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CONSTRUCTION OF FAMILIES OF LONG CONTINUED FRACTIONS REVISITED

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Abstract. In this survey article, we revisit construction of simple continued fractions of quadratic irrationals with long period lengths, which has generated much interest in the relatively recent literature. We show that new and not-so-new results actually follow from results of Perron in the 1950s and from results of this author from over a decade ago. Moreover, we are able to generalize and simplify numerous such results for a better understanding of the phenomenon. This continues work in [7]–[12].

1. Introduction

It is generally acknowledged that Dan Shanks began the search for families of quadratic surds with unbounded continued fraction period length with his discovery in [14]–[15]. Numerous other constructions of explicit constructions of continued fractions has since appeared. However, even some of the most recent contributions such as [3] ostensibly overlooked the contributions of Perron and others from which much of the later results follow. It is the purpose of this paper to exhibit what should be well known and show how some recent results follow from them including some generalizations and simplifications.

2. Notation and Preliminaries

The background for the following together with proofs and details may be found in [4]. Let $\Delta = d^2 D_0$ ($d \in \mathbb{N}$, $D_0 > 1$ squarefree) be the discriminant of a real quadratic order $\mathcal{O}_\Delta = \mathbb{Z} + \mathbb{Z}[\sqrt{\Delta}] = [1, \sqrt{\Delta}]$ in $\mathbb{Q}(\sqrt{\Delta})$, U_Δ the group of units of $\mathcal{O}_\Delta$, and ε_Δ the fundamental unit of $\mathcal{O}_\Delta$. Now we introduce the notation for continued fractions. Let $\alpha \in \mathcal{O}_\Delta$. We denote the simple continued fraction expansion of α (in terms of its *partial quotients*) by:

$$\alpha = \langle q_0; q_1, \ldots, q_n, \ldots \rangle .$$

If α is *periodic*, we use the notation:

$$\alpha = \langle q_0; q_1.q_2.\ldots, q_{k-1}; \overline{q_k, q_{k+1}, \ldots, q_{\ell+k-1}} \rangle ,$$

to denote the fact that $q_n = q_{n+\ell}$ for all $n \geq k$. The smallest such $\ell = \ell(\alpha) \in \mathbb{N}$ is called the *period length* of α. The *convergents* (for $n \geq 0$) of α are denoted by

$$\frac{x_n}{y_n} = \langle q_0; q_1, \ldots, q_n \rangle = \frac{q_n x_{n-1} + x_{n-2}}{q_n y_{n-1} + y_{n-2}}. \tag{2.1}$$

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We will need the following facts, the proofs of which can be found in most standard undergraduate number theory texts (for example see [5], and see [4] for a more advanced exposition).

$$x_j = q_j x_{j-1} + x_{j-2} \qquad (\text{for } j \geq 0 \text{ with } x_{-2} = 0, \text{ and } x_{-1} = 1), \tag{2.2}$$

$$y_j = q_j y_{j-1} + y_{j-2} \qquad (\text{for } j \geq 0 \text{ with } y_{-2} = 1, \text{ and } y_{-1} = 0), \tag{2.3}$$

$$x_j y_{j-1} - x_{j-1} y_j = (-1)^{j-1} \qquad (j \in \mathbb{N}). \tag{2.4}$$

In particular, we will be dealing with $\alpha = \sqrt{D}$ where D is a radicand. In this case, the *complete quotients* are given by $(P_j + \sqrt{D})/Q_j$ where the P_j and Q_j are given by the recursive formulae as follows for any $j \geq 0$ (with $P_0 = 0$ and $Q_0 = 1$):

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{2.5}$$

$$P_{j+1} = q_j Q_j - P_j, \tag{2.6}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{2.7}$$

Thus, we may write:

$$\sqrt{D} = \left\langle q_0; q_1, \ldots, q_n, (P_{n+1} + \sqrt{D})/Q_{n+1} \right\rangle. \tag{2.8}$$

We will also need the following facts for $\alpha = \sqrt{D}$. For any integer $j \geq 0$, and $\ell = \ell(\sqrt{D})$:

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle, \tag{2.9}$$

$$\text{where } q_j = q_{\ell-j} \text{ for } j = 1, 2, \ldots, \ell-1, \text{ and } q_0 = \lfloor \sqrt{D} \rfloor \tag{2.10}$$

$$x_{j\ell-1} = q_0 y_{j\ell-1} + y_{j\ell-2}. \tag{2.11}$$

Also, for any $j \in \mathbb{N}$

$$P_1 = P_{j\ell} = q_0 \qquad \text{and} \qquad Q_0 = Q_{j\ell} = 1, \tag{2.12}$$

$$x_{j-1}^2 - y_{j-1}^2 D = (-1)^j Q_j. \tag{2.13}$$

When ℓ is even,

$$P_{\ell/2} = P_{\ell/2+1} = P_{(2j-1)\ell/2+1} = P_{(2j-1)\ell/2} \text{ and } Q_{\ell/2} = Q_{(2j-1)\ell/2}, \tag{2.14}$$

whereas when ℓ is odd,

$$Q_{(\ell-1)/2} = Q_{(\ell+1)/2}. \tag{2.15}$$

3. Main Results

The first result is due to Perron.

Theorem 3.1. *Given a palindrome $q_1, \ldots, q_{\ell-1}$ of natural numbers for $\ell \geq 2$, there exist integers $u, v, w \in \mathbb{Z}$ such that the following matrix equation holds:*

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix}. \tag{3.1}$$

If we set

$$\tau = \begin{cases} 1 & \text{if } u \equiv vw \equiv 0 \pmod 2, \\ 2 & \text{if } u \equiv vw + 1 \equiv 0 \pmod 2, \end{cases}$$

and either choice of $\tau = 1$ or $\tau = 2$ is allowed if u is odd, then there exists a nonsquare $D \in \mathbb{N}$ such that

$$\frac{\tau - 1 + \sqrt{D}}{\tau} = \langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0 - \tau + 1} \rangle, \tag{3.2}$$

where

$$q_0 = (\tau - 1 + ux - (-1)^\ell vw)/2 \tag{3.3}$$

for some $x \in \mathbb{Z}$. Moreover, when this holds, and x_j/y_j is the j^{th} convergent of $(\tau - 1 + \sqrt{D})/\tau$, then

$$u = y_{\ell-1}, \quad v = y_{\ell-2}, \text{ and } \quad w = x_{\ell-2} - q_0 y_{\ell-2}, \tag{3.4}$$

and

$$D = (\tau q_0 - \tau + 1)^2 + \tau^2 xv - \tau^2(-1)^\ell w^2 = \tag{3.5}$$

$$\left(\frac{\tau}{2}\right)^2 u^2 x^2 + \left(\tau^2 v - \frac{(-1)^\ell}{2} uvw\right) x + \left(\frac{\tau}{2}\right)^2 v^2 w^2 - (-1)^\ell \tau^2 w^2.$$

Proof. See [13]. Also there is a more accessible and recent interpretation in [2]. □

The next result will be useful in the balance of the paper.

Theorem 3.2 (Fundamental Unit Theorem for Quadratic Orders)**.** *Suppose that (3.2) holds. Then*

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{(\tau-1)\tau t + (\tau-1)s + Ds}{\tau^2} & \frac{(\tau-1)s+t}{\tau} \\ \frac{(\tau-1)s+t}{\tau} & s \end{pmatrix}, \tag{3.6}$$

where

$$t^2 - s^2 D = \pm \tau^2,$$

and $(t + s\sqrt{D})/\tau$ is the fundamental unit of the order $\mathbb{Z}[(\tau - 1 + \sqrt{D})/\tau]$.

Proof. See [7]. □

We also need the following, which we proved in [6].

Theorem 3.3. *Suppose that $D \in \mathbb{N}$ is squarefree, $\sigma = 2$ if $D \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise. Then all of the Q_j/σ in the simple continued fraction expansion of ω_D are powers of a single integer $a > 1$ if and only if one of the following holds:*

(a): *$\ell(\omega_D) = 1$ and $D = (\sigma q_0 - \sigma + 1)^2 + \sigma^2$.*
(b): *$\ell(\omega_D) = 2$ and $D = (\sigma q_0 - \sigma + 1)^2 + \sigma^2 a$ with $a q_1 = 2q_0 - \sigma + 1$.*

(c): *$\ell(\omega_D) > 2$ and $D = (ba^n + (a-1)/b)^2 + 4a^n$ where $b \mid (a-1)$ and $b, n \in \mathbb{N}$. In this case, $\ell(\omega_D) = 2n+1$, for $\lfloor \frac{n}{2} \rfloor \geq j \geq 1$:*

$$P_{2j} = \frac{\sigma(ba^n - (a-1)/b)}{2}, \quad Q_{2j} = \sigma a^j, \quad q_{2j} = ba^{n-j},$$

for $\lfloor \frac{n}{2} \rfloor \geq j \geq 0$:

$$P_{2j+1} = \frac{\sigma(ba^n + (a-1)/b)}{2}, \quad Q_{2j+1} = \sigma a^{n-j},$$

and

$$Q_n = Q_{n+1} = \sigma a^{n - \lfloor n/2 \rfloor}.$$

Also, the fundamental unit of $\mathbb{Z}[\omega_D]$ is given by:

$$\varepsilon_{\omega_D} = \left(\frac{\sigma(ba^n + (a-1)/b) + 2\sqrt{D}}{2\sigma} \right) \left(\frac{\sigma(ba^n + a + 1) + 2b\sqrt{D}}{2\sigma a} \right)^n.$$

In [3], Madden develops long continued fractions using a rather complicated process that even entails having zero partial quotients that have to be discarded to get the final continued fraction expansion. In the following, we show how his results follow from the known results, Theorem 3.1–3.3, in a much simpler *and more general* fashion. For instance the development in [3, Section 3, pp. 129–130], there is a development of $\sqrt{D}$ where $D = (b(2bn+1)^k + n)^2 + 2(2bn+1)^k$ for natural numbers b, k, n. We now show how this is merely a special case of a slight variation of Theorem 3.3, seemingly unknown to Madden who does not discuss the nature of the Q_j or P_j in the simple continued fraction expansions of such $\sqrt{D}$.

Theorem 3.4. *If a, b, k are natural numbers with $a \equiv 1 \pmod{2b}$ and*

$$D = \left(ba^k + \frac{a-1}{2b} \right)^2 + 2a^k,$$

then in the simple continued fraction expansion of $\sqrt{D}$, we have the following.

$$P_{2j} = ba^k - \frac{a-1}{2b}, \quad q_{2j} = 2ba^{k-j}, \quad (k \geq j \geq 1), \tag{3.7}$$

$$P_{2j+1} = ba^k + \frac{a-1}{2b}, \quad Q_{2j} = a^j, \quad Q_{2j+1} = 2a^{k-j} \quad (k \geq j \geq 0), \tag{3.8}$$

$$q_{2j+1} = ba^j, \qquad (k > j \geq 0), \tag{3.9}$$

$$q_0 = q_{2k+1} = ba^k + \frac{a-1}{2b} = \frac{q_{4k+2}}{2} = P_1, \tag{3.10}$$

and

$$\ell(\sqrt{D}) = 4k+2. \tag{3.11}$$

Also, if D is squarefree, then the fundamental unit of $\mathbb{Z}[\sqrt{D}]$ is given by:

$$\varepsilon_{4D} = \left(\frac{b^2a^k + (a+1)/2 + b\sqrt{D}}{a} \right)^{2k} \frac{\left(ba^k + (a-1)/(2b) + \sqrt{D} \right)^2}{2}.$$

Proof. Since

$$q_0 Q_0 - P_0 = P_1 = \lfloor \sqrt{D} \rfloor = ba^k + \frac{a-1}{2b},$$

then $Q_1 = 2a^k$ and $q_1 = b$, so

$$P_2 = ba^k - \frac{a-1}{2b}, \quad Q_2 = a, \text{ and } q_2 = 2ba^{k-1}.$$

Thus,

$$P_3 = ba^k + \frac{a-1}{2b}, \quad Q_3 = 2a^{k-1}, \text{ and } q_3 = ba.$$

Continuing in this fashion, we see that we get (3.7)–(3.11). When D is squarefree, the form for the fundamental unit follows from [1, Satz 2, p. 159]. $\square$

For instance, Madden [3, p. 144] looks at $a = 11$ and $b = 1$, so

$$\sqrt{D} = \sqrt{11^{2k} + 12 \cdot 11^k + 25}.$$

In his case, he only looks at the partial quotients q_j. However, by our Theorem 3.4, we see that $P_{2j} = 11^k - 5$ for $k \geq j \geq 1$, $P_{2j+1} = 11^k + 5$ for $k \geq j \geq 0$, $Q_{2j} = 11^j$, and $Q_{2j+1} = 2 \cdot 11^{k-j}$ for $k \geq j \geq 0$. Also, the partial quotients are given by

$$q_{2j} = 2 \cdot 11^{k-j} \text{ for } k \geq j \geq 1, \quad q_{2j+1} = 11^j \text{ for } k > j \geq 0, \tag{3.12}$$

and $q_0 = q_{2k+1} = 11^k + 5 = q_{4k+2}/2 = P_1$. In Madden's case, his methods force him to remove (undefined) zeros from the partial quotients before the correct expansion is achieved. Our method, however, is precise and should be well-known having essentially been discovered by this author over a decade ago. To give more credence to the last allegation, we note that the partial quotients in our Theorem 3.4, which generalized the Madden result, appear (less a factor of 2) in the simple continued fraction of $(1+\sqrt{D})/2$ where

$$D = \left(ba^k + \frac{a-1}{b}\right)^2 + 4a^k,$$

since, by Theorem 3.3, $q_{2j} = ba^{k-j}$ and $q_{2j+1} = ba^j$ for $k > j \geq 0$, with

$$\ell((1+\sqrt{D})/2) = 2k+1.$$

For instance, take $b = 1$, and $a = 11$ then $D = 11^{2k} + 24 \cdot 11^k + 100$, and

$$q_{2j} = 11^{k-j} \text{ for } \lfloor k/2 \rfloor \geq j \geq 1, \quad \text{ and } q_{2j+1} = 11^j \text{ for } \lfloor k/2 \rfloor \geq j \geq 0.$$

Compare with (3.12). The central goal of [3] is to produce a product of matrices

$$\prod_{j=1}^{n} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & 2v - \delta w \\ v & w \end{pmatrix},$$

where $uw - v(2v - \delta w) = (-1)^n$ with $\delta \in \{0, 1\}$, then develop continued fractions with partial quotients based upon the q_j. (Madden uses lower triangular matrices, while we use upper triangular ones.) However, the more than twenty pages of so doing in [3] can be boiled down to an observation from Perron's Theorem 3.1 as follows. Pick any palindrome of natural numbers $q_1, q_2, \ldots, q_{\ell-1}$ and select $q_n = 2q_0$ where q_0 is chosen as in Theorem 3.1. Then,

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2q_0 u + v & u \\ 2q_0 v + w & v \end{pmatrix},$$

and

$$2q_0 uv + v^2 - u(2q_0 v + w) = v^2 - uw = y_{\ell-2}^2 - y_{\ell-1}(x_{\ell-2} - q_0 y_{\ell-2}) =$$

$$y_{\ell-2}^2 - x_{\ell-2} y_{\ell-1} + q_0 y_{\ell-1} y_{\ell-2}.$$

However, by (2.11), $q_0 y_{\ell-1} = x_{\ell-1} - y_{\ell-2}$, so the latter equals

$$y_{\ell-2}^2 - x_{\ell-2} y_{\ell-1} + (x_{\ell-1} - y_{\ell-2}) y_{\ell-2} = x_{\ell-1} y_{\ell-2} - x_{\ell-2} y_{\ell-1} = (-1)^\ell,$$

where the last equality follows from (2.4). Hence, we have accomplished the task. Moreover, this method is more general, apart from being much simpler, than that presented in [3] since it allows us to look at $D \equiv 1 \pmod 4$ which is avoided in [3].

We can even look at the case where D is not squarefree. For instance, consider the following.

Example 3.1. If $D = 245 = 5 \cdot 7^2 \equiv 1 \pmod 4$, then

$$\sqrt{245} = \langle 15; \overline{1, 1, 1, 7, 6, 7, 1, 1, 1, 30}\rangle = \langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0}\rangle,$$

so

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 30 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 101521 & 3312 \\ 66240 & 2161 \end{pmatrix} = \begin{pmatrix} y_\ell & y_{\ell-1} \\ q_0 y_{\ell-2} + x_{\ell-2} & y_{\ell-2} \end{pmatrix},$$

where $y_\ell y_{\ell-2} - (q_0 y_{\ell-2} + x_{\ell-2}) y_{\ell-1} = 101521 \cdot 2161 - 66240 \cdot 3312 = 1 = (-1)^\ell$. Note as well that we may employ Theorem 3.2 to get the fundamental unit:

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 15 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 811440 & 51841 \\ 51841 & 3312 \end{pmatrix} = \begin{pmatrix} sD & t \\ t & s \end{pmatrix},$$

with $t^2 - s^2 D = 51841^2 - 3312^2 \cdot 245 = 1$, where $51841 + 3312\sqrt{245}$ is the fundamental unit of $\mathbb{Z}[\sqrt{245}]$.

However, the idea is to build upon the values of q_j in the simple continued fraction expansion of $\sqrt{245}$ to get infinite families of continued fraction expansions whose period length goes to infinity. We showed how to do this in [8]–[12]. We apply our techniques here to this specific example. Let

$$B_k + A_k\sqrt{245} = (51841 + 3312\sqrt{245})^k,$$

for any $k \in \mathbb{N}$, and set

$$D_k(X) = A_k^2 X^2 + 2B_k + C$$

for any $X \in \mathbb{N}$. Then by [12],

$$\sqrt{D_k(X)} = \langle A_k X + 15; \overline{w_{k-1}, 2(A_k X + 15)}\rangle,$$

where w_{k-1} represents $k-1$ iterations of $1, 1, 1, 7, 6, 7, 1, 1, 1, 30$ followed by one iteration of $1, 1, 1, 7, 6, 7, 1, 1, 1$, and $\ell(\sqrt{D_k(X)}) = 10k$. For instance, if $k = 3$, then $B_3 = 557288527109761$, $A_3 = 35603857991376$, and $X = 1$, then

$$\sqrt{D_3(1)} = \sqrt{1267634703871183234344593143} =$$
$$\langle 35603857991391; \overline{w_2, 71207715982782}\rangle,$$

where $w_2 = 1, 1, 1, 7, 6, 7, 1, 1, 1, 30, 1, 1, 1, 7, 6, 7, 1, 1, 1, 30, 1, 1, 1, 7, 6, 7, 1, 1, 1.$ and $\ell(\sqrt{D_3(1)}) = 30$. Hence, $\lim_{k\to\infty} \ell(\sqrt{D_k(X)}) = \infty$ and if we fix $k \in \mathbb{N}$, then for any $X \in \mathbb{N}$, $\ell(\sqrt{D_k(X)}) = 10k$.

The technique displayed in Example 3.1 comes from [12]. However, we developed numerous other such techniques of different stripes in [7]–[11].

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