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ON A CLASS OF RICCI-RECURRENT MANIFOLDS

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Abstract. Properties of Ricci-recurrent manifolds with some conditions imposed on the Weyl conformal curvature tensor are investigated. The main theorem states: a conformally quasi-recurrent and Ricci-recurrent but not Ricci-parallel manifold of dimension $n > 4$ with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor which is non-conformally related to a conformally symmetric one must be necessary Ricci-generalized pseudosymmetric manifold.

1. Introduction

Let (M, g) be a n-dimensional semi-Riemannian manifold with metric g. A tensor field T of type $(0, q)$ is said to be recurrent ([Rot82a]) if the relation

$$(1) \qquad \nabla_X T(Y_1, \ldots, Y_q)\, T(Z_1, \ldots, Z_q) - T(Y_1, \ldots, Y_q)\, \nabla_X T(Z_1, \ldots, Z_q) = 0$$

holds on (M, g). From the definition it follows that if at a point $x \in M$, $T(x) \neq 0$, then on some neighbourhood of x there exists a unique covector field b satisfying

$$\nabla_X T(Y_1, \ldots, Y_q) = b(X) T(Y_1, \ldots, Y_q).$$

According to Adati and Miyazawa ([AM67]), a manifold (M, g) of dimension $n \geq 4$ is called conformally recurrent if its Weyl conformal curvature tensor C satisfies (1). It is obvious that the class of conformally recurrent manifolds contains all conformally symmetric one, i.e. all manifolds satisfying $\nabla C = 0$.

Investigating conformally flat hypersurfaces immersed in an $(n+1)$-dimensional Euclidean space R. N. Sen and M. C. Chaki ([SC67]) found that if at least $n-1$ principal curvatures are equal to one another and the remaining one is zero, then the Riemann curvature tensor satisfies

$$\begin{aligned}\nabla_Z R(X, Y, V, W) = 2p(Z) R(X, Y, V, W) + \\ p(X) R(Z, Y, V, W) + p(Y) R(X, Z, V, W) + \\ p(V) R(X, Y, Z, W) + p(W) R(X, Y, V, Z)\end{aligned}$$

for some 1-form p. Manifolds satisfying the above condition are called pseudosymmetric (pseudo-symmetric in the sense of M. Chaki) ([Cha87]). We shall call such manifold quasi-recurrent rather than pseudosymmetric since the last notion is used in a different context (see below).

A $(0, 4)$ tensor B is said to be generalized curvature one if it satisfies

$$\begin{aligned}B(X, Y, V, W) = -B(Y, X, V, W) = B(V, W, X, Y), \\ B(X, Y, V, W) + B(X, V, W, Y) + B(X, W, Y, V) = 0.\end{aligned}$$

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If, moreover, the second Bianchi identity

$$\nabla_Z B(X,Y,V,W) + \nabla_V B(X,Y,W,Z) + \nabla_W B(X,Y,Z,V) = 0$$

holds, then B is said to be a proper generalized curvature tensor.

From the results of ([EK93], Proposition 2) it follows

Proposition 1. *Let (M,g) be a semi-Riemannian manifold and B be a generalized curvature tensor on M satisfying*

$$\nabla_{X_5} B(X_1,X_2,X_3,X_4) = \sum_{\sigma} \overset{\sigma}{p}\left(X_{\sigma(5)}\right) B\left(X_{\sigma(1)},X_{\sigma(2)},X_{\sigma(3)},X_{\sigma(4)}\right),$$

for some 1-forms $\overset{\sigma}{p}$, where the sum includes all permutations σ of the set $\{1,2,3,4,5\}$.

Then there exist 1-forms w, p such that

$$\begin{aligned} \nabla_Z B(X,Y,V,W) = w(Z)B(X,Y,V,W)+ \\ p(X)B(Z,Y,V,W) + p(Y)B(X,Z,V,W)+ \\ p(V)B(X,Y,Z,W) + p(W)B(X,Y,V,Z) \end{aligned}$$

holds on M. Moreover, $w = 2p$ if and only if B is a proper generalized curvature tensor.

Thus the weakly symmetric manifolds ([TB92]) as well as the generalized pseudosymmetric one ([Cha94]) are simply quasi-recurrent.

In ([EK93]) it is also proved that conformally flat quasi-recurrent manifold is of quasi-constant curvature and is subprojective in the sense of Kagan ([Kru61]). Finally, the local form of the metric of conformally flat quasi-recurrent manifold was found ([EK93]). Similar results were obtained later by others authors (cf. for example [De00] and [CM97]).

Following Prvanović ([Prv88]), a semi-Riemannian manifold (M,g), $\dim M \geq 4$, will be called conformally quasi-recurrent if its Weyl conformal curvature tensor C satisfies

$$\begin{aligned} \nabla_Z C(X,Y,V,W) = w(Z)C(X,Y,V,W)+ \\ p(X)C(Z,Y,V,W) + p(Y)C(X,Z,V,W)+ \\ p(V)C(X,Y,Z,W) + p(W)C(X,Y,V,Z) \end{aligned} \tag{2}$$

for some 1-forms w, p. The forms w, p will be referred to as fundamental forms or fundamental vectors. In condition considered originally by Prvanović $w = 2p$. However, the last relation together with (2) implies that for the tensor C the second Bianchi identity must hold and, consequently, the manifold is of harmonic conformal curvature.

The aim of this paper is to investigate properties of conformally quasi-recurrent manifolds in the sense of (2) which are simultaneously Ricci-recurrent, i.e. those the Ricci tensor S satisfies

$$\nabla S = b \otimes S$$

for some 1-form b.

For a generalized curvature tensor B define endomorphism $\widetilde{B}(X,Y)$ by

$$g\left(\widetilde{B}(X,Y)V,W\right) = B(X,Y,V,W).$$

Then for a $(0,k)$ tensor field T, $k \geq 1$, and $(0,2)$ tensor field S we define the tensor fields $B \cdot T$ and $Q(S,T)$ by the formulas

$$\begin{aligned} &(B\cdot T)(X_1,\ldots X_k;X,Y) = \\ &\quad -T\left(\widetilde{B}(X,Y)X_1,X_2,\ldots,X_k\right) - \cdots - T\left(X_1,\ldots,X_{k-1},\widetilde{B}(X,Y)X_k\right), \\ &Q(S,T)(X_1,\ldots X_k;X,Y) = \\ &\quad -T\left((X\wedge_S Y)X_1,X_2,\ldots,X_k\right) - \cdots - T\left(X_1,\ldots,X_{k-1},(X\wedge_S Y)X_k\right), \end{aligned}$$

where

$$(X \wedge_S Y)\, Z = S(Y,Z)X - S(X,Z)Y.$$

If the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent then the manifold is said to be Ricci-generalized pseudosymmetric one ([DD91a]). It is obvious that any semisymmetric as well as any Ricci flat manifold is Ricci generalized pseudosymmetric. The manifold (M,g) is Ricci-generalized pseudosymmetric iff the relation

$$R \cdot R = LQ(S,R) \tag{3}$$

holds on the set $\{x \in M, Q(S,R)\,(x) \neq 0\}$, L being a function on M. Note that (3) with $L = 1$ is of particular importance.

All manifolds under consideration are assumed to be smooth Hausdorff connected and their metrics are not assumed to be definite.

2. Preliminary results

Components of the Weyl conformal curvature tensor C are given by

$$\begin{gathered} C_{hijk} = R_{hijk} - \\ \tfrac{1}{n-2}\left(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}\right) + \tfrac{r}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \end{gathered} \tag{4}$$

In the sequel we shall often need the following lemmas:

Lemma 2. *The Weyl conformal curvature tensor satisfies the following well-known relations:*

$$C_{hijk} = -C_{ihjk} = C_{jkhi},\ C^r{}_{rjk} = C^r{}_{irk} = C^r{}_{rjk},$$
$$C_{hijk} + C_{hjki} + C_{hkij} = 0,$$

$$C^r{}_{ijk,r} = \frac{n-3}{n-2}\left[S_{ij,k} - S_{ik,j} - \frac{1}{2(n-1)}\left(g_{ij}r_{,k} - g_{ik}r_{,j}\right)\right], \tag{5}$$

$$\begin{gathered} C_{hijk,l} + C_{hikl,j} + C_{hilj,k} = \\ \tfrac{1}{n-3}(g_{hj}C^r{}_{ikl,r} + g_{hk}C^r{}_{ilj,r} + g_{hl}C^r{}_{ijk,r} - \\ g_{ij}C^r{}_{hkl,r} - g_{ik}C^r{}_{hlj,r} - g_{il}C^r{}_{hjk,r}), \end{gathered} \tag{6}$$

where the comma denotes covariant differentiation with respect to the coordinate vector field.

Lemma 3 ([Wal50], p.26)**.** *On every semi-Riemannian manifold the curvature tensor fulfills the relation*

$$R_{hijk,[lm]} + R_{jklm,[hi]} + R_{lmhi,[jk]} = 0.$$

The following lemma seems to be well-known:

Lemma 4 ([Ols87])**.** *Let M be a Ricci-recurrent manifold such that the set $U = \{x \in M, b(x) \neq 0\}$ is non-empty, b being the recurrence covector of the Ricci tensor. Then the Ricci tensor satisfies*

$$S_{hr}S^r_k = \frac{r}{2}S_{hk}. \tag{7}$$

Lemma 5 ([EK98])**.** *Let A_{lm}, B_{lm}, R_{hijk} be numbers satisfying*

$$A_{lm} = -A_{ml}, \quad B_{lm} = -B_{ml}, \quad R_{hijk} = -R_{ihjk} = R_{jkhi},$$

$$R_{hijk} + R_{hjki} + R_{hkij} = 0,$$

$$\begin{gathered} A_{lm}R_{hijk} + A_{hi}R_{jklm} + A_{jk}R_{lmhi} + B_{hl}R_{mijk} + B_{il}R_{hmjk} + \\ B_{jl}R_{himk} + B_{kl}R_{hijm} - B_{hm}R_{lijk} - B_{im}R_{hljk} - B_{jm}R_{hilk} - B_{km}R_{hijl} + \\ B_{ij}R_{hklm} - B_{ik}R_{hjlm} + B_{hk}R_{ijlm} - B_{hj}R_{iklm} = 0. \end{gathered}$$

Then either $A_{lm} - 2B_{lm} = 0$ or $R_{hijk} = 0$.

Lemma 6 ([Rot82b]). *If w_i, p_i, R_{hijk} are numbers satisfying*

$$w_l R_{hijk} + p_h R_{lijk} + p_i R_{hljk} + p_j R_{hilk} + p_k R_{hijl} = 0,$$

$$R_{hijk} = -R_{ihjk} = R_{jkhi}, \qquad R_{hijk} + R_{hjki} + R_{hkij} = 0,$$

then either each $w_l + 2p_l = 0$ or each $R_{hijk} = 0$.

Lemma 7 ([Pat81]). *On every 4-dimensional semi-Riemannian manifold (M, g) the Weyl conformal curvature tensor satisfies*

$$\begin{aligned} g_{hm}C_{lijk} + g_{lm}C_{ihjk} + g_{im}C_{hljk} + g_{hj}C_{likm} + g_{lj}C_{ihkm} + \\ g_{ij}C_{hlkm} + g_{hk}C_{limj} + g_{lk}C_{ihmj} + g_{ik}C_{hlmj} = 0. \end{aligned} \tag{8}$$

The next lemma shows the difference between 1-forms p and w. Let p_l and w_l be the local components of the 1-forms. Then the local form of (2) is

$$C_{hijk,l} = w_l C_{hijk} + p_h C_{lijk} + p_i C_{hljk} + p_j C_{hilk} + p_k C_{hijl}. \tag{9}$$

Lemma 8. *Suppose that at a point of the manifold M relation (9) holds. Then*

$$p_r C^r_{ijk} = 0, \tag{10}$$

$$w_r C^r_{ijk} = C^r{}_{ijk,r}. \tag{11}$$

Moreover

$$\begin{aligned} C_{hijk,[lm]} = \Delta w_{lm} C_{hijk} + p_{hm} C_{lijk} + p_{im} C_{hljk} + p_{jm} C_{hilk} + p_{km} C_{hijl} - \\ p_{hl} C_{mijk} - p_{il} C_{hmjk} - p_{jl} C_{himk} - p_{kl} C_{hijm}, \end{aligned} \tag{12}$$

where $\Delta w_{lm} = w_{l,m} - w_{m,l}$, $p_{hm} = p_{h,m} - p_h p_m$.

Proof. Contracting (9) with g^{ij} and making use of Lemma 2 we obtain (10). Summing (9) cyclically in (j, k, l), by contraction with g^{hl} and the use of (10), we get (11). Relation (12) is obvious.

Proposition 9. *Let M be a 4-dimensional manifold with nowhere vanishing Weyl conformal curvature tensor C. If C satisfies (9), then M is conformally recurrent manifold.*

Proof. We can suppose $p \neq 0$. Transvecting (8) with p^h and making use of (10) we get $p_m C_{lijk} + p_j C_{likm} + p_k C_{limj} = 0$, which reduce (9) to $\nabla C = (w + 2p) \otimes C$. This completes the proof.

In the sequel we shall often assume the following hypothesis:

(H). *M is a Ricci-recurrent manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor. Moreover, the Weyl conformal curvature tensor satisfies (9), p does not vanishes on a dense subset and the Ricci tensor is not parallel.*

By hypothesis, M admits a covector field b satisfying

$$S_{ij,k} = b_k S_{ij}, \quad S_{ij,kl} = (b_{k,l} + b_k b_l) S_{ij}, \quad S_{ij,[kl]} = \Delta b_{kl} S_{ij}, \tag{13}$$

where $\Delta b_{kl} = b_{k,l} - b_{l,k}$.

Remark. In what follows we can always assume $p(x) \neq 0$ whenever it is necessary, because if $p(x) = 0$, then, in virtue of (H), in each neighbourhood of x we can choose a point, say y, such that $p(y) \neq 0$ and by usual procedure extend our results to the point x.

Lemma 10. *Let M, $\dim M \geq 4$, be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that (10) is satisfied for some non-zero covector p at a point y. Then the scalar curvature of M vanishes at y.*

Proof. The Ricci identity together with (13) and (4) gives

$$\Delta b_{lm} S_{hk} = S_{hr} C^r_{klm} + S_{kr} C^r_{hlm} + \frac{n-3}{2(n-1)(n-2)} r(g_{kl} S_{hm} - g_{km} S_{hl} + g_{hl} S_{km} - g_{hm} S_{kl}), \tag{14}$$

which, by contraction with g_{hk}, yields $\Delta b_{lm} r = 0$. Suppose $r(y) \neq 0$. By transvecting (14) with S^k_x, applying (7) and symmetrizing the resulting equation in (h, x), we obtain

$$S_{hr} C^r_{xlm} + S_{xr} C^r_{hlm} + \frac{n-3}{2(n-1)(n-2)} r(g_{xl} S_{hm} - g_{xm} S_{hl} + g_{hl} S_{xm} - g_{hm} S_{xl}) = 0 \tag{15}$$

whence, by contracting with g^{xl} and transvecting with p^h, by the use of (10), we get $p_r S^r_m = \frac{r}{n} p_m$ at y. Next, transvecting (15) with p^x, we find

$$(S_{hm} - \frac{r}{n} g_{hm}) p_l = (S_{hl} - \frac{r}{n} g_{hl}) p_m.$$

Hence, in virtue of (10), $(S_{hr} C^r_{pqt} - \frac{r}{n} C_{hpqt}) p_m = 0$ at y results. Applying the last equality to (15) we have

$$r(g_{xl} S_{hm} - g_{xm} S_{hl} + g_{hl} S_{xm} - g_{hm} S_{xl}) = 0,$$

which, by transvecting with $S^{xl} g^{hm}$, yields $r = 0$ at y, a contradiction. This completes the proof.

Lemma 10 results in

Proposition 11. *Let M, $\dim M \geq 4$, be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that M admits a covector field p with properties:*
i) p does not vanish on a dense subset of M;
ii) $p_r C^r_{ijk} = 0$ on M.
Then the scalar curvature of M vanishes.

As a consequence of the above Proposition under hypothesis H we have the following frequently used formulas:

$$S_{hr} S^r_k = 0, \qquad S^r_{k,r} = S^r_k b_r = 0$$

Lemma 12. *Under hypothesis (H) relation*

$$2 S_{mr} C^r_{ijk} - S_{mi} \Delta b_{jk} = 0 \tag{16}$$

holds on M.

Proof. By Proposition 11 and (14) we have

$$\Delta b_{lm} S_{hk} = S_{hr} C^r_{klm} + S_{kr} C^r_{hlm} \tag{17}$$

on M. Differentiating covariantly with respect to ∂_z, by the use of (13), (9) and (17), we obtain

$$w_z \Delta b_{lm} S_{hk} + p_r S^r_h C_{zklm} + p_r S^r_k C_{zhlm} + p_k S_{hr} C^r_{zlm} + p_h S_{kr} C^r_{zlm} + p_l \Delta b_{zm} S_{hk} + p_m \Delta b_{lz} S_{hk} = S_{hk} \Delta b_{lm,z}. \tag{18}$$

Summing cyclically the last equation in (h, k, z) and again making use of (17) we get

$$S_{hk} W_{lmz} + S_{kz} W_{lmh} + S_{zh} W_{lmk} = 0,$$

where $W_{lmz} = -\Delta b_{lm,z} + (w_z + p_z) \Delta b_{lm} + p_l \Delta b_{zm} - p_m \Delta b_{zl}$, which, in virtue of Lemma 3, results in

$$\Delta b_{lm,z} = (w_z + p_z) \Delta b_{lm} + p_l \Delta b_{zm} - p_m \Delta b_{zl}. \tag{19}$$

Now, the last result together with (18) yields

$$p_r S^r_h C_{zklm} + p_r S^r_k C_{zhlm} + p_k S_{hr} C^r_{zlm} + p_h S_{kr} C^r_{zlm} = S_{hk} p_z \Delta b_{lm}, \tag{20}$$

whence, by transvection with S^k_v, we obtain

$$p_r S^r_h S_{vs} C^s_{zlm} = p_s S^s_v S_{hr} C^r_{zlm}. \tag{21}$$

On the other side, applying to the left hand side of (12) in turn the Ricci identity, (4) and transvecting with p^h we find

$$\begin{aligned} p_l S_{mr} C^r_{ijk} - p_m S_{lr} C^r_{ijk} + p_r S^r_l C_{mijk} - p_r S^r_m C_{lijk} = \\ (n-2)(p_{rm} p^r C_{lijk} - p_{rl} p^r C_{mijk}). \end{aligned} \tag{22}$$

Moreover, transvecting (22) with S^l_v, making use of (21) and symmetrizing the resulting equation in (m,i), we get $p_{rm} p^r S_{vs} C^s_{ijk} = 0$. Suppose $p_{rm} p^r = 0$. Then (22) gives

$$p_l S_{mr} C^r_{ijk} - p_m S_{lr} C^r_{ijk} + p_r S^r_l C_{mijk} - p_r S^r_m C_{lijk} = 0. \tag{23}$$

Changing in (20) indices (h,k,z,l,m) into (l,m,i,j,k) respectively, then adding to (23) and symmetrizing in (l,i), we get $p_l(2S_{mr}C^r_{ijk} - S_{mi}\Delta b_{jk}) = 0$. On the other side, if $S_{vr}C^r_{ijk} = 0$, then (17) implies $S_{vi}\Delta b_{jk} = 0$. Thus the Lemma is proved.

Lemma 13. *Under hypothesis (H) relations*

$$p_r S^r_h = 0,$$

$$\Delta b = 0,$$

$$S_{mr} C^r_{ijk} = 0$$

hold on M.

Proof. Transvecting (16) with p^i, by the use of (10), we have

$$p_r S^r_m \Delta b_{jk} = 0. \tag{24}$$

Differentiating covariantly (16), in view of (13), (9) and (19), we find

$$2p_r S^r_m C_{lijk} + (p_i S_{ml} - p_l S_{mi})\Delta b_{jk} = 0. \tag{25}$$

Hence, if $\Delta b_{jk} = 0$, then $p_r S^r_m = 0$ holds. On the other hand, if $\Delta b_{jk} \neq 0$, then (24) yields $p_r S^r_m = 0$ again. Then, by the use of (25), we have $(p_i S_{ml} - p_l S_{mi})\Delta b_{jk} = 0$. Transvecting the last result with C^l_{abc}, in virtue of (16) and (10), we get $\Delta b_{jk} = 0$. This completes the proof.

Corollary 14. *Under hypothesis (H)*

$$d(w + 2p) = 0$$

holds on M.

Proof. This results from (12) by the use of Lemmas 3, 13 and 5.

Proposition 15. *Assume that on a manifold $(M,\, g)$ hypothesis (H) is satisfied. If $t = w - 2p = 0$ on $(M,\, g)$, then the manifold is conformally related to a non-conformally flat conformally symmetric one $(M,\, exp\,(2f)\,g)$. Conversely, if $(M,\, g)$ is conformally related to a non-conformally flat conformally symmetric one, then $w - 2p = 0$.*

Proof. It is well known that the Christoffel symbols and the Weyl conformal curvature tensor of the conformally related manifolds $(\overline{M},\, \overline{g}) = (M,\, exp\,(2f)\,g)$ and $(M,\, g)$ are related by

$$\overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \delta^i_j f_k + \delta^i_k f_j - f^i g_{jk}, \qquad \overline{C}_{hijk} = e^{2f} C_{hijk}.$$

By the above formulas and (H) covariant derivatives of C and $\overline{C}$ with respect to the appropriate metrics satisfy

$$\begin{aligned} & e^{-2f}\overline{C}_{hijk;l} = \\ & (w_l - 2f_l)C_{hijk} + \\ & (p_h - f_h)C_{lijk} + (p_i - f_i)C_{hljk} + (p_j - f_j)C_{hilk} + (p_k - f_k)C_{hijl} + \\ & g_{hl}f^r C_{rijk} + g_{il}f^r C_{hrjk} + g_{jl}f^r C_{hirk} + g_{kl}f^r C_{hijr}, \end{aligned} \tag{26}$$

where $f_l = \partial_l f$, $f^l = g^{lr} f_r$.

Suppose $t = w - 2p = 0$. By Corollary 14 we have $dw = dp = 0$. Hence on a neighbourhood of each point there exists a function, say f, such that $f_l = \partial_l f = p_l$ which, in virtue of (10), gives $\nabla\overline{C} = 0$.

On the other hand, by transvecting (26) with p^h, in view of (10) and $\nabla\overline{C} = 0$, we obtain

$$p^r\,(p_r - f_r)\,C_{lijk} + p_l f^r C_{rijk} = 0,$$

which by symmetrization in (l, i) yields $f^r C_{rijk} = 0$. Then relation (26) and Lemma 6 imply $w_l + 2p_l = 4f_l$. But $f^r C_{rijk} = 0$, (10) and (11) result in $C^r_{ijk,r} = 0$ which, together with (6), means that C satisfies the second Bianchi identity. The last property is equivalent to $w = 2p$. This completes the proof.

Lemma 16. *Suppose that hypothesis (H) is satisfied. Then on M relations*

$$t_r C^r_{pqt} = \frac{n-3}{n-2}(S_{pq}b_t - S_{pt}b_q) \tag{27}$$

and

$$\begin{aligned} & t_l C_{hijk} + t_j C_{hikl} + t_k C_{hilj} = \\ & \tfrac{1}{n-2}[g_{hj}(S_{ik}b_l - S_{il}b_k) + g_{hk}(S_{il}b_j - S_{ij}b_l) + g_{hl}(S_{ij}b_k - S_{ik}b_j) - \\ & g_{ij}(S_{hk}b_l - S_{hl}b_k) - g_{ik}(S_{hl}b_j - S_{hj}b_l) - g_{il}(S_{hj}b_k - S_{hk}b_j) \end{aligned} \tag{28}$$

hold, where $t_l = w_l - 2p_l$.

Proof. The first equation is a consequence of (10), (11), (5), (13) and Proposition 11, while the second one results from (6), by the use of (9), (5), (13) and Proposition 11.

Lemma 17. *Under hypothesis (H) we have*

$$\begin{aligned} & t_p(C_{tqlr}C^r_{jih} - C_{tqjr}C^r_{lih}) = \tfrac{3-n}{n-2}[(S_{pq}b_t - S_{pt}b_q)C_{hilj} - \\ & (S_{ji}b_h - S_{jh}b_i)C_{tqlp} + (S_{li}b_h - S_{lh}b_i)C_{tqjp}] + \\ & \tfrac{1}{n-2}[(-g_{hj}S_{il} + g_{hl}S_{ij} + g_{ij}S_{hl} - g_{il}S_{hj})b_r C^r_{pqt} + \\ & (g_{tp}S_{ql} - g_{tl}S_{qp} - g_{qp}S_{tl} + g_{ql}S_{tp})b_r C^r_{jih} - \\ & (g_{tp}S_{qj} - g_{tj}S_{qp} - g_{qp}S_{tj} + g_{qj}S_{tp})b_r C^r_{lih} + \\ & C_{hpqt}(S_{il}b_j - S_{ij}b_l) - C_{ipqt}(S_{hl}b_j - S_{hj}b_l) - \\ & C_{tjih}(S_{ql}b_p - S_{qp}b_l) + C_{qjih}(S_{tl}b_p - S_{tp}b_l) + \\ & C_{tlih}(S_{qj}b_p - S_{qp}b_j) - C_{qlih}(S_{tj}b_p - S_{tp}b_j), \end{aligned} \tag{29}$$

where $t_l = w_l - 2p_l$.

Proof. Transvecting (28) with C^k_{pqt}, in virtue of Lemma 13, we obtain

$$\begin{aligned} & t_l C_{hijr}C^r_{pqt} - t_j C_{hilr}C^r_{pqt} + t_r C^r_{pqt}C_{hilj} = \\ & \tfrac{1}{n-2}[C_{hpqt}(S_{il}b_j - S_{ij}b_l) - C_{ipqt}(S_{hl}b_j - S_{hj}b_l) + \\ & (-g_{hj}S_{il} + g_{hl}S_{ij} + g_{ij}S_{hl} - g_{il}S_{hj})b_r C^r_{pqt}]. \end{aligned} \tag{30}$$

Changing in (30) the indices (h, i, j, p, q, t) into (t, q, p, j, i, h) respectively, we have

$$\begin{aligned} & t_l C_{tqpr}C^r_{jih} - t_p C_{tqlr}C^r_{jih} + t_r C^r_{jih}C_{tqlp} = \\ & \tfrac{1}{n-2}[C_{tjih}(S_{ql}b_p - S_{qp}b_l) - C_{qjih}(S_{tl}b_p - S_{tp}b_l) + \\ & (-g_{tp}S_{ql} + g_{tl}S_{qp} + g_{qp}S_{tl} - g_{ql}S_{tp})b_r C^r_{jih}], \end{aligned} \tag{31}$$

whence, changing the indices j and l,

$$\begin{gathered} t_j C_{tqpr} C^r_{lih} - t_p C_{tqjr} C^r_{lih} + t_r C^r_{lih} C_{tqjp} = \\ \tfrac{1}{n-2}[C_{tlih}(S_{qj}b_p - S_{qp}b_j) - C_{qlih}(S_{tj}b_p - S_{tp}b_j) + \\ (-g_{tp}S_{qj} + g_{tj}S_{qp} + g_{qp}S_{tj} - g_{qj}S_{tp})b_r C^r_{lih}]. \end{gathered} \tag{32}$$

Finally, adding (30) to (32) subtracting (31) and making use of (27) we get (29). This completes the proof.

Lemma 18. *Under hypothesis* (H) *suppose that* $b_l \neq 0$ *at a point of* M*. If* $\dim M > 4$ *then:*

$$t_r S^r_p = w_r S^r_p = 0, \tag{33}$$

$$\operatorname{rank}[S_{ij}] = 1 \tag{34}$$

at the point.

Proof. Transvecting (28) with S^l_p we obtain

$$t_r S^r_p C_{hijk} = \frac{1}{n-2}\left[S_{hp}(S_{ij}b_k - S_{ik}b_j) - S_{ip}(S_{hj}b_k - S_{hk}b_j)\right], \tag{35}$$

whence, by transvecting with t^h and the use of (27), we have

$$(n-4)t_r S^r_p(S_{ij}b_k - S_{ik}b_j) = -(t_r S^r_j b_k - t_r S^r_k b_j)S_{ip}. \tag{36}$$

Transvecting (36) with t^i we find

$$(n-3)t_r S^r_p(t_r S^r_j b_k - t_r S^r_k b_j) = 0. \tag{37}$$

If $t_r S^r_p \neq 0$, then (37) and (36) yield $S_{ij}b_k - S_{ik}b_j = 0$, which, in virtue of (35), implies (33), a contradiction. Thus (33) holds good. Now (34) results from (35) and (33). This completes the proof.

Proposition 19. *Suppose that on a manifold* M*,* $\dim M > 4$*, hypothesis* (H) *is satisfied. If* $b_l \neq 0$ *at a point* $x \in M$*, then on some neighbourhood of* x *there exists null (i.e. isotropic) parallel vector field*

$$v_i = exp[-\frac{1}{2}b]\ k_i, \qquad \partial_i b = b_i, \tag{38}$$

related to the Ricci tensor by

$$S_{ij} = \epsilon\ k_i k_j, \qquad |\epsilon| = 1. \tag{39}$$

Proof. Relation (39) is equivalent to (34). Since the Ricci tensor is recurrent, relations (13) and (39) yield $(k_{i,l} - \frac{1}{2}b_l k_i)k_j + (k_{j,l} - \frac{1}{2}b_l k_j)k_i = 0$. Hence the vector field k is recurrent. By Lemma 13 there exists a function, say b, such that $\partial_i b = b_i$. Moreover k is isotropic by (10) and Proposition 11. Therefore the vector field defined by (38) is isotropic and parallel. This completes the proof.

Lemma 20. *Suppose that on a manifold* M*,* $\dim M > 4$*, hypothesis* (H) *is satisfied. Then*

$$S_{ij}b_r C^r_{lqt} - S_{il}b_r C^r_{jqt} = 0.$$

Proof. We can suppose $b_l \neq 0$. Transvecting (30) with t^h by the use of (27) and (33) we find

$$(n-4)\left[(S_{ij}t_l - S_{il}t_j)\,b_r C^r_{pqt} - \frac{n-3}{n-2}(S_{ij}b_l - S_{il}b_j)(S_{pq}b_t - S_{pt}b_q)\right] = 0.$$

Alternating the last equation in (p,i), multiplying the result by k_m and taking into consideration (39) we obtain

$$(t_l k_j - t_j k_l)\left(S_{im}b_r C^r_{pqt} - S_{pm}b_r C^r_{iqt}\right) = 0,$$

whence, by transvecting with C^l_{abc} and the use of Lemma 13,

$$k_j t_s C^s_{abc}\left(S_{im} b_r C^r_{pqt} - S_{pm} b_r C^r_{iqt}\right) = 0.$$

But $t_s C^s_{abc} = 0$, in virtue of (27) and Lemma 13, implies $S_{ab} b_c C^c_{def} = 0$. Thus the Lemma holds good.

The components $Q(S,C)_{hiqtpj}$ of the tensor field $Q(S,C)$ are given by

$$\begin{aligned} Q(S,C)_{hiqtpj} = {} & S_{hp} C_{jiqt} - S_{hj} C_{piqt} + S_{ip} C_{hjqt} - S_{ij} C_{hpqt} + \\ & S_{qp} C_{hijt} - S_{qj} C_{hipt} + S_{tp} C_{hiqj} - S_{tj} C_{hiqp}. \end{aligned}$$

It is well known that if the scalar curvature vanishes and the rank of the Ricci tensor is one, then

$$Q(S,C) = Q(S,R).$$

Lemma 21. *Suppose that on a manifold M,* $\dim M > 4$, *hypothesis (H) is satisfied. Then*

$$t_r b^r \cdot Q(S,C) = 0. \tag{40}$$

Proof. We can suppose $b_l \neq 0$. Applying Lemma 20 to (29) we obtain

$$\begin{aligned} t_p(C_{tqlr} C^r_{jih} - C_{tqjr} C^r_{lih}) = {} & \tfrac{3-n}{n-2}[(S_{pq} b_t - S_{pt} b_q) C_{hilj} - \\ & (S_{ji} b_h - S_{jh} b_i) C_{tqlp} + (S_{li} b_h - S_{lh} b_i) C_{tqjp}] + \\ & \tfrac{1}{n-2}[(-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}) b_r C^r_{pqt} + \\ & (-g_{tl} S_{qp} + g_{ql} S_{tp}) b_r C^r_{jih} - (-g_{tj} S_{qp} + g_{qj} S_{tp}) b_r C^r_{lih} + \\ & C_{hpqt}(S_{il} b_j - S_{ij} b_l) - C_{ipqt}(S_{hl} b_j - S_{hj} b_l) - \\ & C_{tjih}(S_{ql} b_p - S_{qp} b_l) + C_{qjih}(S_{tl} b_p - S_{tp} b_l) + \\ & C_{tlih}(S_{qj} b_p - S_{qp} b_j) - C_{qlih}(S_{tj} b_p - S_{tp} b_j). \end{aligned} \tag{41}$$

Transvecting (41) with t^l, making use of (27), (33) and Lemma 13, then alternating the obtained equation in (p,j) and making use of (39) we get (40). This completes the proof.

Lemma 22. *Suppose that on a manifold M,* $\dim M > 4$, *hypothesis (H) is satisfied and*

$$t_r b^r = 0. \tag{42}$$

Then

$$(n-4)\, b_r b^r = 0, \tag{43}$$

$$t_l C_{hijk} + t_j C_{hikl} + t_k C_{hilj} = 0, \tag{44}$$

and

$$t_p\left(C_{tqlr} C^r_{jih} - C_{tqjr} C^r_{lih}\right) = 0 \tag{45}$$

are satisfied.

Proof. Assume $t_r b^r = 0$. Transvecting (41) with t^t, then making use of (27), (33) and Lemma 13 we obtain

$$\begin{aligned} & \tfrac{n-3}{n-2}\left[b_h\left(S_{il} b_j - S_{ij} b_l\right) - b_i\left(S_{hl} b_j - S_{hj} b_l\right)\right] + \\ & \tfrac{1}{n-3}\left(t_j b_r C^r_{lih} - t_l b_r C^r_{jih}\right) + \\ & \tfrac{1}{n-2} b_r b^r\left(-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}\right) = 0, \end{aligned} \tag{46}$$

whence, by transvecting with $b^j b^i$ we find (43).

Transvecting (28) with b^k we easily get

$$t_l b_r C^r_{jih} - t_j b_r C^r_{lih} = \frac{1}{n-2}\left[b_h\left(S_{il} b_j - S_{ij} b_l\right) - b_i\left(S_{hl} b_j - S_{hj} b_l\right)\right]$$

which, combined with (43), (46) and (39) yields

$$(n-4)\left(S_{al}b_j - S_{aj}b_l\right)\left(S_{bh}b_i - S_{bi}b_h\right) = 0.$$

But, in view of Lemma 13, the last relation gives $b_r C^r_{jih} = 0$. Now (44) and (45) are simply consequences of (28) and (29) respectively. Thus the Lemma is proved.

We end this section with well-known.

Lemma 23 (cf. [Rot82a], [Rot74], Lemma 4 and [Wal50], p. 45)**.** *Suppose that at a point of a manifold M, $\dim M \geq 4$, relations*

$$Q(S,C) = 0, \tag{47}$$

$$S_{ij} = \epsilon\; k_i k_j, \qquad |\epsilon| = 1,$$

and

$$r = 0$$

are satisfied. Then

$$k_l C_{hijk} + k_j C_{hikl} + k_k C_{hilj} = 0 \tag{48}$$

and

$$R_{hijk} = k_i k_j T_{hk} - k_i k_k T_{hj} + k_h k_k T_{ij} - k_h k_j T_{ik}, \tag{49}$$

where $T_{ij} = z^r z^s R_{rijs}$, $z^r k_r = 1$.

3. Main results

Proposition 24. *Suppose that on a manifold M, $\dim M > 4$, hypothesis (H) is satisfied. If at $x \in M$, $t_r b^r(x) \neq 0$ (hence $b_l(x) \neq 0$), then on some neighbourhood of x the Riemann-Christoffel curvature tensor has the form (49).*

Proof. By (39), (40), (10) and Proposition 11 the assumption of Lemma 23 are satisfied on some neighbourhood of x. This completes the proof.

Corollary 25. *There do not exist quasi-recurrent manifolds which are Ricci-flat.*

Proof. On arbitrary quasi-recurrent and Ricci-flat manifold M relation (9) holds. Suppose that the curvature tensor of M does not vanish on an open subset $U \subset M$. Let V_r, $r = 1, \ldots, n$ be a family of non-conformally flat manifolds with recurrent curvature tensor and nowhere vanishing Ricci tensors indexed by r. Then the product manifold $U \times (\times_{r=1}^{n} V_r)$, $n > 1$, would be a conformally quasi-recurrent and Ricci-recurrent manifold and $\operatorname{rank}[S_{ij}] \geq 2$, a contradiction.

As a consequence of Propositions 11, 19 and 24 we obtain

Corollary 26. *Suppose that on a manifold M, $\dim M > 4$, hypothesis (H) and $Q\,(S,C) = 0$ hold. If $b_l \neq 0$ at a point $x \in M$, then on some neighbourhood of x the metric g is of the Walker type. Moreover, M is semi-symmetric, i.e. $R \cdot R = 0$.*

On the other hand, if $Q\,(S,C) \neq 0$, then (45), Lemma 13 and Proposition 11 result in

Corollary 27. *Let on a manifold M, $\dim M > 4$, hypothesis (H) be satisfied. Suppose moreover that $Q\,(S,C) \neq 0$ and $w - 2p \neq 0$. Then*

$$R \cdot R = Q(S,R).$$

In virtue of Proposition 15 we conclude with the following

Theorem 28. *Let M, $\dim M > 4$, be a manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor.*

Suppose that M is conformally quasi-recurrent with recurrent but non-parallel Ricci tensor and, moreover, is non-conformally related to a conformally symmetric manifold.

Then M must be necessary Ricci-generalized pseudosymmetric manifold.

4. Remarks on the existence

There exist Ricci-generalized pseudosymmetric manifolds which are neither conformally quasi-recurrent nor Ricci-recurrent in an essential way.

Example. Let M, $\dim M > 4$, be a Ricci-recurrent manifold which is simultaneously conformally recurrent. Then from results of ([Rot82a], [Rot74], see also [EK91]) it follows that on a neighbourhood of a generic point relations (2) and $R \cdot R = Q(S, R) = 0$ are satisfied.

Example. The warped product $M_1 \times_F M_2$ of a 2-dimensional manifold $(M_1, \overline{g})$ and a manifold of constant curvature $(M_2, \widetilde{g})$, $\dim M_2 \geq 2$, is a manifold satisfying $R \cdot R = Q(S, R)$ under some metric condition imposed on F ([DD91b], Corollary 5.1), however it could not be a non-conformally recurrent conformally quasi-recurrent and Ricci-recurrent one.

Example. Let $(M, \widetilde{g})$ be a hypersurface of a semi-Riemannian manifold (N, g) of constant curvature K, $\dim N = n + 1$, $n \geq 3$, $\widetilde{g}$ being the metric tensor of M induced from g. Then the condition

$$\widetilde{R} \cdot \widetilde{R} = Q(\widetilde{S}, \widetilde{R}) - \frac{n-2}{n(n+1)} K Q(\widetilde{g}, \widetilde{C})$$

holds on $(M, \widetilde{g})$ ([DV91]).

It is clear that $\widetilde{R} \cdot \widetilde{R} = Q(\widetilde{S}, \widetilde{R})$ implies $KQ(\widetilde{g}, \widetilde{C}) = 0$, whence $K\widetilde{C} = 0$ results. Thus $K = 0$ is the only case when hypersurfaces satisfying the required condition can exist.

Example. The warped product $M_1 \times_F M_2$, where M_1 is an open interval in R, M_2 is a locally symmetric Einstein space of non-constant curvature and $F = p^2(x^1)$, $p''(x^1) \neq 0$,

$$\widetilde{K} + (n-1)(n-2)(p'(x^1))^2 + (n-1)p(x^1)p''(x^1) = 0,$$

$\widetilde{K}$ being the scalar curvature of M_2, is a non-conformally recurrent (hence non-conformally flat) conformally quasi-recurrent manifold. The fundamental forms w, p satisfy $w = 2p$ and $dp \neq 0$. Hence, in virtue of Corollary 14, the manifold is not Ricci-recurrent.

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