Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 165–168
www.emis.de/journals
ISSN 1786-0091

STATISTICAL CONVERGENCE OF WALSH-FOURIER SERIES

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Dedicated to Professor William R. Wade on his 60th birthday

Abstract. This is a brief and concise account of the basic concepts and results on statistical convergence, strong Cesàro summability and Walsh-Fourier series. To emphasize the significance of statistical convergence, for example we mention the fact that the one-dimensional Walsh-Fourier series of an integrable (in Lebesgue's sense) function may be divergent almost everywhere, but it is statistically convergent almost everywhere. The case of multi-dimensional Walsh-Fourier series is also considered. For future research, we raise two open problems and formulate two conjectures.

1. **Statistical convergence of sequences.** The concept of *statistical convergence* was first introduced and studied by Fast [2] in 1951. We note that in the first edition (1935) of the book "Trigonometric Series" by Anthony Zygmund one can find two theorems involving the concept of almost convergence (see [12, Vol. 2, on pp. 181 and 188]), which turned out to be equivalent to the concept of statistical convergence.

A sequence $(s_k : k = 1, 2, \ldots)$ of real or complex numbers is said to be statistically convergent to some finite number L, if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} n^{-1}|\{k \le n : |s_k - L| > \varepsilon\}| = 0, \tag{1.1}$$

where by $k \le n$ we mean $k = 1, 2, \ldots, n$, and by $|\mathcal{S}|$ the cardinality of the set $\mathcal{S} \subseteq \mathcal{P}$, the set of positive integers. Clearly, the statistical limit L is uniquely determined.

The following concept is due to Fridy [3]. A sequence (s_k) is said to be *statistically Cauchy* if for every $\varepsilon > 0$ there exists an integer m such that

$$\lim_{n\to\infty} n^{-1}|\{k \le n : |s_k - s_m| > \varepsilon\}| = 0;$$

and he proved that (s_k) is statistically convergent if and only if it is statistically Cauchy.

The concept of statistical convergence can be reformulated in terms of natural density. To this end, we recall (see, e.g. [6, on p. 290]) that the *natural* (or asymptotic) *density* of a set $\mathcal{S} \subseteq \mathcal{P}$ is defined by

$$d(\mathcal{S}) := \lim_{n\to\infty} n^{-1}|\{k \le n : k \in \mathcal{S}\}|,$$

2000 *Mathematics Subject Classification.* Primary 42C10; Secondary 40A05, 40G05.

Key words and phrases. Statistical convergence, almost convergence, natural density, strong Cesàro summability, Walsh-Fourier series, W-continuity.

This research was supported by the Hungarian National Foundation for Scientific Research under Grant T 046 192.

provided that this limit exists. Now, definition (1.1) can be equivalently rewritten in the following form: for every $\varepsilon > 0$ we have

$$d(\{k \in \mathcal{P} : |s_k - L| > \varepsilon\}) = 0.$$

Connor [1] proved the so-called *decomposition theorem*: A sequence (s_k) is statistically convergent to a limit L if and only if there exists a sequence (t_k) which is convergent to L in the ordinary sense:

$$\lim_{k\to\infty} t_k = L \tag{1.2}$$

and

$$d(\{k \in \mathcal{P} : s_k \neq t_k\}) = 0. \tag{1.3}$$

Moreover, if (s_k) is bounded, then (t_k) is also bounded.

Taking into account that a set $\mathcal{S} \subseteq \mathcal{P}$ has natural density 0 if and only if its complement $\mathcal{S}^c := \mathcal{P}\backslash\mathcal{S}$ has natural density 1, by (1.2) and (1.3) the following statement is obvious. A sequence (s_k) is statistically convergent to a limit L if and only if there exists an increasing sequence $(k_\ell : \ell = 1, 2, \ldots) \subseteq \mathcal{P}$ of density 1 such that

$$\lim_{\ell\to\infty} s_{k_\ell} = L. \tag{1.4}$$

If was Zygmund [12, Vol. 2, on p. 181] who introduced the concept of *almost convergence* in the sense of (1.4). By virtue of the decomposition theorem, the concepts of statistical convergence and almost convergence are equivalent.

The concept of statistical convergence can be extended to multiple sequences of real or complex numbers in a straightforward way. For instance, a double sequence $(s_{j,k} : j, k = 1, 2, \ldots)$ of real or complex numbers is said to be statistically convergent to a finite number L if for every $\varepsilon > 0$ we have

$$\lim_{m,n\to\infty} m^{-1}n^{-1}|\{j \leq m \text{ and } k \leq n : |s_{j,k} - L| > \varepsilon\}| = 0,$$

where m and n tend to infinity independently of one another. Theorems are valid for the statistical convergence of multiple sequences which are analogous to those for single sequences presented above. (See [5] for details.)

2. **Strong Cesàro summability of sequences.** This concept goes back to Hardy and Littlewood [4]. Let p be a positive real number. A sequence (s_k) of real or complex numbers is said to be *strongly p-Cesàro summable* to a finite limit L if

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |s_k - L|^p = 0.$$

The following theorem (see, e.g. [12, Vol. 2, (7.2) Theorem on p. 181, where the term ‘summable H_p’ is used]) expresses the intimate connection between the concepts of strong Cesàro summability and statistical convergence:

(i) If (s_k) is strongly p-Cesàro summable to L for some $p > 0$, then (s_k) is statistically convergent to L.
(ii) Conversely, if (s_k) is statistically convergent to L and is bounded, then (s_k) is strongly p-Cesàro summable for every $p > 0$.

An analogous theorem is valid for multiple sequences. (See again [5] for details.)

3. **Walsh-Fourier series.** Concerning the Walsh system $(w_j(x) : j = 0, 1, \ldots)$, $x \in [0, 1)$, in the Paley enumeration, we refer to the monograph [10, on pp. 1 and 2]. Given a function f integrable in Lebesgue's sense on the interval $[0, 1)$, in symbol: $f \in L^1[0, 1)$, its Walsh-Fourier series is defined by

$$f(x) \sim \sum_{j=0}^{\infty} \hat{f}(j) w_j(x), \quad \text{where} \quad \hat{f}(j) := \int_0^1 f(x) w_j(x) dx. \tag{3.1}$$

Denote the partial sums of the series in (3.1) by

$$s_n(f; x) := \sum_{j=0}^{n-1} \hat{f}(j) w_j(x), \quad n \in \mathcal{P}.$$

It was Stein [11] (see also [10, Theorem 14 on p. 298]) who first proved that there exists a function $f \in L^1[0, 1)$ whose Walsh-Fourier series is divergent almost everywhere.

The multiple Walsh-Fourier series on the N-dimensional cube $[0, 1)^N$, $N \in \mathcal{P}$, can be defined analogously. For simplicity in writing, let $N = 2$. The double Walsh-Fourier series of a function $f \in L^1[0, 1)^2$ is defined by

$$f(x, y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \hat{f}(j, k) w_j(x) w_k(y), \tag{3.2}$$

where

$$\hat{f}(j, k) := \int_0^1 \int_0^1 f(x, y) w_j(x) w_k(y) dx dy.$$

Denote the rectangular partial sums of the double series in (3.2) by

$$s_{m,n}(f; x, y) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \hat{f}(j, k) w_j(x) w_k(y), \quad (m, n) \in \mathcal{P}^2.$$

The convergence of series (3.2) is meant in Pringsheim's sense, that is, the convergence of the double sequence $(s_{m,n}(f; x, y) : m, n = 1, 2, \ldots)$ as both m and n tend to infinity, independently of one another.

Let $f \in L^1[0, 1)$. The strong p-Cesàro summability of the sequence $(s_n(f; x) : n = 1, 2, \ldots)$ of the partial sums of the series in (3.1) to $f(x)$ at almost every point $x \in [0, 1)$ was proved by Schipp [9] in the case $p = 2$; and by Rodin [7] in the general case $0 < p < \infty$ (even in the more general case of BMO means). In the case of N-dimensional Walsh-Fourier series, $N \in \mathcal{P}$, Rodin [8] proved the following theorem: If $f \in L^1(\log^+ L)^{N-1}[0, 1)^N$, then the N-multiple sequence of the rectangular partial sums $(s_{n_1,\ldots,n_N}(f; x_1, \ldots, x_N) : n_1, \ldots, n_N = 1, 2, \ldots)$ of the Walsh-Fourier series of f are strongly p-Cesàro summable to $f(x_1, \ldots, x_N)$ at almost every point $(x_1, \ldots, x_N) \in [0, 1)^N$ for any $0 < p < \infty$.

Combining these results with those in Section 2 yields the following remarkable result.

Theorem. *If $f \in L^1(\log^+ L)^{N-1}[0, 1)^N$, then the Walsh-Fourier series of f is statistically convergent to $f(x_1, \ldots, x_N)$ at almost every point $(x_1, \ldots, x_N) \in [0, 1)^N$.*

In particular, if $f \in L^1[0, 1)$, then the Walsh-Fourier series of f is statistically convergent almost everywhere.

Two questions arise naturally.

(i) What is the situation if $f \in L^1[0, 1)^N$, but $f \notin L^1(\log^+ L)^{N-1}[0, 1)^N$? On the basis of a result of Jessen, Marcinkiewicz and Zygmund (see, e.g. [12, Vol. 2, (2.14) Theorem on p. 308]), we formulate the following conjecture.

Conjecture 1. Given any function $\varphi(u)$, positive and increasing on the interval $(0, \infty)$ and $o(u \log^{N-1} u)$ as $u \to \infty$, there exists a function $f \geq 0$ such that $f \in L^1[0,1)^N$ and $\varphi(f) \in L^1[0,1)^N$, but the multiple Walsh-Fourier series of f is nowhere statistically convergent.

(ii) What is the situation if f is a W-continuous function? We recall (see [10, on p. 11 for the case $N = 1$]) that a real-valued function f defined on $[0,1)^N$ is said to be W-continuous there if it is continuous from the (product) dyadic topology to the usual topology.

Conjecture 2. If f is a W-continuous function on $[0,1)^N$, then the Walsh-Fourier series of f is statistically convergent to $f(x_1, \ldots, x_N)$ uniformly in

$$(x_1, \ldots, x_N) \in [0,1)^N.$$

According to Section 2, in order to justify Conjecture 2 it would be enough to prove that every W-continuous function on $[0,1)^N$ is strongly p-Cesàro summable to f uniformly on $[0,1)^N$ for some $0 < p < \infty$. We guess that this claim is true even for every $0 < p < \infty$.

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