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COEFFICIENT PROBLEMS FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING HADAMARD PRODUCTS

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ABSTRACT. For $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, let $MS(\Phi, \Psi; \lambda, \alpha, \beta)$ be the class of functions defined in the open unit disc D by

$$\left|\arg\left(\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z g'(z) + (1-\lambda)g(z)}\right)\right| < \frac{\pi\alpha}{2}, \quad z \in D$$

where $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ is analytic function and satisfies

$$\left|\arg\left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)}\right)\right| < \frac{\pi\beta}{2}, \quad z \in D$$

for some $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$. For $f \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$ and given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, a sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ when $\mu \geq 1$.

1. Introduction

Let A denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are in the open unit disk $D = \{z : |z| < 1\}$. Further, let S denote the class of functions which are univalent in D.

Let the function $f(z)$ be defined by (1.1). Also let the function $g(z)$ be defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.2}$$

Then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.3}$$

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Fekete and Szegő [8] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ when μ is real. For various subclasses of S, sharp upper bound for functional $|a_3 - \mu a_2^2|$ has been studied by many different authors including [1–7, 9–11, 13–17, 19–20].

In this paper we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when f belonging to the class of functions defined as follows:

Definition 1.1. Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, and let $f \in A$. Then $f \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$ if and only if

$$\left|\arg\left(\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z g'(z) + (1-\lambda)g(z)}\right)\right| < \frac{\pi\alpha}{2}, \tag{1.4}$$

with $g \in A$ and satisfies

$$\left|\arg\left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)}\right)\right| < \frac{\pi\beta}{2}, \quad z \in D \tag{1.5}$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$.

Note that $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, \beta) = K(\beta)$ the class of close-to-convex functions defined in [3] and $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, 1) = K(1)$ is the class of normalised close-to-convex functions defined by Kaplan [12]. Whereas,

$$MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, \alpha, \beta) = K(\alpha, \beta)$$

is the class of normalised close-to-convex functions defined in [7].

2. Main Result

In order to derive our main results, we have to recall here the following lemma.

Lemma 2.1 ([18])**.** *Let $h \in P$ that is, h be analytic in D and be given by $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$, and $\operatorname{Re} h(z) > 0$ for $z \in D$, then*

$$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}. \tag{2.1}$$

Theorem 2.2. *Let $f(z)$ be given by (1.1). If $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$; $0 < \alpha \leq 1$, $\beta \geq 1$, $0 \leq \lambda \leq 1$ and $3\eta\mu \geq 2\delta^2 + 4\delta\gamma_2$ where $\delta = \lambda_2 - \gamma_2$, $\eta = \lambda_3 - \gamma_3$ and $\mu \geq 1$, then we have the sharp inequality*

$$\begin{aligned} |a_3 - \mu a_2^2| \leq\; & \frac{\beta^2}{3\eta\delta^2}[3\mu\eta - 2\delta(\delta + 2\gamma_2)] \\ & + \frac{\alpha(\alpha\delta + 2\beta(1+\lambda))(3\mu(1+2\lambda) - 2(1+\lambda)^2)}{3\delta(1+\lambda)^2(1+2\lambda)}. \end{aligned} \tag{2.2}$$

Proof. Let $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$. It follows from (1.4) that

$$\lambda z^2 f''(z) + zf'(z) = q^{\alpha}(z)[\lambda z g'(z) + (1-\lambda)g(z)], \tag{2.3}$$

for $z \in D$, with $q \in P$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$2a_2(1+\lambda) = b_2(1+\lambda) + \alpha q_1 \tag{2.4}$$

and

$$3(1+2\lambda)a_3 = \alpha q_2 + \alpha q_1 b_2(1+\lambda) + \frac{\alpha(\alpha-1)}{2}q_1^2 + b_3(1+2\lambda). \tag{2.5}$$

Also, it follows from (1.5) that

$$g(z) * \Phi(z) = (g(z) * \Psi(z))p^{\beta}(z), \tag{2.6}$$

where $p \in P$ with $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ for $z \in D$. Thus equating coefficients, we obtain

$$\delta b_2 = \beta p_1 \tag{2.7}$$

and

$$\eta b_3 = \beta\left(p_2 + \frac{\beta(\delta + 2\gamma_2) - \delta}{2\delta}p_1^2\right). \tag{2.8}$$

From (2.4), (2.5), (2.7) and (2.8) we have

$$\begin{aligned} a_3 - \mu a_2^2 = {} & \frac{\alpha}{3(1+2\lambda)}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\beta}{3\eta}\left(p_2 - \frac{p_1^2}{2}\right) \\ & + \frac{\alpha^2 q_1^2\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}}{12(1+\lambda)^2(1+2\lambda)} \\ & + \frac{\alpha\beta p_1 q_1\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}}{6\delta(1+\lambda)(1+2\lambda)} \\ & + \frac{\beta^2 p_1^2\{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\}}{12\eta\delta^2}. \end{aligned} \tag{2.9}$$

□

Assume that $a_3 - \mu a_2^2$ positive. Thus we now estimate $\mathrm{Re}(a_3 - \mu a_2^2)$ by applying the same technique done by London [16]. And so from (2.9) and by using lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \le r \le 1$, $0 \le R \le 1$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le 2\pi$, we obtain

$$\begin{aligned} \mathrm{Re}(a_3 - \mu a_2^2) = {} & \frac{\alpha}{3(1+2\lambda)}\,\mathrm{Re}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\alpha^2\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}\,\mathrm{Re}\,q_1^2}{12(1+\lambda)^2(1+2\lambda)} \\ & + \frac{\beta}{3\eta}\,\mathrm{Re}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\alpha\beta\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}\,\mathrm{Re}\,p_1 q_1}{6\delta(1+\lambda)(1+2\lambda)} \\ & + \frac{\beta^2\{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\}\,\mathrm{Re}\,p_1^2}{12\eta\delta^2} \\ \le {} & \frac{2\alpha}{3(1+2\lambda)}(1-R^2) + \frac{\alpha^2\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}R^2\cos 2\phi}{3(1+\lambda)^2(1+2\lambda)} \\ & + \frac{2\beta}{3\eta}(1-r^2) + \frac{2\alpha\beta\{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}rR\cos(\theta+\phi)}{3\delta(1+\lambda)(1+2\lambda)} \\ & + \frac{\beta^2\{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\}r^2\cos 2\theta}{3\eta\delta^2} \\ \le {} & \frac{2\alpha}{3(1+2\lambda)}(1-R^2) + \frac{\alpha^2\{3\mu(1+2\lambda) - 2(1+\lambda)^2\}R^2}{3(1+\lambda)^2(1+2\lambda)} \\ & + \frac{2\beta}{3\eta}(1-r^2) + \frac{2\alpha\beta\{3\mu(1+2\lambda) - 2(1+\lambda)^2\}rR}{3\delta(1+\lambda)(1+2\lambda)} \end{aligned}$$

$$+\frac{\beta^2\{3\mu\eta-(2\delta^2+4\gamma_2\delta)\}r^2}{3\eta\delta^2}=G(r,R).$$

Letting α, β and μ fixed and differentiating $G(r,R)$ partially when $0<\alpha\le 1$, $\beta\ge 1$, and $\mu\ge 1$ we observe that

$$\begin{aligned}
&G_{rr}G_{RR}-(G_{rR})^2\\
&\quad=\frac{16\alpha\beta}{9\eta(1+2\lambda)}+\frac{4\alpha^2\beta^2(3\mu\eta-(2\delta^2+4\gamma_2\delta))(3\mu(1+2\lambda)-2(1+\lambda)^2)}{9\delta^2\eta(1+\lambda)^2(1+2\lambda)}\\
&\qquad-\frac{8\alpha^2\beta(3\mu(1+2\lambda)-2(1+\lambda)^2)}{9\eta(1+\lambda)^2(1+2\lambda)}-\frac{8\alpha\beta^2(3\mu\eta-(2\delta^2+4\gamma_2\delta))}{9\delta^2\eta(1+2\lambda)}\\
&\qquad\qquad-\frac{4\alpha^2\beta^2(3\mu(1+2\lambda)-2(1+\lambda)^2)^2}{9\delta^2(1+\lambda)^2(1+2\lambda)^2}<0.
\end{aligned}$$

Therefore, the maximum of $G(r,R)$ occurs on the boundaries. Thus the desired inequality follows by observing that $G(r,R)\le G(1,1)=\dfrac{\beta^2}{3\eta\delta^2}[3\mu\eta-2\delta(\delta+2\gamma_2)]$

$$+\frac{\alpha(\alpha\delta+2\beta(1+\lambda))(3\mu(1+2\lambda)-2(1+\lambda)^2)}{3\delta(1+\lambda)^2(1+2\lambda)}. \tag{2.10}$$

The equality is attained when choosing $p_1=q_1=2i$ and $p_2=q_2=-2$ in (2.9).

Remark 1. Letting $\lambda=0$ in Theorem 2.2, we have the result given by Darus and Thomas [7].

Remark 2. Letting $\Phi(z)=\dfrac{z}{(1-z)^2}$, $\Upsilon(z)=\dfrac{z}{1-z}$, $\lambda=0$ and $\alpha=1$ in Theorem 2.2, we have the result given by Jahangiri [11].

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