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ON 2×2 MATRICES OVER C^*-ALGEBRAS

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ABSTRACT. We characterize C^*-algebras isomorphic to $M_2(\mathbf{C})$ and C^*-algebras containing $M_2(\mathbf{C})$ as a unital C^*-subalgebra. $*$-isomorphisms between the full 2×2 matrix algebras over C^*-algebras are also discussed.

1. Introduction

Applying an orthonormal basis, it could be shown that the algebra $M_2(\mathbf{C})$ of 2×2 matrices with entries in $\mathbf{C}$ together with the conjugate transpose operation is $*$-isomorphic to the algebra $B(\mathbf{C}^2)$ of all linear operators on the two dimensional complex Hilbert space $\mathbf{C}^2$ together with the Hilbert adjoint operation. Identifying these $*$-algebras, $M_2(\mathbf{C})$ equipped with the operator norm is a C^*-algebra. This C^*-algebra is the most elementary example of a non-commutative C^*-algebra and provide us significant counterexamples in many areas of Banach algebra theory [3].

For a C^*-algebra $\mathcal{A}$, let $M_2(\mathcal{A})$ denote the C^*-algebra of 2×2 matrices with entries in $\mathcal{A}$. Note that $M_2(\mathcal{A})$ is $*$-isomorphic to the spatial tensor product $\mathcal{A} \otimes M_2(\mathbf{C})$. In fact if $\{e_{ij}\}$ is the standard basis for $M_2(\mathbf{C})$, then every element of the algebraic tensor product A and $M_2(\mathbf{C})$ is of the form $\sum\limits_{1 \leq i,j \leq 2} a_{ij} \otimes e_{ij}$ in which the a_{ij}'s are unique and $\sum\limits_{1 \leq i,j \leq n} a_{ij} \otimes e_{ij} \mapsto [a_{ij}]$ is a $*$-isomorphism from the algebraic tensor product of A and $M_2(\mathbf{C})$ onto $M_2(\mathcal{A})$. In addition this algebraic tensor product is already complete with respect to the spatial C^*-norm [3, page 190].

It is well-known that if C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are $*$-isomorphic, then $M_2(\mathcal{A})$ and $M_2(\mathcal{B})$ are also $*$-isomorphic. But there exist two non-isomorphic unital C^*-algebra $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \simeq M_2(\mathcal{A}) \simeq M_2(\mathcal{B})$.

For example, consider

$$\mathcal{A} = \{T \oplus T; T \in B(H)\} + K(H \oplus H)$$

and

$$\mathcal{B} = \{T \oplus T \oplus 0; T \in B(H), 0 \in B(H_0)\} + K(H \oplus H \oplus H_0)$$

where H is a separable infinite dimensional Hilbert space, H_0 is one dimensional, $B(H_1)$ and $K(H_1)$ denote the algebra of bounded and compact linear operators on the Hilbert space H_1, respectively [4].

If both $\mathcal{A}$ and $\mathcal{B}$ belong however to one of the following class of C^*-algebras, $M_2(\mathcal{A}) \simeq M_2(\mathcal{B})$ implies that $\mathcal{A} \simeq \mathcal{B}$:

(i) Commutative C^*-algebras, since the center of $M_2(\mathcal{A})$ is

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathcal{A} \right\};$$

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(ii) UHF algebras [2, Theorem 1];
(iii) perturbed block diagonal algebras [5].

We should mention that there are two non-isomorphic C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $K(H) \subseteq \mathcal{A}_i$ and $\frac{\mathcal{A}_i}{K(H)} \simeq M_2(\mathbf{C})$, $i = 1, 2$ [1].

2. C^*-algebras containing $M_2(\mathbf{C})$

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra containing $M_2(\mathbf{C})$ as a unital C^*-subalgebra. Then $\mathcal{A} \simeq M_2(\mathcal{B})$ for some C^*-algebra $\mathcal{B}$.*

Proof. Suppose that $\{e_{ij}\}_{1 \leq i,j \leq 2}$ is the standard system of matrix units of $M_2(\mathbf{C})$ and $\mathcal{B} = e_{11}\mathcal{A}e_{11}$. Then

$$\phi \colon \mathcal{A} \longrightarrow M_2(e_{11}\mathcal{A}e_{11})$$

defined by

$$\phi(a) = \begin{bmatrix} e_{11}ae_{11} & e_{11}ae_{21} \\ e_{12}ae_{11} & e_{12}ae_{21} \end{bmatrix}$$

and

$$\psi \colon M_2(e_{11}\mathcal{A}e_{11}) \longrightarrow \mathcal{A}$$

defined by

$$\psi\left(\begin{bmatrix} e_{11}ae_{11} & e_{11}be_{11} \\ e_{11}ce_{11} & e_{11}de_{11} \end{bmatrix}\right) = e_{11}ae_{11} + e_{11}be_{12} + e_{21}ce_{11} + e_{21}de_{12}$$

are $*$-homomorphisms which are each other's inverse. □

3. C^*-algebras isomorphic to $M_2(\mathbf{C})$

Theorem 3.1. *A unital C^* algebra $\mathcal{A}$ is $*$-isomorphic to $M_2(\mathbf{C})$ iff there exists a projection $p \in \mathcal{A}$ such that*

$$(*) \qquad p\mathcal{A}p = \mathbf{C}p, (1-p)\mathcal{A}(1-p) = \mathbf{C}(1-p), (1-p)\mathcal{A}p \neq 0, p\mathcal{A}(1-p) \neq 0$$

Proof. If $\mathcal{A} = M_2(\mathbf{C})$, then

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is an appropriate projection.

Conversely, suppose that p is a projection satisfying $(*)$. For $0 \neq u \in p\mathcal{A}(1-p)$ we clearly have $u = pu(1-p)$ and so $uu^* \in p\mathcal{A}p$ and $u^*u \in (1-p)\mathcal{A}(1-p)$. Hence there exists $r > 0$ such that $uu^* = rp$ and $u^*u = r(1-p)$. Replacing u by $\frac{u}{\sqrt{r}}$ we may assume that $uu^* = p$ and $u^*u = 1-p$. If $a \in p\mathcal{A}(1-p)$, then there is $\lambda \in \mathbf{C}$ such that $a = a(1-p) = a(u^*u) = (au^*)u = (\lambda p)u = \lambda u$.

Similarly, if $b \in (1-p)\mathcal{A}p$, we have $b = \mu u^*$ for some $\mu \in \mathbf{C}$.

Since $\mathcal{A} = p\mathcal{A}p \oplus (1-p)\mathcal{A}p \oplus p\mathcal{A}(1-p) \oplus (1-p)\mathcal{A}(1-p)$, every $x \in \mathcal{A}$ is of the form $\lambda_1 p + \lambda_2 u + \lambda_3 u^* + \lambda_4(1-p)$; $\lambda_i \in \mathbf{C}, 1 \leq i \leq 4$.

It is straightforward to show that $\phi : \mathcal{A} \longrightarrow M_2(\mathbf{C})$ defined by

$$\phi(x) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$$

is a $*$-isomorphism. □

Remark 3.2. $p\mathcal{A}(1-p) = 0$ iff $(1-p)\mathcal{A}p = 0$. If this happens and $p\mathcal{A}p = \mathbf{C}p$ and $(1-p)\mathcal{A}(1-p) = \mathbf{C}(1-p)$, we obviously have

$$\mathcal{A} = \mathbf{C}p \oplus \mathbf{C}(1-p) \simeq \mathbf{C}^2 \simeq \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

4. Isomorphisms between C^*-algebras $\mathcal{A}$ and $M_2(\mathcal{A})$

It is known that every C^*-algebra $\mathcal{A}$ with a projection p could be embedded in $M_2(\mathcal{A})$. Indeed $\phi\colon \mathcal{A} \longrightarrow M_2(\mathcal{A})$ defined by

$$\phi(a) = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}$$

is an injective $*$-homomorphism. We are however interested in C^*-algebras $\mathcal{A}$ for which $\mathcal{A} \simeq M_2(\mathcal{A})$:

Definition 4.1. A projection p in a unital C^*-algebra $\mathcal{A}$ is called halving if $p \sim 1$ and $1 - p \sim 1$; i.e. there are partial isometries $u, v \in \mathcal{A}$ such that $p = uu^*, 1 - p = vv^*, u^*u = 1 = v^*v$.

Theorem 4.2. *If $\mathcal{A}$ is a unital C^*-algebra containing a halving projection p, then $\mathcal{A} \simeq M_2(\mathcal{A})$. (See also [6, Corollary 5.3.6])*

Proof. In the notation above, it is straightforward to show that

$$a \mapsto \begin{bmatrix} u^*papu & u^*pa(1-p)v \\ v^*(1-p)apu & v^*(1-p)a(1-p)v \end{bmatrix}$$

is an isomorphism between $\mathcal{A}$ and $M_2(\mathcal{A})$. $\square$

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References

[1] H. Behncke and H. Leptin. C^*-algebras with a two point dual. *J. Func. Anal.*, 10(3):330–335, 1972.
[2] J. Glimm. On a certain class of operator algebras. *Trans. Amer. Math. Soc.*, 95:318–340, 1960.
[3] G.J. Murphy. *Operator Theory and C^*-algebras*. Acad. Press, 1990.
[4] J. Plastiras. C^*-algebras isomorphic after tensoring. *Proc. Amer. Math. Soc.*, 66(2):276–278, 1977.
[5] J. Plastiras. Compact perturbations of certain von Neumann algebras. *Trans. Amer. Math. Soc.*, 234(2), 1977.
[6] N.E. Wegge-Olsen. *K-Theory and C^*-Algebras*. Oxford Univ. Press, 1993.

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