a_{kn+2}, \ldots\} = \frac{q_{kn}x - p_{kn}}{p_{kn-1} - q_{kn-1}x}$$

where $\frac{p_{kn-1}}{q_{kn-1}} = \{0; a_1, a_2, \ldots, a_{kn-1}\}$ and $\frac{p_{kn}}{q_{kn}} = \{0; a_1, a_2, \ldots, a_{kn}\}$. In particular, for $x = \alpha_{j_1,j_2,\ldots,j_n} = \left\{0; \overline{j_1, j_2, \ldots, j_n}\right\}$,

$$\{0; j_{kn+1}, j_{kn+2}, \ldots\} = \left\{0; \overline{j_1, j_2, \ldots, j_n}\right\} = x = \frac{q_{kn}x - p_{kn}}{p_{kn-1} - q_{kn-1}x}$$

where $\frac{p_{kn-1}}{q_{kn-1}} = \{0; j_1, j_2, \ldots, j_{kn-1}\}$ and $\frac{p_{kn}}{q_{kn}} = \{0; j_1, j_2, \ldots, j_{kn}\}$. Solving for x, the result follows. □

5. Gradient of Parts

Theorem 17. *For n odd, $G^n(x)$ consists of negatively sloped parts. For n even, $G^n(x)$ consists of positively sloped parts.*

Proof. By Theorems 10 and 12, for $x \in I_{j_1,j_2,\ldots,j_n}$,

$$G^n(x) = \frac{q_n x - p_n}{p_{n-1} - q_{n-1}x}.$$

Hence

$$\frac{dG^n_{j_1,j_2,\ldots,j_n}(x)}{dx} = \frac{p_{n-1}q_n - p_n q_{n-1}}{(p_{n-1} - q_{n-1}x)^2} = \frac{(-1)^n}{(p_{n-1} - q_{n-1}x)^2} \tag{13}$$

by (4). The denominator is never zero, so by (13) the slope of $G^n_{j_1,j_2,\ldots,j_n}$ is negative for n odd and positive for n even, and so our theorem is proved. □

Note that this means that the sign of the slope of a particular part of $G^n(x)$ is not dependent on x but only on n.

Example 18. Figure 1a shows the negatively sloped parts within G whilst Figure 1b shows the positively sloped parts of G^2.

Theorem 19. *The magnitude of the slope of any part of G^n is greater than 1.*

Proof. We showed in (13) that within the domain of any particular part of G^n, where we have p_{n-1}, q_{n-1}, p_n and q_n constant, that

$$\frac{dG^n(x)}{dx} = \frac{(-1)^n}{(p_{n-1} - q_{n-1}x)^2}. \tag{14}$$

If n is even we have, for x in $I_{j_1,j_2,\ldots,j_n}$, $x > \frac{p_n}{q_n}$, and so

$$p_{n-1} - q_{n-1}x < \frac{p_{n-1}q_n - q_{n-1}p_n}{q_n} = \pm\frac{1}{q_n}$$

by (4). So $(p_{n-1} - q_{n-1}x)^2 < \frac{1}{q_n^2} < 1$.

If n is odd we have, for x in $I_{j_1,j_2,\ldots,j_n}$, $x > \frac{p_n+p_{n-1}}{q_n+q_{n-1}}$, and so

$$\begin{aligned} p_{n-1} - q_{n-1}x &< \frac{p_{n-1}q_n + p_{n-1}q_{n-1} - p_nq_{n-1} - p_{n-1}q_{n-1}}{q_n + q_{n-1}} \\ &= \pm\frac{1}{q_n + q_{n-1}} \end{aligned}$$

by (4). So $(p_{n-1} - q_{n-1}x)^2 < \frac{1}{(q_n+q_{n-1})^2} < 1$. Hence for all n, $\left|\frac{dG^n(x)}{dx}\right| > 1$ for all x in $I_{j_1,j_2,\ldots,j_n}$. □

6. Symmetry Partners

We now wish to identify parts from $G^n(x)$ that represent symmetry partners.

Definition 20. The parts $G^n_{i_1,\ldots,i_n}$ and $G^t_{m_1,\ldots,m_t}$ are said to represent *symmetry partners* if there is a $h \in [0,1]$ such that for all $x \in I_{i_1,\ldots,i_n}$ and $(2h - x) \in I_{m_1,\ldots,m_t}$ where $I_{i_1,\ldots,i_n}$ and $I_{m_1,\ldots,m_t}$ are the same length,

$$G^n_{i_1,\ldots,i_n}(x) = G^t_{m_1,\ldots,m_t}(2h - x).$$

Moreover h is said to be the *centre of symmetry.*

Our aim is to show that $h = \frac{1}{2}$ is the only case for which symmetry occurs. Thus, if a part from G^n has a symmetry partner in G^{n+1} with centre of symmetry $h = \frac{1}{2}$, then $G^n_{i_1,\ldots,i_n}(x) = G^{n+1}_{m_1,\ldots,m_{n+1}}(1 - x)$ for some positive integer values $i_1, \ldots, i_n$ and $m_1, \ldots, m_{n+1}$. Additionally we will show that for $h = \frac{1}{2}$ we have $m_1 = 1$, $m_2 = i_1 - 1$, $m_{r+1} = i_r$, $r > 1$. Note that Definition 20 implies that symmetry partners are the mirror image of each other around some centre of symmetry.

The previous section identified a necessary condition for symmetry partners to exist - they must first be width partners. We will demonstrate that the converse is not necessarily true. So what other necessary conditions must exist for width partners to also be symmetry partners? This we now explore.

Theorem 21. *Let $G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h - x)$ be symmetry partners with centre of symmetry h and $k > n$. Let also $\frac{p_t}{q_t}$ (for $t \le n$) be the t^{th} convergent of all values within $I_{i_1,\ldots,i_n}$, and $\frac{p'_t}{q'_t}$ (for $t \le k$) be the t^{th} convergent of all values within $I_{m_1,\ldots,m_k}$. Then the following are true:*

i) $q_{n-1} = q'_{k-1}$

ii) $p'_{k-1} = 2hq_{n-1} - p_{n-1}$, and

iii) $k = n + 2m + 1$ for some $m = 0, 1, 2, \ldots$

Proof. Since $G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h-x)$ are symmetry partners, they must have slopes with opposite signs. Therefore by Theorem 17, k and n cannot both be even or odd. It follows that $k = n + 2m + 1$ for some $m = 0, 1, 2, \ldots$ establishing iii).

From (14) within the domain of $G^n_{i_1,\ldots,i_n}$,

$$\frac{dG^n_{i_1,\ldots,i_n}(x)}{dx} = \frac{(-1)^n}{(p_{n-1} - q_{n-1}x)^2}.$$

Similarly, within the domain of $G^k_{m_1,\ldots,m_k}(y)$ where $y = (2h - x)$

$$\begin{aligned}\frac{dG^k_{m_1,\ldots,m_k}(y)}{dy} &= \frac{(-1)^{n+1}}{\left(p'_{k-1} - q'_{k-1}y\right)^2} \\ &= \frac{(-1)^{n+1}}{\left(p'_{k-1} - q'_{k-1}(2h-x)\right)^2}.\end{aligned}$$

But this means that, for symmetry to be in existence,

$$(p_{n-1} - q_{n-1}x)^2 = \left(p'_{k-1} - q'_{k-1}(2h-x)\right)^2.$$

That is,

$$p_{n-1} - q_{n-1}x = \pm\left(p'_{k-1} - q'_{k-1}(2h-x)\right). \tag{15}$$

Consider the two cases of (15):

Case 1:

$$p_{n-1} - q_{n-1}x = p'_{k-1} - q'_{k-1}(2h-x). \tag{16}$$

Since within $I_{i_1,\ldots,i_n}$, p_{n-1} and q_{n-1} are fixed, and within $I_{m_1,\ldots,m_k}$, p'_{k-1} and q'_{k-1} are fixed, we can equate coefficients of powers of x in (16). That is,

$$p_{n-1} = p'_{k-1} - 2hq'_{k-1}$$

and

$$q_{n-1} = -q'_{k-1}.$$

These are impossible equalities since q_{n-1} and q'_{k-1} are both positive.

Case 2:

$$q_{n-1}x - p_{n-1} = p'_{k-1} - q'_{k-1}(2h-x). \tag{17}$$

Since within $I_{i_1,\ldots,i_n}$, p_{n-1} and q_{n-1} are fixed, and within $I_{m_1,\ldots,m_k}$, p'_{k-1} and q'_{k-1} are fixed, we can equate coefficients of powers of x in (17). That is,

$$p_{n-1} = 2hq'_{k-1} - p'_{k-1}$$

and

$$q_{n-1} = q'_{k-1}.$$

thereby establishing i) and ii). ∎

Corollary 22. *Let $G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h-x)$ be symmetry partners with centre of symmetry h, and $k>n$. Let also $\frac{p_t}{q_t}$ be the t^{th} convergent of all values within $I_{i_1,\ldots,i_n}$ and $\frac{p'_t}{q'_t}$ be the t^{th} convergent of all values within $I_{m_1,\ldots,m_k}$. Then $q_n = q'_k$ where $k = n+2m+1$ for some $m = 0,1,2,\ldots$*

Proof. From Theorem 21, $k = n+2m+1$ for some positive integer m. Also since $G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h-x)$ are symmetry partners the widths of their domains are identical. Hence

$$W_{i_1,\ldots,i_n} = W_{m_1,\ldots,m_k}.$$

That is,

$$\begin{aligned}\frac{1}{q_n\left(q_n+q_{n-1}\right)} &= \frac{1}{q'_k\left(q'_k+q'_{k-1}\right)} \text{ by (11)}\\ &= \frac{1}{q'_k\left(q'_k+q_{n-1}\right)} \text{ by Theorem 21.}\end{aligned}$$

Therefore

$$\left(q'_k\right)^2 + q_{n-1}q'_k - \left(q_n^2 + q_nq_{n-1}\right) = 0. \tag{18}$$

Solving for q'_k in (18), we have,

$$q'_k = \frac{-q_{n-1} \pm \left(q_{n-1}+2q_n\right)}{2}.$$

Since q'_k must be positive, we have $q'_k = q_n$. □

Theorem 23. *Let $G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h-x)$ be symmetry partners with centre of symmetry h, and $k>n$. Let also $\frac{p_t}{q_t}$ be the t^{th} convergent of all values within $I_{i_1,\ldots,i_n}$ and $\frac{p'_t}{q'_t}$ be the t^{th} convergent of all values within $I_{m_1,\ldots,m_k}$. Then the following must be true*
i) $p'_k = 2hq_n - p_n$
ii) $h = \frac{1}{2}$.

Proof. Since $G^n_{i_1,\ldots,i_n}$ and $G^k_{m_1,\ldots,m_k}$ are symmetry partners, h lies equidistant between $\frac{p_n}{q_n}$ and $\frac{p'_k}{q'_k}$ on the x-axis. Accordingly h is rational, and

$$\begin{aligned}\frac{p'_k}{q'_k} &= 2h - \frac{p_n}{q_n}\\ &= \frac{2hq_n - p_n}{q_n}.\end{aligned} \tag{19}$$

By Theorem 22, $q'_k = q_n$. Substituting in (19) gives

$$p'_k = 2hq_n - p_n. \tag{20}$$

which establishes part i). Accordingly, since p'_k is an integer, $2hq_n$ must also be an integer. It follows that h in its reduced form has a denominator that is a factor of $2q_n$ and by i) of $2q_{n-1}$. Since $q_n = q_{n-2} + a_nq_{n-1}$, the denominator of h must also be a factor of $2q_{n-2}, 2q_{n-3}, \ldots, 2q_0 = 2$. Thus the denominator of h is 1 or 2, and $2h$ is an integer. Since by Definition 20, $h \in [0,1]$, the only possible solutions are

$h = 0, 1$ or $\frac{1}{2}$. But G^k has domain $[0,1]$ therefore the only non-trivial case is $h = \frac{1}{2}$. The only domain that can be mirrored at $x = 0$ is itself. Similarly for $x = 1$. $\square$

We are now able to demonstrate our main result:

Main Theorem. *The graph of G^n over $\left[0, \frac{1}{2}\right]$ is symmetric to the graph of G^{n+1} over $\left[\frac{1}{2}, 1\right]$.*

This requires that for the graph of G^n, each part to the left of $x = \frac{1}{2}$ possesses a symmetry partner. This symmetry partner is found equidistant from but to the right of $x = \frac{1}{2}$ in G^{n+1}. It follows that the only part that does not possess a symmetry partner is the first part found in G, that is, G_1.

Lemma 24. *If $x = \{0; a_1, a_2, \ldots\}$, then $1 - x = \{0; 1, a_1 - 1, a_2, a_3, \ldots\}$.*

Proof. Let $\beta = \{0; a_2, a_3, \ldots\}$. Now

$$x = \{0; a_1, a_2, \ldots\} = \frac{1}{a_1 + \beta}.$$

Therefore

$$1 - x = \frac{a_1 + \beta - 1}{a_1 + \beta}.$$

Now

$$\begin{aligned}\{0; 1, a_1 - 1, a_2, a_3, \ldots\} &= \cfrac{1}{1 + \cfrac{1}{(a_1 - 1) + \beta}} \\ &= \frac{a_1 + \beta - 1}{a_1 + \beta} \\ &= 1 - x.\end{aligned}$$

$\square$

Theorem 25. *$G^n_{i_1,\ldots,i_n}(x)$ and $G^k_{m_1,\ldots,m_k}(2h - x)$ with $k > n$, are symmetry partners if and only if*
i) $h = \frac{1}{2}$
ii) $k = n + 1$
iii) $m_1 = 1$, $m_2 = i_1 - 1 > 0$, $m_{t+2} = i_{t+1}$ for $0 < t < n$.

Proof. a) *"If"*. Let $h = \frac{1}{2}$, $k = n + 1$ and $m_1 = 1$, $m_2 = i_1 - 1 > 0$, $m_{t+2} = i_{t+1}$ for $0 < t < n$. By Lemma 24, if $x = \{0; i_1, \ldots, i_n, a_{n+1}, a_{n+2}, \ldots\}$ then

$$1 - x = \{0; 1, i_1 - 1, i_2, i_3, \ldots, i_n, a_{n+1}, a_{n+2}, \ldots\}.$$

It follows by Theorem 3 that

$$\begin{aligned}G^{n+1}_{1,i_1-1,i_2,\ldots,i_n}(1 - x) &= \{0; a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \\ &= G^n_{i_1,\ldots,i_n}(x).\end{aligned}$$

b) *"Only if"*. For $x = \{0; i_1, \ldots, i_n, a_{n+1}, a_{n+2}, \ldots\}$, let

$$G^n_{i_1,\ldots,i_n}(x) \text{ and } G^k_{m_1,\ldots,m_k}(2h - x)$$

be symmetry partners. Then by Theorem 23, $h = \frac{1}{2}$ establishing i).

Now by Theorem 21, for some $l \geq 0$,

$$G^n_{i_1,\ldots,i_n}(x) = G^{n+2l+1}_{1,i_1-1,i_2,\ldots,i_n,a_{n+1},a_{n+2},\ldots,a_{n+2l}}(1 - x).$$

Since by Theorem 3

$$G^{n}_{i_1,\ldots,i_n}(x) = \{0; a_{n+1}, a_{n+2}, \ldots\} \text{ and}$$
$$G^{n+2l+1}_{1,i_1-1,i_2,\ldots,i_n,a_{n+1},a_{n+2},\ldots,a_{n+2l}}(1-x) = \{0; a_{n+2l+1}, a_{n+2l+2}, \ldots\}$$

we have $G^{n}_{i_1,\ldots,i_n}(x) = G^{n+2l+1}_{1,i_1-1,i_2,\ldots,i_n,a_{n+1},a_{n+2},\ldots,a_{n+2l}}(1-x)$ only when $l = 0$, establishing ii).

But this means that $1 - x \in I_{1,i_1-1,i_2,\ldots,i_n}$ establishing iii). □

We summarise some useful findings through the following theorem.

Theorem 26. *For $n > 0$ and $j_1 > 1$, let $G^{n}_{j_1,j_2,\ldots j_n}(x)$ and $G^{n+1}_{1,j_1-1,j_2,\ldots j_n}(1-x)$ represent symmetry partners in the Gauss Map and its iterates. Let p_n and q_n be associated with x and p'_n and q'_n be associated with $1-x$. Then*
i) $q'_{i+1} = q_i$ for $i \geq -1$
ii) $p'_{i+1} = q_i - p_i$ for $i \geq 0$
iii) $p_i = q'_{i+1} - p'_{i+1}$ for $i \geq 0$.

Proof. We note by Lemma 24 that if

$$x = \{a_0; a_1, a_2, \ldots\} = \{0; j_1, \ldots, j_n, a_{n+1}, a_{n+2}, \ldots\},$$

then

$$1 - x = \left\{a'_0; a'_1, a'_2, \ldots\right\} = \{0; 1, j_1 - 1, j_2, j_3, \ldots, j_n, a_{n+1}, a_{n+2}, \ldots\}.$$

We proceed by induction on i.

i) We have

$$q'_0 = 1 = q_{-1}.$$
$$q'_1 = 1 = q_0.$$
$$q'_2 = (j_1 - 1)\, q'_1 + q'_0 = j_1 = q_1.$$

Suppose i) holds for $i = -1, 0, 1, \ldots, m$. By Theorem 25, (2) and the induction hypothesis

$$q'_{m+2} = a'_{m+2} q'_{m+1} + q'_m = a_{m+1} q_m + q_{m-1} = q_{m+1},$$

establishing i).

ii) We have

$$p'_1 = 1 = q_0 - p_0.$$
$$p'_2 = (j_1 - 1)\, p'_1 + p'_0 = j_1 - 1 = q_1 - p_1.$$

Suppose ii) holds for all $i = 0, 1, \ldots, m$. By Theorem 25, (3) and the induction hypothesis

$$\begin{aligned} p'_{m+2} &= a'_{m+2} p'_{m+1} + p'_m \\ &= a_{m+1}(q_m - p_m) + (q_{m-1} - p_{m-1}) \\ &= (a_{m+1} q_m + q_{m-1}) - (a_{m+1} p_m + p_{m-1}) \\ &= q_{m+1} - p_{m+1}, \end{aligned}$$

establishing ii).

iii) This follows by i) and ii). □

Two examples of this symmetry are shown at Figure 1. A and B represent the second and third parts of G. $A^{'}$ and $B^{'}$ are their respective symmetry partners, namely $G^2_{1,1}$ and $G^2_{1,2}$, found in G^2. Note that the centre of symmetry is $x = \frac{1}{2}$ which agrees with our earlier results.

Remark 27. We have shown that merely finding two parts that possess domains with the same width is not a sufficient condition for symmetry. In Example 9 we showed that $G^2_{59,1}$ and G_{84} are width partners. However by Theorem 25 they are not symmetry partners.

We conclude with a result that links the fixed points of symmetry partners.

Theorem 28. *Let $\alpha_{j_1,j_2,\ldots,j_n}$ be the fixed point of $G^n_{j_1,j_2,\ldots,j_n}$, where $j_1 > 1$. Then the fixed point of $G^{n+1}_{1,\ j_1-1,\ j_2,\ldots,j_n}$, designated as $\alpha_{1,\ j_1-1,\ j_2,\ldots,j_n}$, is*

$$\begin{aligned}\alpha_{1,\ j_1-1,\ j_2,\ldots,j_n} &= \frac{q_{n-1} - p_{n-1} - q_n + \sqrt{\left(q_{n-1} - p_{n-1} - q_n\right)^2 + 4q_{n-1}\left(q_n - p_n\right)}}{2q_{n-1}} \\ &= \left\{0; \overline{1,\ j_1 - 1,\ j_2, \ldots, j_n}\right\}\end{aligned}$$

where $\frac{p_{n-1}}{q_{n-1}} = \{0; j_1, j_2, \ldots, j_{n-1}\}$ *and* $\frac{p_n}{q_n} = \{0; j_1, j_2, \ldots, j_n\}$.

Proof. By Theorem 14 the fixed point of $G^{n+1}_{1,\ j_1-1,\ j_2,\ldots,j_n}$ is

$$\alpha_{1,\ j_1-1,\ j_2,\ldots,j_n} = \frac{p'_n - q'_{n+1} + \sqrt{\left(p'_n - q'_{n+1}\right)^2 + 4q'_n p'_{n+1}}}{2q'_n}$$

where

$$\frac{p'_{n-1}}{q'_{n-1}} = \{0; 1, j_1 - 1, j_2, \ldots, j_{n-1}\}$$

and

$$\frac{p'_n}{q'_n} = \{0; 1, j_1 - 1, j_2, \ldots, j_n\}\,.$$

As $G^n_{j_1,j_2,\ldots,j_n}$ and $G^{n+1}_{1,\ j_1-1,\ j_2,\ldots,j_n}$ are symmetry partners, the result follows from Theorem 26. □

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