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FIXED POINTS THEOREMS FOR α-FUZZY MONOTONE MAPS

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Abstract. In this note, we prove the existence of least and greatest fixed points of α-fuzzy monotone maps.

1. Introduction

The theory of fuzzy sets was initiated by L.A. Zadeh [12]. In [13], L.A. Zadeh introduced the notion of fuzzy order and similarity which was developed by several authors (see for example [1, 3, 10, 11, 13]). On the other hand, fixed points theorems in fuzzy setting have been established by lots of authors (see for example [1, 2, 4, 5, 6, 7, 8, 10, 9]). In [2], I. Beg proved the existence of fixed point of fuzzy monotone maps, by using Claude Ponsard's definition of order (see [3]). Recently, we have introduced in [10], the notion of α-fuzzy order. An important property of this order is due to the fact that the greatest and the least elements of a subset are unique when they exist. This uniqueness fails in the case of Claude Ponsard's order (see [3]). In this note, we first prove the existence of least and greatest fixed points of α-fuzzy monotone maps defined on α-fuzzy ordered complete sets (see Theorems 3.1 and 3.4). Secondly, we give a computation of the least fixed point of α-fuzzy order continuous maps (see Theorem 4.3).

2. Preliminaries

Let X be a nonempty set. A fuzzy subset A of X is characterized by its membership function $A\colon X \to [0,1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy subset A for each $x \in X$.

Definition 2.1 ([10])**.** Let X be a nonempty set and $\alpha \in]0,1]$. An α-fuzzy order on X is a fuzzy subset r_α of $X \times X$ satisfying the following three properties:

(i) for all $x \in X$, $r_\alpha(x,x) = \alpha$ (α-fuzzy reflexivity);
(ii) for all $x, y \in X$, $r_\alpha(x,y) + r_\alpha(y,x) > \alpha$ implies $x = y$ (α-fuzzy antisymmetry);
(iii) for all $x, z \in X$, $r_\alpha(x,z) \geq \sup_{y\in X} [\min\{r_\alpha(x,y), r_\alpha(y,z)\}]$ (α-fuzzy transitivity).

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The pair (X, r_α), where r_α is a α-fuzzy order on X is called a r_α-fuzzy ordered set. An α-fuzzy order r_α is said to be total if for all $x \neq y$ we have either $r_\alpha(x,y) > \frac{\alpha}{2}$ or $r_\alpha(y,x) > \frac{\alpha}{2}$. A r_α-fuzzy ordered set X on which the order r_α is total is called r_α-fuzzy chain.

Let (X, r_α) be a nonempty r_α-fuzzy ordered set and A be a subset of X. An element u of X is said to be a r_α-upper bound of A if $r_\alpha(x,u) > \frac{\alpha}{2}$ for all $x \in A$. If x is a r_α-upper bound of A and $x \in A$, then it is called a greatest element of A. An element l of X is said to be a r_α-lower bound of A if $r_\alpha(l,x) > \frac{\alpha}{2}$ for all $x \in A$. If l is a r_α-lower bound of A and $l \in A$, then it is called a least element of A. As usual, $\sup_{r_\alpha}(A)$=the least element of r_α-upper bounds of A (if it exists), $\inf_{r_\alpha}(A)$=the greatest element of r_α-lower bounds of A (if it exists).

Example 1. Let $X = \{0,1,2\}$ and r_α be the α-fuzzy order relation defined on X by:

$$r_\alpha(0,0) = r_\alpha(1,1) = r_\alpha(2,2) = \alpha,$$
$$\begin{cases} r_\alpha(0,2) = 0.55\alpha \\ r_\alpha(2,0) = 0.1\alpha \end{cases}$$
$$\begin{cases} r_\alpha(2,1) = 0.2\alpha \\ r_\alpha(1,2) = 0.6\alpha \end{cases}$$
$$\begin{cases} r_\alpha(1,0) = 0.7\alpha \\ r_\alpha(0,1) = 0.15\alpha. \end{cases}$$

As properties of r_α, we have $\inf_{r_\alpha}(X) = 0$ and $\sup_{r_\alpha}(X) = 2$.

Next, we shall compare the Zadeh-Venugopalan fuzzy order with the α-fuzzy order. First, note that for $\alpha = 1$ the Definition 2.1 was introduced by Zadeh in [13]. Later on, Venugopolan defined in [11] the upper bound of a subset A of a Zadeh's fuzzy ordered set (X, r) as follows: An element u of X is said to be a r-upper bound of A if $r(x,u) > \frac{1}{2}$ for all $x \in A$.

In this way we extended in [10] the Zadeh-Venugopalan fuzzy order [11, 13] to α fuzzy order. It is clear that the fuzzy transitivity is the same for Zadeh-Venugopalan's fuzzy order and α-fuzzy order. In addition, if we denote by A the set of all α-fuzzy orders (for $\alpha \in]0,1]$) and by B the set of all Zadeh-Venugopalan's fuzzy order, then we get

$$A = \bigcup_{\alpha \in]0,1]} \alpha B$$

where

$$\alpha B = \{\alpha r : \text{ such that } r \text{ is a Zadeh-Venugopalan's fuzzy order } \}.$$

In order to compare the α-fuzzy order to Claude Ponsard's fuzzy order [3], we give the following definitions.

Definition 2.2. Let X be a nonempty set. A fuzzy order relation on X is a fuzzy subset R of $X \times X$ satisfying the following three properties

(i) for all $x \in X$, $r(x,x) \in [0,1]$ (f-reflexivity);
(ii) for all $x, y \in X$, $r(x,y) + r(y,x) > 1$ implies $x = y$ (f-antisymmetry);
(iii) for all $(x,y,z) \in X^3$, $[r(x,y) \geq r(y,x)$ and $r(y,z) \geq r(z,y)]$ implies

$$r(x,z) \geq r(z,x)$$

(f-transitivity).

A nonempty set X with fuzzy order r defined on it, is called r-fuzzy ordered set. We denote it by (X, r). A r-fuzzy order is said to be total if for all $x \neq y$ we have either $r(x, y) > r(y, x)$ or $r(y, x) > r(x, y)$. A r-fuzzy ordered set on which the r-fuzzy order is total is called r-fuzzy chain.

Let A be a nonempty subset of X. We say that $u \in X$ is a r-upper bound of A if $r(y, u) \geq r(x, u)$ for all $y \in A$. A r-upper bound u of A with $u \in A$ is called a greatest element of A. An element l of X is called a r-lower bound of A if $r(l, x) \geq r(l, x)$ for all $x \in A$. A r-lower bound of A with $l \in A$ is called a least element of A. An $m \in A$ is called a maximal element of A if there is no $x \neq m$ in A for which $r(m, x) \geq r(x, m)$. Similarly, we can define minimal element of A. The supremum and the infimum are defined as follows: $\sup_r(A)=$ the unique least element of r-upper bounds of A (if it exists), $\inf_r(A)=$ the unique greatest element of r-upper bounds of A (if it exists),

Note that if a subset A of a Claude Ponsard's ordered set (X, r) has two least or greatest elements, say, u and v, then $r(u, v) = r(v, u)$. In general we have not $u = v$.

Example 2. Let $X = [0, 1]$ and r be the Claude Ponsard fuzzy order defined on X by:

$$r(x, x) = 1 \text{ and } r(x, y) = 0 \text{ if } x \neq y \text{ for all } x, y \in X.$$

Then, each element $x \in X$ is an r-upper bound and a lower bound of X and a least and greatest element of X. On the other hand, in α-fuzzy ordered sets the greatest and least elements of a subset are unique when they exist. Because if a subset A of an α-fuzzy ordered set (X, r_α) has two least or greatest elements, say, u and v, then $r_\alpha(u, v) + r_\alpha(v, u) > \frac{\alpha}{2}$. By α-fuzzy transitivity, we get $u = v$.

Next, we recall some definitions and results for subsequent use.

Definition 2.3. Let (X, r_α) be a nonempty r_α-fuzzy ordered set and $f \colon X \to X$ a map. We say that f is r_α-fuzzy monotone if for all $x, y \in X$ with $r_\alpha(x, y) > \frac{\alpha}{2}$, then $r_\alpha(f(x), f(y)) > \frac{\alpha}{2}$.

An element x of X is said to be a fixed point of a map $f \colon X \to X$ if $f(x) = x$. The set of all fixed points of f is denoted by $Fix(f)$.

Definition 2.4 ([10])**.** Let (X, r_α) be a nonempty r_α-fuzzy ordered set. The inverse α-fuzzy relation s_α of r_α is defined by $s_\alpha(x, y) = r_\alpha(y, x)$, for all $x, y \in X$.

Let (X, r_α) be a nonempty r_α-fuzzy ordered set and s_α its inverse α-fuzzy relation. Then, by [9, Proposition 3.5] s_α is an α-fuzzy order. Also, if f is r_α-fuzzy monotone, then f is s_α-fuzzy monotone.

In [9], we proved the following lemma.

Lemma 2.5. *Let (X, r_α) be a r_α-fuzzy order set and s_α be the inverse fuzzy order relation of r_α. Then,*

(i) If a nonempty subset A of X has a r_α-supremum, then A has a s_α-infimum and $\sup_{r_\alpha}(A) = \inf_{s_\alpha}(A)$.

(ii) If a nonempty subset A of X has a r_α-infimum, then A has a s_α-supremum and $\inf_{r_\alpha}(A) = \sup_{s_\alpha}(A)$.

The following α-fuzzy Zorn's Lemma is given in [10].

Lemma 2.6. *Let (X, r_α) be a nonempty α-fuzzy ordered sets. If every nonempty r_α-fuzzy chain in X has a r_α-upper bound, then X has a maximal element.*

In this note, we shall need the following definition.

Definition 2.7. Let (X, r_α) be a nonempty r_α-fuzzy ordered set. We say that (X, r_α) is a r_α-fuzzy ordered complete set if every nonempty r_α-fuzzy chain in X has a r_α-supremum.

3. Least and greatest fixed points for α-fuzzy monotone maps

In this subsection, we shall establish the existence of a least and a greatest fixed points in nonempty α-fuzzy ordered complete sets. First, we shall prove the following:

Theorem 3.1. *Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set. Let $f\colon X \to X$ be a r_α-fuzzy monotone map. Assume that there exists some $a \in X$ with $r_\alpha(a, f(a)) > \frac{\alpha}{2}$. Then, f has a least fixed point in the subset*

$$\left\{x \in X : r_\alpha(a, x) > \frac{\alpha}{2}\right\}.$$

To prove Theorem 3.1, we need following lemmas:

Lemma 3.2. *Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set. Let $f\colon X \to X$ be a r_α-fuzzy monotone map such that $r_\alpha(x, f(x)) > \frac{\alpha}{2}$ for all $x \in X$. Then, f has a fixed point.*

Proof. From Lemma 2.6, the fuzzy ordered set (X, r_α) has a maximal element, a, say. By our hypothesis, we have $r_\alpha(a, f(a)) > \frac{\alpha}{2}$. From this and since a is maximal, so $f(a) = a$. □

Lemma 3.3. *Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set. Let $f\colon X \to X$ be a r_α-fuzzy monotone map such that there is some $a \in X$ with $r_\alpha(a, f(a)) > \frac{\alpha}{2}$. Then, f has a fixed point in the subset $\left\{x \in X : r_\alpha(a, x) > \frac{\alpha}{2}\right\}$.*

Proof. Let A be the subset of X defined by

$$A = \left\{x \in X : r_\alpha(x, f(x)) > \frac{\alpha}{2} \text{ and } r_\alpha(a, x) > \frac{\alpha}{2}\right\}.$$

As $a \in A$, then A is nonempty.

Claim 1. We have: $f(A) \subset A$. Indeed, from the definition of A, we have

$$r_\alpha(x, f(x)) > \frac{\alpha}{2}, \text{ for all } x \in A \tag{2.1}$$

and

$$r_\alpha(a, x) > \frac{\alpha}{2}, \text{ for all } x \in A. \tag{2.2}$$

By our hypothesis we know that

$$r_\alpha(a, f(a)) > \frac{\alpha}{2}. \tag{2.3}$$

From (2.1) and (2.2) and α-fuzzy transitivity, we obtain

$$r_\alpha(a, f(x)) > \frac{\alpha}{2}, \text{ for all } x \in A. \tag{2.4}$$

On the other hand, by using (2.1) and (2.2) and r_α-fuzzy monotonicity of f, we get

$$r_\alpha(f(x), f(f(x))) > \frac{\alpha}{2}, \text{ for all } x \in A \tag{2.5}$$

and

$$r_\alpha(f(a), f(x)) > \frac{\alpha}{2}, \text{ for all } x \in A. \tag{2.6}$$

Combining (2.4) and (2.5) and α-fuzzy transitivity, we deduce that we have

$$r_\alpha(a, f(f(x))) > \frac{\alpha}{2}, \text{ for all } x \in A. \tag{2.7}$$

Then, by using (2.5) and (2.7), we conclude that we have $f(A) \subset A$.

Claim 2. Every nonempty r_α-fuzzy chain in A has a r_α-supremum in A. Indeed, let $\mathcal{C}$ be a r_α-fuzzy chain in A and s be its r_α-supremum in X. Then,

$$r_\alpha(c, s) > \frac{\alpha}{2}, \text{ for all } c \in \mathcal{C}. \tag{2.8}$$

As f is r_α-fuzzy monotone, then

$$r_\alpha(f(c), f(s)) > \frac{\alpha}{2}, \text{ for all } c \in \mathcal{C}. \tag{2.9}$$

On the other hand, we know that $\mathcal{C} \subset A$. So,

$$r_\alpha(c, f(c)) > \frac{\alpha}{2}, \text{ for all } c \in \mathcal{C}. \tag{2.10}$$

From (2.9) and (2.10) and α-fuzzy transitivity, we get

$$r_\alpha(c, f(s)) > \frac{\alpha}{2}, \text{ for all } c \in \mathcal{C}. \tag{2.11}$$

It follows that $f(s)$ is a r_α-upper bound of $\mathcal{C}$. From this and as $s = \sup_{r_\alpha}(\mathcal{C})$, we deduce that

$$r_\alpha(s, f(s)) > \frac{\alpha}{2}. \tag{2.12}$$

Now, let $c \in \mathcal{C}$ be given. As $\mathcal{C} \subset A$, then

$$r_\alpha(a, c) > \frac{\alpha}{2}. \tag{2.13}$$

Hence, from (2.8) and (2.13), we obtain $r_\alpha(a, s) > \frac{\alpha}{2}$. From this and (2.12), we conclude that $s \in A$.

Claim 3. We have: $Fix(f) \cap A \neq \emptyset$. Indeed, by Claims 1 and 2, $f(A) \subset A$ and every nonempty r_α-fuzzy chain in A has a r_α-supremum in A. From Lemma 3.2, we deduce that there exists $b \in A$ such that $f(b) = b$. Thus, $b \in Fix(f) \cap A$. □

Now, we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let A and B be the two subsets of X defined by

$$A = \left\{ x \in X : r_\alpha(x, f(x)) > \frac{\alpha}{2} \text{ and } r_\alpha(a, x) > \frac{\alpha}{2} \right\}.$$

and

$$B = \{x \in A : x \text{ is a } r_\alpha\text{-lower bound of } Fix(f) \cap A\}.$$

As $a \in B$, then the subset B is nonempty.

Claim 1. We have: $f(B) \subseteq B$. Indeed, if $x \in B$, then $r_\alpha(x, y) > \frac{\alpha}{2}$ for all $y \in Fix(f) \cap A$. As f is r_α-fuzzy monotone, so $r_\alpha(f(x), y) > \frac{\alpha}{2}$ for all $y \in Fix(f) \cap A$. Therefore, $f(x)$ is r_α-lower bound of $Fix(f) \cap A$. Since $B \subset A$, then $f(B) \subset f(A)$. On the other hand, by the Claim 1 of Lemma 3.3, we know that $f(A) \subset A$. Then, $f(B) \subset A$ and our Claim is proved.

Claim 2. Every nonempty r_α-fuzzy chain in B has a r_α-supremum in B. Indeed, let $\mathcal{C}$ be a r_α-fuzzy chain in B and let s be its r_α-supremum in (X, r_α). Let $y \in Fix(f) \cap A$. Then, $r_\alpha(c, y) > \frac{\alpha}{2}$ for all $c \in \mathcal{C}$. Thus y is a r_α-upper bound of $\mathcal{C}$. Therefore, $r_\alpha(s, y) > \frac{\alpha}{2}$ for all $y \in Fix(f) \cap A$. On the other hand by the Claim 2 of Lemma 3.3, we have $s \in A$. Hence, $s \in B$.

Claim 3. The map f has a least fixed point in $\left\{x \in X : r_\alpha(a,x) > \frac{\alpha}{2}\right\}$. Indeed, from Claim 2 above and Lemma 3.2, there exists $l \in B$ such that $f(l) = l$. Now, it is easy to see that the set of all fixed points of f in the subset $\left\{x \in X : r_\alpha(a,x) > \frac{\alpha}{2}\right\}$ is $Fix(f) \cap A$. Since $B \subset A$, so $r_\alpha(a,l) > \frac{\alpha}{2}$. So, $l \in Fix(f) \cap A$. Therefore, l is a least element of $Fix(f) \cap A$. □

Combining Lemma 2.5 and Theorem 3.1, we obtain the following:

Theorem 3.4. *Let (X, r_α) be a nonempty r_α-fuzzy ordered set with the property that every nonempty r_α-chain has a r_α-infimum. Let $f\colon X \to X$ be a r_α-fuzzy monotone map. Assume that there exists some $a \in X$ with $r_\alpha(f(a),a) > \frac{\alpha}{2}$. Then, f has a greatest fixed point in the subset $\left\{x \in X : r_\alpha(x,a) > \frac{\alpha}{2}\right\}$.*

Proof. By our hypothesis, every nonempty r_α-fuzzy chain has a r_α-infimum. Then from Lemma 2.5, every nonempty s_α-fuzzy chain has a s_α-supremum. Furthermore, we know that the map f is s_α-fuzzy monotone. Let A be the following subset defined by

$$A = \left\{x \in X : r_\alpha(x,a) > \frac{\alpha}{2}\right\}.$$

Then, we have

$$A = \left\{x \in X : s_\alpha(a,x) > \frac{\alpha}{2}\right\}.$$

Hence, all conditions of Theorem 3.1 are satisfied. Therefore, f has a least fixed point, l, say in $\left\{x \in X : s_\alpha(a,x) > \frac{\alpha}{2}\right\}$. Thus, l is a greatest fixed point of f in the subset $\left\{x \in X : r_\alpha(x,a) > \frac{\alpha}{2}\right\}$. □

4. Least fixed point for α-fuzzy order continuous maps

In this section, we shall give a computation of the least fixed point for r_α-fuzzy order continuous maps defined on a nonempty α-fuzzy ordered complete set.

Definition 4.1. Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set. A map $f\colon X \to X$ is said to be r_α-fuzzy order continuous if for every nonempty r_α-fuzzy chain $\mathcal{C}$ of X, we have $f(sup_{r_\alpha}(\mathcal{C})) = \sup_{r_\alpha}(f(\mathcal{C}))$.

Remark 4.2. Every r_α-fuzzy order continuous map defined on a nonempty r_α-fuzzy ordered complete set is r_α-fuzzy monotone.

Next, we shall show the following:

Theorem 4.3. *Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set with a least element l. Let $f\colon X \to X$ be a r_α-fuzzy order continuous map. Then, f has a least fixed point a. Furthermore, $a = \sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\})$.*

Proof. Let (X, r_α) be a nonempty r_α-fuzzy ordered complete set and l its least element. Let $f\colon X \to X$ be a r_α-fuzzy order continuous map. From Theorem 3.1, the map f has least fixed point a (say). In what follows we shall show that it is equal to $\sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\})$.

Claim 1. We have: $f(a) = \sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\})$. Indeed, since l is the least element of (X, r_α), then $r_\alpha(l, f(l)) > \frac{\alpha}{2}$. As f is r_α-fuzzy monotone, so $r_\alpha(f(l), f^2(l)) > \frac{\alpha}{2}$. By induction, we obtain $r_\alpha(f^n(l), f^{n+1}(l)) > \frac{\alpha}{2}$ for all $n \in \mathbf{N}$. Thus the set $\{f^n(l) : n \in \mathbf{N}\}$ is a r_α-fuzzy chain. As f is r_α-fuzzy order continuous then

$$f(a) = f(\sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\})) = \sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\}) = a.$$

Claim 2. The element a is the least fixed point of f. Indeed, if x is a fixed point of f, so $r_\alpha(l, x) > \frac{\alpha}{2}$. As f is r_α-fuzzy monotone then $r_\alpha(f(l), x) > \frac{\alpha}{2}$. By induction, we obtain $r_\alpha(f^n(l), x) > \frac{\alpha}{2}$ for all $n \in \mathbf{N}$. Therefore, x is a r_α-upper bound of the r_α-fuzzy chain $\{f^n(l) : n \in \mathbf{N}\}$. As $a = \sup_{r_\alpha}(\{f^n(l) : n \in \mathbf{N}\})$, then $r_\alpha(a, x) > \frac{\alpha}{2}$. Thus, a is the least fixed point of f. □

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