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COMMON FIXED POINT THEOREMS FOR E-CONTRACTIVE OR E-EXPANSIVE MAPS IN UNIFORM SPACES

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Abstract. The main purpose of this paper is to obtain several common fixed point theorems for contractive or expansive self-mappings of uniform spaces.

1. Introduction

The theory of fixed point or common fixed point for contractive or expansive self-mappings of complete metric spaces has been well developed ([2], [3], [4], [5], [8]). Recently, O. Kada, T. Suzuki and W. Takahashi [4] have introduced the concept of a W-distance on metric spaces and have generalized some important results in non-convex minimizations and in fixed point theory for both W-contractive or W-expansive maps.

On the other hand, it has always been tempting to generalize certain existence fixed or common fixed point theorems to uniform spaces. Following ideas in [4], J.R. Montes and J.A. Charris established, in [7], some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform spaces. In this paper, we give many common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the concept of an A-distance or an E-distance.

The paper is divided into three sections. In section 2 we introduce the concept of an A-distance and an E-distance. In section 3 (resp. 4) we prove some common fixed point theorems for A (resp. E)-contractive maps (resp. for E-expansive maps). We begin by recalling some basic concepts of the theory of uniform spaces needed in the sequel. For more information we refer the reader to the book by N. Bourbaki [1], chapter II.

We call uniform space (X, ϑ) a nonempty set X endowed of an uniformity ϑ, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x, x)/x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V-close, and a sequence (x_n) in X is a Cauchy sequence for ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X/(x, y) \in V\}$ when V runs over ϑ.

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X, i.e., if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y, x)/(x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close,

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then for our purpose, we assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they always refer to the topological space $(X, \tau(\vartheta))$.

2. A (resp. E)-distance

Definition 2.1. Let (X, ϑ) be a uniform space. A function $p\colon X \times X \longrightarrow \mathbb{R}^+$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.2. Let (X, ϑ) be a uniform space. A function $p\colon X \times X \longrightarrow \mathbb{R}^+$ is said to be an E-distance if

(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y, z \in X$.

Examples.

1. Let (X, ϑ) be a uniform space and let d be a distance on X. Clearly (X, ϑ_d) is a uniform space where ϑ_d is the set of all subsets of $X \times X$ containing a "band" $B_\varepsilon = \{(x, y) \in X^2 / d(x, y) < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X, ϑ).

2. Recently, J.R. Montes and J.A. Charris introduced the concept of W-distance on uniform spaces. Every W-distance p is an E-distance since it satisfies (p_1), (p_2) and the following condition: for all $x \in X$, the function $p(x, .)$ is lower semi-continuous.

3. Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the function p defined as follows

$$p(x, y) = \begin{cases} y, & y \in [0, 1[, \\ 2y, & y \in [1, +\infty[. \end{cases}$$

It is easy to see that the function p is an E-distance on (X, ϑ_d) but it is not an W-distance on (X, ϑ_d) since the function $p(x, .)\colon X \longrightarrow \mathbb{R}^+$ is not lower semi-continuous at 1.

The following lemma contains some useful properties of A-distances. It is stated in [4] for metric spaces and in [7] for uniform spaces. The proof is straightforward.

Lemma 2.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds*

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.*
(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to z.*
(c) *If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then (x_n) is a Cauchy sequence in (X, ϑ).*

Let (X, ϑ) be a uniform space with an A-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting

Definition 2.3. Let (X, ϑ) be a uniform space and p be an A-distance on X.

(1) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n \to \infty} p(x_n, x) = 0$.
(2) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n \to \infty} x_n = x$ with respect to $\tau(\vartheta)$.

(3) $f\colon X \longrightarrow X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies

$$\lim_{n\to\infty} p(f(x_n), f(x)) = 0.$$

(4) $f\colon X \longrightarrow X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\vartheta)$.

(5) X is said to be p-bounded if $\delta_p(X) = \sup\{p(x, y)/x, y \in X\} < \infty$.

Remark. Let (X, ϑ) be a Hausdorff uniform space and let (x_n) be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Lemma 2.1($b$) then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. Therefore S-completeness implies p-Cauchy completeness.

3. Common fixed point theorems for A(resp. E)-contractive maps

In the sequel we involve a nondecreasing function $\psi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

(ψ_1) For each $t \in]0, +\infty[$, $0 < \psi(t)$.

(ψ_2) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \in]0, +\infty[$

It is easy to see that under the above properties, ψ satisfies also $\psi(t) < t$ for each $t > 0$.

Main results

Theorem 3.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Suppose X is p-bounded and S-complete. Let f and g be commuting p-continuous or $\tau(\vartheta)$-continuous self-mappings of X such that*

(1) *$f(X) \subseteq g(X)$,*

(2) *$p(f(x), f(y)) \leq \psi(p(g(x), g(y)))$, $\quad \forall x, y \in X$.*

Then f and g have a common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Choose $x_2 \in X$ such that $f(x_1) = g(x_2)$. In general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. We have

$$\begin{array}{rcl} p(f(x_n), f(x_{n+m})) & \leq & \psi(p(g(x_n), g(x_{n+m}))) \\ & = & \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ & \leq & \psi^2(p(g(x_{n-1}), g(x_{n+m-1}))) \\ & = & \psi^2(p(f(x_{n-2}), f(x_{n+m-2}))) \\ & \vdots & \\ & \leq & \psi^n(p(f(x_0), f(x_m))) \\ & \leq & \psi^n(\delta_p(X)). \end{array}$$

where $\delta_p(X) = sup\{p(x, y)/x, y \in X\}$. Then, by ($\psi_2$) and lemma 2.1($c$), we deduce that the sequence $(f(x_n))$ is a p-Cauchy sequence. Since X is S-complete, $\lim_{n\to\infty} p(f(x_n), u) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(g(x_n), u) = 0$. The assumption that f and g are p-continuous implies $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0$. Since $fg = gf$, it follows that $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, and Lemma 2.1($a$) then gives $f(u) = g(u)$. Also $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. From (2), it follows

$$p(f(u), f(f(u))) \leq \psi(p(g(u), g(f(u)))) = \psi(p(f(u), f(f(u)))) < p(f(u), f(f(u)))$$

which is a contradiction. Thus $p(f(u), f(f(u))) = 0$. Suppose that $p(f(u), f(u)) \neq 0$. Also from (2), we have

$$p(f(u), f(u)) \leq \psi(p(g(u), g(u))) = \psi(p(f(u), f(u))) < p(f(u), f(u))$$

a contradiction. Thus $p(f(u), f(u)) = 0$. Now we have $p(f(u), f(u)) = 0$ and $p(f(u), f(f(u))) = 0$, lemma 2.1(a) then gives $f(f(u)) = f(u)$. Hence $g(f(u)) = f(f(u)) = f(u)$, and therefore $f(u)$ is a common fixed point of f and g. The proof is similar when f and g are $\tau(\vartheta)$-continuous since S-completeness implies p-Cauchy completeness (remark 2.1). □

Clearly, one would ask whether the common fixed point is unique. This will be happen if we assume that the function p is an E-distance.

Theorem 3.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is p-bounded and S-complete. Let f and g be commuting p-continuous or $\tau(\vartheta)$-continuous self-mappings of X such that*

(1) $f(X) \subseteq g(X)$,
(2) $p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X$.

Then f and g have a unique common fixed point.

Proof. Since an E-distance function p is an A-distance, f and g have a common fixed point. Suppose that there exists $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. If $p(u, v) \neq 0$, then

$$p(u, v) = p(f(u), f(v)) \leq \psi(p(g(u), g(v))) = \psi(p(u, v)) < p(u, v)$$

which is a contradiction. Thus $p(u, v) = 0$. Similarly, we show that $p(v, u) = 0$. Consequently, by (p_2), we have $p(u, u) \leq p(u, v) + p(v, u)$ and therefore $p(u, u) = 0$. Now we have $p(u, u) = 0$ and $p(u, v) = 0$, which implies $u = v$. □

Example. Let $X = [0, 1]$ and $d(x, y) = |x - y|$ the usual metric. Let f and g defined by

$$fx = \begin{cases} x^2, & x \in [0, \frac{1}{2}[, \\ 0, & x \in [\frac{1}{2}, 1]. \end{cases} \qquad gx = \begin{cases} x, & x \in [0, \frac{1}{2}[, \\ 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Consider the functions p and ψ defined as follows

$$\psi(x) = \begin{cases} x^2, & x \in [0, \frac{1}{2}[, \\ \frac{1}{2}x, & x \in [\frac{1}{2}, 1]. \end{cases}$$

And

$$p(x, y) = \begin{cases} y, & y \in [0, \frac{1}{2}[, \\ 1, & y \in [\frac{1}{2}, 1]. \end{cases}$$

On the one hand, the function p is an E-distance but not a W-distance and X is S-complete. Moreover f, g are commuting, p-continuous and

$$d(f(\frac{1}{3}), f(\frac{1}{4})) = \frac{7}{144} > \psi(d(g(\frac{1}{3}), g(\frac{1}{4}))) = \psi(\frac{1}{12}) = \frac{1}{144}$$

which implies that $d(f(x), f(y)) \leq \psi(d(g(x), g(y)))$ does not hold for all $x, y \in X$. On the other hand, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X$$

and 0 is the unique common fixed point of f and g.

Letting $g = \mathrm{Id}_X$, the identity, gives a generalization of ψ-contraction in metric spaces, which is given in [9](page 39) as problem 1.4

Corollary 3.1. *Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is p-bounded and S-complete. Let f be a p-continuous or $\tau(\vartheta)$-continuous self-mapping of X such that*

$$p(f(x),f(y)) \leq \psi(p(x,y)), \quad \forall x,y \in X$$

Then f has a unique fixed point.

Also for $f = \mathrm{Id}_X$, we get the following result

Corollary 3.2. *Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is p-bounded and S-complete. Let g be a surjective p-continuous or $\tau(\vartheta)$-continuous self-mapping of X such that*

$$p(x,y) \leq \psi(p(g(x),g(y))), \quad \forall x,y \in X$$

Then g has a unique fixed point.

In 1986, G. Jungck [3] introduced the notion of compatible maps in metric spaces. This concept was frequently used to prove existence theorems in common fixed point theory. We formulate this concept in the setting of uniform spaces as follows

Definition 3.1. Let (X,ϑ) be a Hausdorff uniform space and p be an A-distance on X. Two self-mappings f and g of X are said to be p-compatible if, for each sequence (x_n) of X such that $\lim_{n\to\infty} p(f(x_n),u) = \lim_{n\to\infty} p(g(x_n),u) = 0$ for some $u \in X$, one has $\lim_{n\to\infty} p(f(g(x_n)),g(f(x_n))) = 0$.

Examples.

1. Let $X = [0,+\infty[$ and $p(x,y) = max\{x,y\}$.
The function p is an A-distance. Let f and g be a self-mappings of X defined by

$$f(x) = 2x \text{ and } g(x) = 3x$$

it is easy to see that

$$\lim_{n\to\infty} p(f(x_n),u) = 0 \text{ implies } \lim_{n\to\infty} x_n = 0 \text{ and } u = 0$$

and

$$\lim_{n\to\infty} p(g(x_n),u) = 0 \text{ implies } \lim_{n\to\infty} x_n = 0 \text{ and } u = 0$$

Therefore

$$\lim_{n\to\infty} p(f(g(x_n)),g(f(x_n))) = \lim_{n\to\infty} 6x_n = 0$$

Thus f and g are p-compatible.

2. Let $X = [0,+\infty[$ and $p(x,y) = y$. The function p is an A-distance. Let f and g be self-mappings of X defined by

$$f(x) = 2x \text{ and } g(x) = 3x$$

Consider the sequence (x_n) with $x_n = 1$, $n = 1,2,\ldots$. We have

$$\lim_{n\to\infty} p(f(x_n),0) = 0 \text{ and } \lim_{n\to\infty} p(g(x_n),0) = 0$$

but

$$\lim_{n\to\infty} p(f(g(x_n)),g(f(x_n))) = \lim_{n\to\infty} 6x_n = 6$$

Which implies that f and g are not p-compatible.

Remark. Obviously, in the setting of metric spaces two commuting maps are compatible. However, in our setting this implication does not hold as it is shown in the above example 3.1.2.

Theorem 3.3. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is p-bounded and S-complete. Let f and g be p-compatible, p-continuous or $\tau(\vartheta)$-continuous self-mappings of X such that*

(1) $f(X) \subseteq g(X)$,
(2) $p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X$.

Then f and g have a unique common fixed point.

Proof. As in the proof of theorem 3.1, $\lim_{n\to\infty} p(f(x_n), u) = \lim_{n\to\infty} p(g(x_n), u) = 0$, for some $u \in X$. Since f and g are p-continuous, one has $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0$. The assumption that f and g are compatible, gives $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$. Moreover, we have

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u))$$

On letting $n \longrightarrow \infty$ and using lemma 2.1(a), we obtain $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$. Now we have $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$, and Lemma 2.1($b$) then gives $f(u) = g(u)$. The rest is the same as given for theorem 3.1 (resp. theorem 3.2 for uniqueness). □

4. Common fixed point theorems for E-expansive maps

In this section, we involve a nondecreasing function $\phi \colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the following conditions

(ϕ_1) For each $t > 0$, $t < \phi(t)$,
(ϕ_2) For any decreasing sequence (t_n) in $\mathbb{R}^+$, if

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \phi(t_n) = t, \text{ for some } t \in \mathbb{R}^+$$

then $t = 0$.

Main results

Theorem 4.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is S-complete. Let f and g be commuting p-continuous or $\tau(\vartheta)$-continuous self-mappings of X such that*

(1) $g(X) \subseteq f(X)$,
(2) $\phi(p(g(x), g(y))) \leq p(f(x), f(y)), \quad \forall x, y \in X$.

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $g(x_0) = f(x_1)$. Choose $x_2 \in X$ such that $g(x_1) = f(x_2)$. In general, choose $x_n \in X$ such that $g(x_{n-1}) = f(x_n)$. Consider the sequence $y_n = p(g(x_n), g(x_{n+1}))$, $n = 0, 1, \ldots$. We wish to show that $\lim_{n\to\infty} y_n = 0$. Indeed, we have

$$\begin{aligned} y_{n+1} = p(g(x_{n+1}), g(x_{n+2})) \quad &< \phi(p(g(x_{n+1}), g(x_{n+2})) \\ &\leq p(f(x_{n+1}), f(x_{n+2})) = p(g(x_n), g(x_{n+1}))) \\ &< y_n \end{aligned}$$

and $y_n < \phi(y_n) \leq y_{n-1} < \phi(y_{n-1})$, which implies that (y_n) and $(\phi(y_n))$ are decreasing and then $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} \phi(y_n)$ exist. Therefore, on letting $n \longrightarrow +\infty$, we obtain $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \phi(y_n) = t$, for some $t \in \mathbb{R}^+$. Condition (ϕ_2) then gives $t = 0$. Hence $\lim_{n\to\infty} p(g(x_n), g(x_{n+1})) = 0$.

Now we wish to show that the sequence $(g(x_{2n}))$ is a p-Cauchy sequence. Suppose that $(g(x_{2n}))$ is not a p-Cauchy sequence. Then there exists a positive number ε

such that, for each positive integer $2k$, there exist integers $2n(k)$ and $2m(k)$ such that $2k \leq 2n(k) < 2m(k)$ and $p(g(x_{2n(k)}), g(x_{2m(k)})) \geq \varepsilon$.

For each integer $2k$, let $2m(k)$ denotes the smallest integer satisfying the last two inequalities. Then $p(g(x_{2n(k)}), g(x_{2m(k)-2})) < \varepsilon$. From (p_1), we get

$$\begin{aligned}\varepsilon &\leq p(g(x_{2n(k)}), g(x_{2m(k)}))\\ &\leq p(g(x_{2n(k)}), g(x_{2m(k)-2})) + p(g(x_{2m(k)-2}), g(x_{2m(k)-1}))\\ &\quad + p(g(x_{2m(k)-1}), g(x_{2m(k)-2})),\end{aligned}$$

which gives $\lim_{k\to\infty} p(g(x_{2n(k)}), g(x_{2m(k)})) = \varepsilon$, since $\lim_{n\to\infty} p(g(x_n), g(x_{n+1})) = 0$.

On the other hand, we have

$$\begin{aligned}p(g(x_{2n(k)-1}), g(x_{2m(k)-1})) &= p(f(x_{2n(k)}), f(x_{2m(k)}))\\ &\geq \phi(p(g(x_{2n(k)}), g(x_{2m(k)})))\\ &> p(g(x_{2n(k)}), g(x_{2m(k)}))\end{aligned}$$

and

$$\begin{aligned}p(g(x_{2n(k)-1}), g(x_{2m(k)-1})) &\leq p(g(x_{2n(k)-1}), g(x_{2n(k)}))\\ &\quad + p(g(x_{2n(k)}), g(x_{2m(k)-2}))\\ &\quad + p(g(x_{2m(k)-2}), g(x_{2m(k)-1})).\end{aligned}$$

Putting $k \longrightarrow \infty$, we obtain $\lim_{k\to\infty} \phi(p(g(x_{2n(k)}), g(x_{2m(k)}))) = \varepsilon$.

Now we have

$$\lim_{k\to\infty} p(g(x_{2n(k)}), g(x_{2m(k)})) = \lim_{k\to\infty} \phi(p(g(x_{2n(k)}), g(x_{2m(k)}))) = \varepsilon.$$

By (ϕ_2), we get $\varepsilon = 0$, which is a contradiction. Thus the sequence $(g(x_{2n}))$ is a p-Cauchy sequence and therefore $(g(x_n))$ is a p-Cauchy sequence. The S-completeness of X implies $\lim_{n\to\infty} p(g(x_n), u) = 0$, for some $u \in X$. Also $\lim_{n\to\infty} p(f(x_n), u) = 0$. Since g is p-continuous, one has $\lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0$. Commutativity of f and g and p-continuity of f imply $\lim_{n\to\infty} p(g(f(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$. Lemma 2.1$(a)$ then gives $f(u) = g(u)$.

Also $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. From (2), it follows

$$p(f(u), f(f(u))) = p(g(u), g(g(u))) < \phi(p(g(u), g(g(u)))) \leq p(f(u), f(f(u)))$$

which is a contradiction. Thus $p(f(u), f(f(u))) = 0$. Suppose that $p(f(u), f(u)) \neq 0$. Also from (2), we have

$$p(f(u), f(u)) = p(g(u), g(u)) < \phi(p(g(u), g(u))) \leq p(f(u), f(u))$$

a contradiction. Thus $p(f(u), f(u)) = 0$. Now we have $p(f(u), f(u)) = 0$ and $p(f(u), f(f(u))) = 0$, and lemma 2.1(a) then gives $f(f(u)) = f(u)$. Hence $g(f(u)) = f(f(u)) = f(u)$, and therefore $f(u)$ is a common fixed point of f and g. Suppose that there exists $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. If $p(u, v) \neq 0$, then

$$p(u, v) = p(g(u), g(v)) < \phi(p(g(u), g(v))) \leq p(f(u), f(v)) = p(u, v)$$

which is a contradiction. Thus $p(u, v) = 0$. Similarly, we show that $p(v, u) = 0$. Consequently, by (p_2), we have $p(u, u) \leq p(u, v) + p(v, u)$ and therefore $p(u, u) = 0$. Now we have $p(u, u) = 0$ and $p(u, v) = 0$, which implies $u = v$. The proof is similar when f and g are $\tau(\vartheta)$-continuous since S-completeness implies p-Cauchy completeness (Remark 2.1). $\square$

Example. Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Let f and g defined by

$$g(x) = \begin{cases} \frac{1}{3}x, & x \in [0, 1[, \\ 0, & x \geq 1; \end{cases} \qquad f(x) = \begin{cases} \frac{1}{2}x, & x \in [0, 1[, \\ 1, & x \geq 1. \end{cases}$$

Consider the functions p and ψ defined as follows

$$\phi(x) = \begin{cases} \frac{7}{6}x, & x \in [0, 1[, \\ x + 1, & x \geq 1, \end{cases} \quad \text{and } p(x, y) = y.$$

It is easy to see that p is an E-distance and X is S-complete. Moreover f, g are commuting, p-continuous and

$$\phi(p(g(x), g(y))) \leq p(f(x), f(y))), \quad \forall x, y \in X$$

and 0 is the unique common fixed point of f and g.

Letting $f = \mathrm{Id}_X$ (resp. $g = \mathrm{Id}_X$), we get the following results

Corollary 4.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is S-complete. Let g be a p-continuous or $\tau(\vartheta)$-continuous self-mapping of X such that*

$$\phi(p(g(x), g(y))) \leq p(x, y), \quad \forall x, y \in X$$

Then g has a unique fixed point.

Corollary 4.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is S-complete. Let f be a surjective p-continuous or $\tau(\vartheta)$-continuous self-mapping of X such that*

$$\phi(p(x, y)) \leq p(f(x), f(y)), \quad \forall x, y \in X.$$

Then f has a unique fixed point.

Also for p-compatible maps, we have the following result

Theorem 4.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is S-complete. Let f and g be p-compatible, p-continuous or $\tau(\vartheta)$-continuous self-mappings of X such that*

(1) $g(X) \subseteq f(X)$,
(2) $\phi(p(g(x), g(y))) \leq p(f(x), f(y)), \quad \forall x, y \in X$.

Then f and g have a unique common fixed point.

Proof. The proof is almost the same as that of theorem 4.1 by utilizing a similar argument of theorem 3.3. □

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