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EXPLODED AND COMPRESSED SPACES

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Abstract. Continuing the theory of exploded and compressed numbers the paper contains five parts. Part 1.: Introduction which contains the most important rules of computation with exploded and compressed numbers. Part 2.: This part contains the concept of explosion and compression of k-dimensional Euclidean space R^k extending the concepts of traditional linear operations, inner product, norm and metric. The elements of exploded space $\overbracket{R^k}$ (super-line, super-plane) are introduced. The concepts of super- and sub-functions were introduced in [1]. Here we extend them for the case of several variables. Part 3.: Descriptions of lux phenomena which show the visible parts of objects in the exploded spaces. Part 4.: The beginning of analysis of functions with several variables defined on the exploded space. Part 5.: A few words on the geometry of the exploded three dimensional space with an interesting open problem for the traditional three dimensional space.

1. Introduction

In [1] we introduced the set of exploded real numbers $\overbracket{R}$ with same equality and ordering relations, familiar on the set of real numbers R such that R is a real subset of $\overbracket{R}$. Any real number was considered as an exploded real number by the explosion

$$\overbracket{x} = \operatorname{area\,th} x, \quad x \in (-1,1), \tag{1.1}$$

that is they are the explodeds of real numbers with an absolute value less than 1. The exploded real numbers $u = \overbracket{x}$ with $x \in (-1,1)$ were called visible exploded real numbers while in the case $x \in R\backslash(-1,1)$, the exploded real numbers were called invisible exploded real numbers. The invisible $\overbracket{-1}$ and $\overbracket{1}$ were called negative and positive discriminators, respectively. For any exploded real number $u = \overbracket{x}$, where $x \in R$, the number x was called the compressed of u denoted by $\underbracket{u}$, that is

$$u = \overbracket{(\underbracket{u})}, \quad u \in \overbracket{R}. \tag{1.2}$$

On the other hand, the identity

$$x = \underbracket{(\overbracket{x})}, \quad x \in R \tag{1.3}$$

can be used, too. The set of the compresseds of real numbers was denoted by $\underbracket{R}$. Clearly, $\underbracket{R} = (-1,1)$. (1.1) and (1.3) yield

$$\underbracket{x} = \operatorname{th} x, \quad x \in R. \tag{1.4}$$

The concept of neighbourhood together with the concept of convergence was extended for the set $\overbracket{R}$. (See [1], Definition 1.9 and Definition 1.17 with Theorem

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2.47.) Moreover, the set $\overbracket{R}$ is a field with the super-addition defined by

$$\overbracket{x} \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\oplus}}\!\!-} \overbracket{y} = \overbracket{x+y}, \quad x, y \in R \tag{1.5}$$

and the super-multiplication defined by

$$\overbracket{x} \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odot}}\!\!-} \overbracket{y} = \overbracket{x \cdot y}, \quad x, y \in R \tag{1.6}$$

such that the field $(R, +, \cdot)$ is isomorphic with the field $(\overbracket{R}, \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\oplus}}\!\!-}, \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odot}}\!\!-})$. (See [1] Theorem 1.38.) The operations super-subtraction

$$\overbracket{x} \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\ominus}}\!\!-} \overbracket{y} = \overbracket{x-y}, \quad x, y \in R \tag{1.7}$$

and super-division

$$\overbracket{x} \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odiv}}\!\!-} \overbracket{y} = \overbracket{\left(\frac{x}{y}\right)}, \quad x, y \in R, y \neq 0 \tag{1.8}$$

were introduced, too. Moreover, we can use

$$u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\oplus}}\!\!-} v = \overbracket{\underbracket{u} + \underbracket{v}}, \quad u, v \in \overbracket{R}, \text{ (See [1], (1.27).)} \tag{1.9}$$

$$u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\ominus}}\!\!-} v = \overbracket{\underbracket{u} - \underbracket{v}}, \quad u, v \in \overbracket{R}, \text{ (See [1], (1.30).)} \tag{1.10}$$

and

$$u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odot}}\!\!-} v = \overbracket{\underbracket{u} \cdot \underbracket{v}}, \quad u, v \in \overbracket{R}.\text{(See [1], (1.33).)} \tag{1.11}$$

The convergence $\lim_{n\to\infty} u_n = u_0$ $(u_n, u_0 \in \overbracket{R})$ means that for any positive $\varepsilon (> 0)$ there exists a real number ν such that if $n > \nu$ then $|u_n \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\ominus}}\!\!-} u_0| < \varepsilon$ holds. (See [1], Theorem 2.46.) If $u_n, u_0 \in R$, then we get back to the familiar definition of convergence. (See [1], Theorem 2.47.) If $u_n \in R$ then the traditional $\lim_{n\to\infty} u_n = \infty$ is equivalent with $\lim_{n\to\infty} u_n = \overbracket{1}$. Generally, the latter limit has a meaning for the sequences of invisible exploded real numbers, too. By (1.1) and (1.4) the identity (1.9) yields

Theorem 1.12. *If* $u, v \in R$ *and* $|\operatorname{th} u + \operatorname{th} v| < 1$ *then* $u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\oplus}}\!\!-} v \in R$ *and* $u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\oplus}}\!\!-} v = \operatorname{area\,th}(\operatorname{th} u + \operatorname{th} v)$.

By (1.1) and (1.4) the identity (1.11) yields

Theorem 1.13. *If* $u, v \in R$ *then* $u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odot}}\!\!-} v \in R$ *and* $u \mathbin{-\!\!\overset{\vee}{\underset{\wedge}{\odot}}\!\!-} v = \operatorname{area\,th}(\operatorname{th} u \cdot \operatorname{th} v)$.

Theorem 3.7 in [1] says that the field $(\underbracket{R}, \oplus, \odot)$ with

$$\xi \oplus \eta = \frac{\xi + \eta}{1 + \xi \cdot \eta}, \quad \xi, \eta \in \underbracket{R} \tag{1.14}$$

and

$$\xi \odot \eta = \operatorname{th}(\operatorname{area\,th} \xi \cdot \operatorname{area\,th} \eta), \quad \xi, \eta \in \underbracket{R} \tag{1.15}$$

is isomorphic with the field $(R, +, \cdot)$. Hence

$$\xi \ominus \eta = \frac{\xi - \eta}{1 - \xi \cdot \eta}, \quad \xi, \eta \in \underbracket{R}. \tag{1.16}$$

The set $\overbracket{R}$ has the same ordering which is usual in R. Namely, for any $x, y \in R$

$$\overbracket{x} < \overbracket{y} \quad \text{if and only if} \quad x < y. \tag{1.17}$$

The concepts of super- and sub-functions were also introduced in [1]. (See Part 4.) Namely, if f is a traditional (one-variable) function then

$$\operatorname{spr} f(u) = \overbracket{f(\underbracket{u})}, \quad \underbracket{u} \in D_f \subseteq R \tag{1.18}$$

and if F is an (one-variable) function with definition-domain $D_F \subseteq \overbracket{R}$ then

$$\operatorname{sub} F(x) = \underbracket{F(\overbracket{x})}, \quad \overbracket{x} \in D_F. \tag{1.19}$$

2. Exploded and compressed k-dimensional spaces

Considering the familiar k-dimensional Euclidean space R^k with its traditional linear operations, inner product, norm and metric, we define the exploded k-dimensional space as follows:

$$\overbracket{R^k} = \{U = (u_1, u_2, u_3, \ldots, u_k) : u_i \in \overbracket{R},\ i = 1, 2, 3, \ldots k\}. \tag{2.1}$$

If $V = (v_1, v_2, v_3, \ldots, v_k) \in \overbracket{R^k}$, we say that $U = V$ if and only if $u_i = v_i$, $i = 1, \ldots k$. Denoting $x_i = \underbracket{u_i}$, $(i = 1, \ldots, k)$, the point

$$\underbracket{U} = (x_1, x_2, x_3, \ldots, x_k), \quad x_i \in R, \quad i = 1, 2, 3, \ldots, k \tag{2.2}$$

is called the compressed of U. On the other hand, if $X = (x_1, x_2, x_3, \ldots, x_k) \in R^k$ then

$$\overbracket{X} = (\overbracket{x_1}, \overbracket{x_2}, \overbracket{x_3}, \ldots, \overbracket{x_k}) \tag{2.3}$$

is called the exploded of X. By (1.2) and (2.2) we have

$$U = \overbracket{(\underbracket{U})}, \quad U \in \overbracket{R^k}. \tag{2.4}$$

Similarly, (1.3) and (2.3) yield

$$X = \underbracket{(\overbracket{X})}, \quad X \in R^k. \tag{2.5}$$

Clearly, $U = V$ if and only if $\underbracket{U} = \underbracket{V}$. By (1.1), (2.3) shows that $\overbracket{X} \in R^k$ if and only if $x_i \in \underbracket{R}$, $i = 1, 2, 3, \ldots, k$. Hence,

$$R^k \subset \overbracket{R^k}, \quad k = 1, 2, 3, \ldots. \tag{2.6}$$

For any subset $S \subset \overbracket{R^k}$, the set

$$\underbracket{S} = \{\underbracket{U} \in R^k : U \in S\} \tag{2.7}$$

is called the compressed set of S. Hence, (1.4), (2.2) and (2.7) yield

$$\underbracket{R^k} = \{X \in R^k : |x_i| < 1, \quad i = 1, 2, 3, \ldots, k\}. \tag{2.8}$$

For any subset $S \subset R^k$, the set

$$\overbracket{S} = \{\overbracket{X} \in \overbracket{R^k} : X \in S\} \tag{2.9}$$

is called the exploded set of S. Hence (2.7) with (2.4) and (2.9) with (2.5) yield

$$S = \overbracket{(\underbracket{S})}, \quad S \subset \overbracket{R^k} \tag{2.10}$$

and

$$S = \underbracket{(\overbracket{S})}, \quad S \subset R^k \tag{2.11}$$

respectively. The most important exploded and compressed sets are mentioned in the following

Definition 2.12.

- The exploded set of a line of the space R^k is called a super-line. Its compressed set is called a sub-line.

- The exploded set of a plane of the space R^k is called a super-plane. Its compressed set is called a sub-plane.
- The exploded set of a sphere of the space R^k is called a super-sphere. Its compressed set is called a sub-sphere.

Definition 2.13. The set $S \subset \overbracket{R^k}$ is called bounded if there exists a positive exploded real number b such that for any $U = (u_1, u_2, u_3, \dots, u_k)$ belonging to S, the inequalities $-b \le u_i \le b$, $i = 1, 2, 3, \dots, k$ hold. (The concept of a positive exploded number and the sign "-" for exploded real numbers was introduced in Definitions 1.8 and 2.32 of [1], respectively. We mention that Definition 1.22 in [1] is special case of the present Definition 2.13.)

Remark 2.14. With respect to (2.6) we have that R^k is bounded with $b = \overbracket{1}$. Moreover, by Theorem 1.11 of [1] we have that the set S is bounded if and only if the set $\underbracket{S}$ is bounded with a bound $b \in R^+$. Consequently, every super-sphere is bounded.

Now we extend the concepts of super- and sub-functions.

First of all we say that a function given by the equation

$$y = f(X), \quad X = (x_1, x_2, x_3, \dots, x_k), \tag{2.15}$$

is a traditional function if its domain $D_f \subseteq R^k$ and range $R_f \subseteq R$ where

$$R_f = \{y \in R : y = f(X) \quad \text{with} \quad X \in D_f\}. \tag{2.16}$$

For any traditional function f we define its super-function denoted by $\operatorname{spr} f$ as follows:

$$D_{\operatorname{spr} f} = \{U \in \overbracket{R^k} : \underbracket{U} \in D_f\} \tag{2.17}$$

and

$$\operatorname{spr} f(U) = \overbracket{f(\underbracket{U})}. \tag{2.18}$$

Hence,

$$R_{\operatorname{spr} f} = \{v \in \overbracket{R} : v = \operatorname{spr} f(U) \quad \text{with} \quad U \in D_{\operatorname{spr} f}\}. \tag{2.19}$$

For any function F with $D_F \subseteq \overbracket{R^k}$ and $R_F \subseteq \overbracket{R}$ we define its sub-function, denoted by $\operatorname{sub} F$ as follows:

$$D_{\operatorname{sub} F} = \{X \in R^k : \overbracket{X} \in D_F\} \tag{2.20}$$

and

$$\operatorname{sub} F(X) = \underbracket{F(\overbracket{X})}. \tag{2.21}$$

Hence,

$$R_{\operatorname{sub} F} = \{y \in R : y = \operatorname{sub} F(X) \quad \text{with} \quad X \in D_{\operatorname{sub} F}\}. \tag{2.22}$$

Clearly,

$$D_{\operatorname{sub} F} = \underbracket{D_F} \quad \text{and} \quad R_{\operatorname{sub} F} = \underbracket{R_F}. \tag{2.23}$$

Theorems 4.11 and 4.14 in [1] can easily be extended for the functions of several variables so, without any proof, we use that

$$F = \operatorname{spr}(\operatorname{sub} F) \tag{2.24}$$

and for any traditional function

$$f = \operatorname{sub}(\operatorname{spr} f) \tag{2.25}$$

holds. Moreover, we remark on the basis of (2.24) that every traditional function is a super-function, too.

Considering the inner product of

$$X = (x_1, x_2, x_3, \dots, x_k) \text{ and } Y = (y_1, y_2, y_3, \dots, y_k)$$

$$X \cdot Y = \sum_{i=1}^{k} x_i y_i, \quad X, Y \in R^k \tag{2.26}$$

as a traditional function f of X with parameter Y and using $x_i = \underbracket{u_i}$, $y_i = \underbracket{v_i}$, $i = 1, 2, 3, \dots, k$, by (2.18), (2.26), (1.5) and (1.6) we obtain

$$\begin{aligned} \operatorname{spr} f(U) = {} & (u_1 \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} v_1) \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} (u_2 \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} v_2) \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} \\ & (u_3 \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} v_3) \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} \dots \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} (u_k \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} v_k). \end{aligned} \tag{2.27}$$

Moreover, if $X = Y$, (2.26) gives the traditional norm of X

$$\|X\|_{R^k} = \sqrt{\sum_{i=1}^{k} x_i^2} \tag{2.28}$$

can be considered as a traditional function f of X with $D_f = R^k$ and $R_f = R_0^+$ (where R_0^+ denotes the set of non-negative real numbers). Applying the traditional power-function p_α (see (4.16) in [1]) we have the one variable square root function $p_{\frac{1}{2}}$ that is

$$p_{\frac{1}{2}}(x) = \sqrt{x} \quad \text{with} \quad D_{p_{\frac{1}{2}}} = R_{p_{\frac{1}{2}}} = R_0^+.$$

So, the super-square root function is

$$\operatorname{spr} p_{\frac{1}{2}}(u) = \overbracket{(\sqrt{\underbracket{u}})} \quad \text{with} \quad u \in \overbracket{R_0^+}. \tag{2.29}$$

Now, we introduce the following operations based on (1.5) and (1.6). Having the elements of $\overbracket{R^k}$, $U = (u_1, u_2, u_3, \dots, u_k)$, $V = (v_1, v_2, v_3, \dots, v_k)$ and $W = (w_1, w_2, w_3, \dots, w_k)$ the super-addition $U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} V$ is an element of $\overbracket{R^k}$ such that

$$U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} V = (u_1 \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} v_1, u_2 \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} v_2, u_3 \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} v_3, \dots, u_k \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} v_k). \tag{2.30}$$

Considering number $c \in \overbracket{R}$ the super-multiplication $c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} U$ is an element of $\overbracket{R^k}$ such that

$$c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} U = (c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} u_1, c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} u_2, c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} u_3, \dots, c \overset{\vee}{\underset{\wedge}{-\!\odot\!-}} u_k). \tag{2.31}$$

By the identities (1.9) and (1.11) the following theorems are obvious.

Theorem 2.32. *For any $U, V, W \in \overbracket{R^k}$ the identities*

$$U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} V = V \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} U \tag{2.33}$$

$$(U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} V) \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} W = U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} (V \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} W) \tag{2.34}$$

$$U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} O = U \quad \textit{with} \quad O = (0, 0, 0, \dots, 0) \in \overbracket{R^k} \tag{2.35}$$

and

$$U \overset{\vee}{\underset{\wedge}{-\!\oplus\!-}} (-U) = O \quad \textit{with} \quad -U = (-u_1, -u_2, -u_3, \dots, -u_k) \tag{2.36}$$

hold.

Theorem 2.37. *For any c, c_1 and $c_2 \in \overbracket{R}$ and $U, V \in \overbracket{R^k}$ the identities*

$$\overbracket{1} \mathbin{-\!\!\odot\!\!-} U = U \tag{2.38}$$

$$(c_1 \mathbin{-\!\!\odot\!\!-} c_2) \mathbin{-\!\!\odot\!\!-} U = c_1 \mathbin{-\!\!\odot\!\!-} (c_2 \mathbin{-\!\!\odot\!\!-} U) \tag{2.39}$$

$$c \mathbin{-\!\!\odot\!\!-} (U \mathbin{-\!\!\oplus\!\!-} V) = (c \mathbin{-\!\!\odot\!\!-} U) \mathbin{-\!\!\oplus\!\!-} (c \mathbin{-\!\!\odot\!\!-} V) \tag{2.40}$$

and

$$(c_1 \mathbin{-\!\!\oplus\!\!-} c_2) \mathbin{-\!\!\odot\!\!-} U = (c_1 \mathbin{-\!\!\odot\!\!-} U) \mathbin{-\!\!\oplus\!\!-} (c_2 \mathbin{-\!\!\odot\!\!-} U) \tag{2.41}$$

hold.

Considering Theorems 2.32 and 2.37 we say that $\overbracket{R^k}$ is a super-linear space over the field $\overbracket{R}$. Applying (2.2), (2.3), (1.9) and (1.11), (2.30) and (2.31) yield the identities

$$U \mathbin{-\!\!\oplus\!\!-} V = \overbracket{\underbracket{U} + \underbracket{V}} \quad U, V \in \overbracket{R^k}, \tag{2.42}$$

where "+" is the familiar addition of vectors in R^k and

$$c \mathbin{-\!\!\odot\!\!-} U = \overbracket{\underbracket{c} \cdot \underbracket{U}} \quad c \in \overbracket{R} \quad \text{and} \quad U \in \overbracket{R^k} \tag{2.43}$$

where is "$\cdot$" the familiar multiplication of vectors by a number.

Assuming that $U, V \in R^k$, by (2.30) Theorem 1.12 shows that under certain conditions $U \mathbin{-\!\!\oplus\!\!-} V \in R^k$ while the identities (1.9) and (2.42) show that $U \mathbin{-\!\!\oplus\!\!-} V$ may be outside R^k. On the other hand, if $c \in R$ and $U \in R^k$ then (2.31), (2.43) and Theorem 1.13 show that $c \mathbin{-\!\!\odot\!\!-} U \in R^k$.

Definition 2.44. The super-sum of exploded real numbers $a_1, \ldots, a_n$ will be signed by

$$\operatorname{spr}\sum_{i=1}^{n} a_i = a_1 \mathbin{-\!\!\oplus\!\!-} \ldots \mathbin{-\!\!\oplus\!\!-} a_n.$$

Refering back to (2.26) and (2.27) we give

Definition 2.45. For any pair $U, V \in \overbracket{R^k}$, their super-inner product is defined by the super-sum

$$U \mathbin{-\!\!\odot\!\!-} V = \operatorname{spr}\sum_{i=1}^{k} (u_i \mathbin{-\!\!\odot\!\!-} v_i).$$

Hence, the identity (1.11) and Definition 2.44 by (1.5), (2.2) and (2.26) yield the identity

$$U \mathbin{-\!\!\odot\!\!-} V = \overbracket{\underbracket{U} \cdot \underbracket{V}} \quad U, V \in \overbracket{R^k}. \tag{2.46}$$

Theorem 2.47. *For any $U, V, W \in \overbracket{R^k}$ and $c \in \overbracket{R}$ the properties*

$$U \mathbin{-\!\!\odot\!\!-} V = V \mathbin{-\!\!\odot\!\!-} U \tag{2.48}$$

$$(U \mathbin{-\!\!\oplus\!\!-} V) \mathbin{-\!\!\odot\!\!-} W = (U \mathbin{-\!\!\odot\!\!-} W) \mathbin{-\!\!\oplus\!\!-} (V \mathbin{-\!\!\odot\!\!-} W) \tag{2.49}$$

$$(c \mathbin{-\!\!\odot\!\!-} U) \mathbin{-\!\!\odot\!\!-} V = c \mathbin{-\!\!\odot\!\!-} (U \mathbin{-\!\!\odot\!\!-} V) \tag{2.50}$$

$$U \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} U \geq 0, \quad U \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} U = 0 \quad \textit{if and only if} \quad U = O \tag{2.51}$$

are valid.

Proof. Applying Definition 2.45 with (2.46) and having the familiar properties of inner product of vectors in R^k, properties (2.48) - (2.51) are obtained. In detail: (2.48) is an immediate consequence of (2.46); (2.49) is obtained by (2.42), (1.5) and (2.46); (2.50) is obtained by (2.43), (1.6) and (2.46); and finally (2.51) is obtained by (2.3), (2.4), (1.1) and (2.46). $\square$

After Theorem 2.47 we can say that $\overline{R^k}$ is a super-Euclidean space. With respect to (2.29) and (2.51) we can give the following

Definition 2.52. For any $U \in \overline{R^k}$, its super-norm is

$$\|U\|_{\overline{R^k}} = \operatorname{spr} p_{\frac{1}{2}}(U \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} U).$$

Hence, (2.46) with (1.3) yields the identity

$$\|U\|_{\overline{R^k}} = \overline{\|\underline{U}\|_{R^k}}, \quad U \in \overline{R^k}. \tag{2.53}$$

Remark 2.54. Considering $F(U) = \|U\|_{\overline{R^k}}$ with $D_F = \overline{R^k}$ and $R_F = \overline{R_0^+}$ applying (2.20) and (2.21) by (2.53), (2.5) and (2.4) we obtain that $\operatorname{sub} F(X) = \|X\|_{R^k}$.

Theorem 2.55. *For any $U, V \in \overline{R^k}$ the Cauchy-type inequality*

$$|U \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} V| \leq \|U\|_{\overline{R^k}} \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} \|V\|_{\overline{R^k}} \tag{2.56}$$

holds.

Proof. Starting from (2.46), after (2.36) in [1], we apply the well-known Cauchy-inequality with Definition 1.7 in [1], by (1.6) we have

$$|u \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} v| = \overline{|\underline{U} \cdot \underline{V}|} \leq \overline{\|\underline{U}\|_{R^k} \cdot \|\underline{V}\|_{R^k}} = \overline{\|\underline{U}\|_{R^k}} \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} \overline{\|\underline{V}\|_{R^k}}$$

Hence, (2.53) gives (2.56). $\square$

Theorem 2.57. *For any $U, V \in \overline{R^k}$ and $c \in \overline{R}$ the properties*

$$\|U\|_{\overline{R^k}} \geq 0 \textit{ and } \|U\|_{\overline{R^k}} = 0 \textit{ if and only if } U = 0, \tag{2.58}$$

$$\|c \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} U\|_{\overline{R^k}} = |c| \mathbin{\overset{\vee}{\underset{\wedge}{-\odot-}}} \|U\|_{\overline{R^k}} \tag{2.59}$$

and

$$\|U \mathbin{\overset{\vee}{\underset{\wedge}{-\oplus-}}} V\|_{\overline{R^k}} \leq \|U\|_{\overline{R^k}} \mathbin{\overset{\vee}{\underset{\wedge}{-\oplus-}}} \|V\|_{\overline{R^k}} \tag{2.60}$$

are valid.

Proof. Applying Definition 2.52 with (2.53) and (2.2) as well as having the familiar properties of norm (2.28), properties (2.58) - (2.60) are obtained. In detail: (2.58) is a consequence of (2.3) and (1.1); (2.59) is obtained by (2.43), (1.2) and (1.6); (2.60) is obtained by (2.42) and (1.5). $\square$

After Theorem 2.57 we can say that $\overline{R^k}$ is a super-normed space. The inequality (2.60) is called a Minkowski-type inequality.

Definition 2.61. For any pair $U, V \in \overbracket{R^k}$ their super-distance is

$$d_{\overbracket{R^k}}(U,V) = \|U \ominus V\|_{\overbracket{R^k}}$$

where the super-difference of U and V is based on (2.30) and (2.31) such that

$$U \ominus V = U \oplus (\overbracket{-1} \odot V). \tag{2.62}$$

Hence, (2.42), (2.43) with (1.3) yield the identity

$$U \ominus V = \overbracket{\underbracket{U} - \underbracket{V}}, \quad U, V \in \overbracket{R^k}, \tag{2.63}$$

where "-" is the familiar difference of vectors in R^k. The identity (2.63) with (2.5) gives

$$\|U \ominus V\|_{\overbracket{R^k}} = \overbracket{\|\underbracket{U} - \underbracket{V}\|_{R^k}}. \tag{2.64}$$

Having the familiar

$$d_{R^k}(\underbracket{U}, \underbracket{V}) = \|\underbracket{U} - \underbracket{V}\|_{R^k} \tag{2.65}$$

by (2.64) Definition 2.61 gives

$$d_{\overbracket{R^k}}(U,V) = \overbracket{d_{R^k}(\underbracket{U}, \underbracket{V})}, \quad U, V \in \overbracket{R^k}. \tag{2.66}$$

Theorem 2.67. *For any U, V and $W \in \overbracket{R^k}$ the properties*

$$d_{\overbracket{R^k}}(U,V) = d_{\overbracket{R^k}}(V,U)$$

$$d_{\overbracket{R^k}}(U,V) \geq 0 \text{ and } d_{\overbracket{R^k}}(U,V) = 0 \text{ if and only if } U = V$$

and

$$d_{\overbracket{R^k}}(U,V) \leq d_{\overbracket{R^k}}(U,W) \oplus d_{\overbracket{R^k}}(W,V)$$

are valid.

Proof. Applying Definition 2.61 with (2.66) and having the familiar properties of traditional distance (2.65), the first two properties are trivial. For the last by (2.66), Definition 1.7 in [1], (1.5) and (2.66) again, we can write

$$d_{\overbracket{R^k}}(U,V) = \overbracket{d_{R^k}(\underbracket{U}, \underbracket{V})} \leq \overbracket{d_{R^k}(\underbracket{U}, \underbracket{W}) + d_{R^k}(\underbracket{W}, \underbracket{V})} =$$
$$= \overbracket{d_{R^k}(\underbracket{U}, \underbracket{W})} \oplus \overbracket{d_{R^k}(\underbracket{W}, \underbracket{V})} = d_{\overbracket{R^k}}(U,W) \oplus d_{\overbracket{R^k}}(W,V).$$

□

After Theorem 2.67 we can say that $\overbracket{R^k}$ is a super-metrical space. The third property in Theorem 2.67 may be called a super-triangle inequality. (See [1], (2.38).)

Returning back to Definition 2.12 we characterize some sets mentioned there by equations or inequalities.

Example 2.68. It is known that a line of the space R^k is characterized by the equation

$$X = X_0 + t \cdot E, \quad t \in R \tag{2.69}$$

where $X_0, E \in R^k$ are given such that $\|E\|_{R^k} = 1$. Denoting by S the set of X given by the equation (2.69) and considering (2.9), by (1.5) and (1.6) we have

$$\overbracket{X} = \overbracket{X_0} \oplus (\overbracket{t} \odot \overbracket{E}). \tag{2.70}$$

So, denoting $\overbracket{X} = U$, $\overbracket{X_0} = U_0$, $\overbracket{t} = \tau$ and $V = \overbracket{E}$ we have the equation of super-line

$$U = U_0 \mathbin{\overset{\vee}{\underset{\wedge}{\oplus}}} (\tau \mathbin{\overset{\vee}{\underset{\wedge}{\odot}}} V), \quad \tau \in \overbracket{R} \tag{2.71}$$

where by (2.53) we have

$$\|V\|_{\overbracket{R^k}} = \overbracket{1}. \tag{2.72}$$

Moreover, by Definition 2.61, (2.59), (2.72) and (1.6)

$$d_{\overbracket{R^k}}(U, U_0) = |\tau|. \tag{2.73}$$

Similarly to (2.70) the equation of sub-line which is the compressed of the line given by (2.69) is

$$\underbracket{X} = \underbracket{X_0} \oplus (\underbracket{t} \odot \underbracket{E}), \quad \underbracket{t} \in \underbracket{R}, \tag{2.74}$$

where the sub-addition and sub-multiplication are mentioned under (1.14) and (1.15).

Example 2.75. It is known that a plane of the space R^k is characterized by the equation

$$(X - X_0) \cdot N = 0 \tag{2.76}$$

where X_0, $N \in R^k$ such that $\|N\|_{R^k} = 1$. Denoting by S the set of X given by the equation (2.76) and considering (2.9) by (2.62), (1.5), (1.6), (2.46) and (1.1) we have for the points of super-plane $\overbracket{S}$

$$(\overbracket{X} \mathbin{\overset{\vee}{\underset{\wedge}{\ominus}}} \overbracket{X_0}) \mathbin{\overset{\vee}{\underset{\wedge}{\odot}}} \overbracket{N} = 0. \tag{2.77}$$

So, denoting $\overbracket{X} = U$, $\overbracket{X_0} = U_0$ and $\overbracket{N} = M$ we have the equation of super-plane

$$(U \mathbin{\overset{\vee}{\underset{\wedge}{\ominus}}} U_0) \mathbin{\overset{\vee}{\underset{\wedge}{\odot}}} M = 0 \tag{2.78}$$

where by (2.53) we have that

$$\|M\|_{\overbracket{R^k}} = \overbracket{1}. \tag{2.79}$$

Similarly to (2.77) the equation of sub-plane $\underbracket{S}$ which is the compressed of the plane S given by (2.76) is

$$(\underbracket{X} \ominus \underbracket{X_0}) \odot \underbracket{N} = 0. \quad \text{(See (2.2), (1.15) and (1.16).)} \tag{2.80}$$

Example 2.81. If $X_0 \in R^k$ and $r \in R_+$ then the sphere and open ball with centre X_0 and radius r can be described by the equation

$$d_{R^k}(X, X_0) = r \tag{2.82}$$

and the inequality

$$d_{R^k}(X, X_0)) < r, \tag{2.83}$$

respectively. Considering (2.9) we denote the super-sphere and the super-ball with $U_0 = \overbracket{X_0}$ and $\rho = \overbracket{r}$ by $S_{U_0}(\rho)$ and $G_{U_0}(\rho)$, respectively. Using (2.82) and (2.83) by (2.66) we obtain

$$S_{U_0}(\rho) = \{U \in \overbracket{R^k} : d_{\overbracket{R_k}}(U, U_0) = \rho, \quad \rho \in \overbracket{R^+}\} \tag{2.84}$$

and

$$G_{U_0}(\rho) = \{U \in \overbracket{R^k} : d_{\overbracket{R^k}}(U, U_0) < \rho, \quad \rho \in \overbracket{R^+}\}. \tag{2.85}$$

Remark 2.86. Referring back (2.10) the familiar lines, planes spheres and balls can be considered as exploded sets of their compressed sets. Moreover, (2.11) shows that they behave as "super-lines", "super-planes", "super-spheres" and "super-balls" with respect to the compressed k-dimensional space $\underbracket{R^k}$ mentioned under (2.8). On the other hand, the sub-line given by (2.74), the sub-plane given by (2.80), sub-sphere $\underbracket{\underbracket{S_{U_0}(\rho)}}$ and the sub-ball $\underbracket{\underbracket{S_{U_0}(\rho)}}$ behave as "line", "plane", "sphere" and "ball" in $\underbracket{R^k}$. So, $\underbracket{R^k}$ is a model of R^k while R^k is a model of $\overbracket{R^k}$.

Closing part 2 for any $1 \leq \ell < k$ we identify the element $(u_1, u_2, \ldots, u_\ell) \in \overbracket{R^\ell}$ with $(u_1, u_2, \ldots, u_\ell, 0, \ldots, 0) \in \overbracket{R^k}$ so, $\overbracket{R^\ell} \subset \overbracket{R^k}$. Clearly, $\overbracket{R^\ell}$ is a subspace of $\overbracket{R^k}$.

3. The lux and sub-lux phenomena

Definition 3.1. If S is a subset of $\overbracket{R^k}$ then the segments $S_{\text{lux}} = S \cap R^k$ and $S_{\text{sublux}} = S \cap \underbracket{R^k}$ are called lux phenomenon and sub-lux phenomenon of the set S, respectively. In the case $k = 1, 2$ and 3 the lux (and sub-lux) phenomena are called road, window and box phenomena, respectively.

Clearly, if $S \subseteq R^k$ then $S_{\text{lux}} = S$. Especially, $R^k_{\text{lux}} = R^k$. Moreover, R is the road phenomenon of $\overbracket{R^3}$. The subsets of R, R^2 and R^3 are their own road, window and box phenomena, respectively. By Definition 2.12 the elements of $\overbracket{R}$ form a super-line which coincides with $\overbracket{R^2}$. Moreover, the points of $\overbracket{R^2}$ form a super-plane in the space $\overbracket{R^3}$. These relationships can be studied on the compressed three dimensional space $\underbracket{R^3}$ which is the cube model of the traditional three dimensional space R^3: By Fig.3.2 the space $\underbracket{R}$ can be considered as the compressed of the real

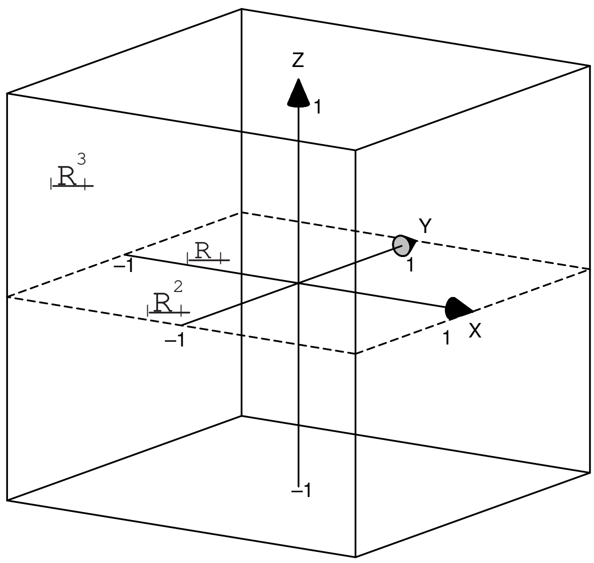


Figure 3.2.

axis identified by R. Similarly, space $\overbracket{R}$ can be considered the exploded of the real axis. Shortly, we can speak of compressed and exploded real axes, respectively.

Example 3.3. The elements of sequence

$$u_n = \overbracket{\left(\frac{n+(-1)^n}{n}\right)} \quad , \quad n = 1, 2, 3, \ldots$$

form a subset of $\overbracket{R}$. Its road phenomenon is the set of elements of sub-sequence

$$u_{2\ell-1} = \overbracket{\left(\frac{2\ell-2}{2\ell-1}\right)} \quad = \operatorname{area\,th}\big(1 - \frac{1}{2\ell-1}\big), \quad 1 = 1, 2, \ldots \text{(see (1.1))}.$$

Clearly, $\lim_{n\to\infty} u_n = \overbracket{1}$, (if $n > \frac{1}{\operatorname{th}\varepsilon}$ then $|u_n \ominus \overbracket{1}| < \varepsilon$) and, consequently, $\lim_{\ell\to\infty} u_{2\ell-1} = \overbracket{1}$, (traditionally, $\lim_{\ell\to\infty} u_{2\ell-1} = \infty$). The elements of subsequence $\{u_{2\ell}\}_{\ell=1}^{\infty}$ are invisible exploded real numbers and they do not belong to the road phenomenon although $\lim_{\ell\to\infty} u_{2\ell} = \overbracket{1}$, of course.

Example 3.4. The points $(u, v) \in R^2$ satisfying equation $v = 2 \cdot u$ form a line in R^2 and their set is the window phenomenon of the set S of points $(u, v) \in \overbracket{R^2}$ satisfying the equation

$$v = (\overbracket{2} \odot u) \odot (\overbracket{1} \oplus \operatorname{spr} p_2(u)), \quad \text{(see Example 4.51 in [1])}. \tag{3.5}$$

Moreover, if $u \in \overbracket{R}\backslash R$ the points (u, v) are invisible. On the other hand, the points $(x, y) \in R^2$ satisfying the equation $y = 2 \cdot x$ form the compressed set of the set L of points $(\overbracket{x}, \overbracket{y}) \in \overbracket{R^2}$ satisfying the equation

$$\overbracket{y} = \overbracket{2} \odot \overbracket{x} \tag{3.6}$$

because denoting $u = \overbracket{x}$ and $v = \overbracket{y}$, by (2.2), (1.3), (3.6) and (1.6) we can write $\underbracket{(u, v)} = (x, \underbracket{(\overbracket{y})}) = (x, \underbracket{\overbracket{2} \odot \overbracket{x}}) = (x, 2 \cdot x)$ for any $x \in R$. Hence, (2.10) and Definition 2.12 say that the points $(u, v) \in \overbracket{R^2}$ form a super-line. (We can check that by the equation (2.70) with $X_0 = (0, 0)$, $E = \big(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\big)$ and $t = x$.) Of course, the line $\ell = \{(u, v) \in R^2 : v = 2 \cdot u\}$ is not the window phenomenon of super-line L given by (3.6) because the equation of window phenomenon of super-line is

$$v = \operatorname{area\,th}(2 \operatorname{th} u), \quad |u| < \operatorname{area\,th} \frac{1}{2} \left(= \overbracket{\left(\frac{1}{2}\right)} \;\right). \tag{3.7}$$

(See Theorem 1.12 where the identity $\overbracket{2} \odot u = u \oplus u$ is used.) If $|u| \geq \operatorname{area\,th} \frac{1}{2}$ then the points of super-line L are invisible in R^2. Using Theorem 4.50 in [1] we have that super-linear function $v = \overbracket{2} \odot u$ is continuous on $\overbracket{R}$. Hence,

$$\lim_{u \to -\operatorname{area\,th}\frac{1}{2}} v(u) = \overbracket{-1} \text{ and } \lim_{u \to \operatorname{area\,th}\frac{1}{2}} v(u) = \overbracket{1},$$

see Definition 4.48 in [1]. (The right-hand-side limit of the first and the left-hand-side limit of the second can be checked by (3.7).)

If we compress the line ℓ given by the equation $y = 2 \cdot x, (x \in R)$ we have a sub-line with the equation $\eta = \frac{2\xi}{1+\xi^2}$, $(\xi \in \underbracket{R})$. (See (3.4) in [1]). Moreover, $y = \frac{2x}{1+x^2}$ is the subfunction of the function $v = F(u)$ given by (3.5) because (2.21) shows that

$$\operatorname{sub} F(x) = \underline{F(\overline{x})} = \underline{(\overline{2} \odot \overline{x}) \oslash (\overline{1} \oplus \operatorname{spr} p_2(\overline{x}))} = \frac{2x}{1+x^2}.$$

In the square model $\underline{R}^2$ (see Fig.3.2) the following figure shows the relationship of set S given by (3.5) to its window phenomenon which is the line ℓ by the relationship of the set $\{(x,y) \in R^2 : y = \frac{2x}{1+x^2}\}$ to its sub-window phenomenon which is a sub-line compressed of ℓ. Moreover, we can show the window phenomenon S_{window} which is the compressed of super-line L such that $L_{\text{window}} \neq \ell$.

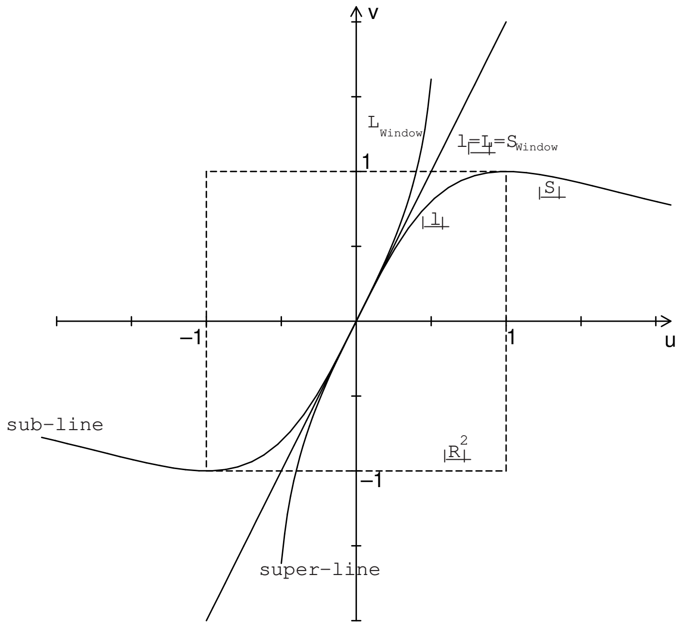


FIGURE 3.8.

Example 3.9. Applying (2.76) with $X_0 = (0,0,0)$ and $N = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ we have the plane

$$S = \{X = (x,y,z) \in R^3 : x, y \in R \quad \text{and} \quad z = x + y\}. \tag{3.10}$$

Using (2.3) with $u = \overline{x}$, $v = \overline{y}$, $w = \overline{z}$ and applying (1.9), Definition 2.12 say that

$$\overline{S} = \{U = (u,v,w) \in \overline{R} : u, v \in \overline{R} \quad \text{and} \quad w = u \oplus v\} \tag{3.11}$$

is a super-plane. Moreover, with respect to (2.80) and (1.14), we have the sub-plane

$$\underline{S} = \{(\xi, \eta, \zeta) \in \underline{R}^3 : \xi, \eta \in \underline{R} \quad \text{and} \quad \zeta = \frac{\xi + \eta}{1 + \xi \cdot \zeta}\}. \tag{3.12}$$

Easy to see that the line ℓ given by the equation

$$X = t \cdot E \quad (E = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}) \quad \text{and} \quad -\infty < t < \infty)$$

coincides with S. Hence, (2.70) and (2.71) yield that the super-line $\overline{\ell}$ having the equation

$$U = \tau \odot V \quad (V = \overline{E} \quad \text{and} \quad \tau \in \overline{R}) \tag{3.13}$$

coincides with the super-plane $\overbracket{S}$. The sub-line

$$\underbracket{\ell} = \{(\xi, \eta, \zeta) \in \underbracket{R^3} : \xi = \eta \in \underbracket{R} \quad \text{and} \quad \zeta = \frac{2\xi}{1+\xi^2}\}$$

coincides with sub-plane $\underbracket{S}$.

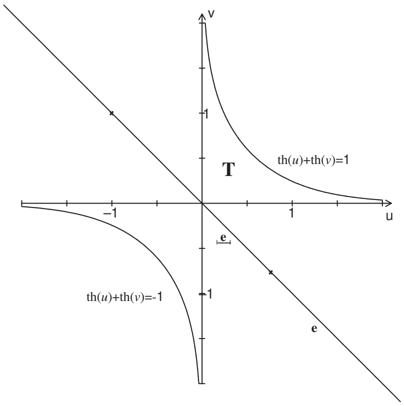
It is obvious that line e given by the equation-system

$$x = s, \quad y = -s, \quad z = 0, \quad s \in R$$

coincides with plane S. Moreover, the super-line $\overbracket{e}$ (where $s \in \overbracket{R}$) and sub-line $\underbracket{e}$ (where $s \in \underbracket{R}$) coincide with super- and sub-plane $\overbracket{S}$ and $\underbracket{S}$, respectively. Let

$$T = \{(u, v) \in R^2 : |\operatorname{th} u + \operatorname{th} v| < 1\}$$

which is shown in the following figure.



By (3.11) Theorem 1.12 says that $(u, v, w) \in \overbracket{S}_{\text{box}}$ if and only if $(u, v) \in T$. Hence, $\overbracket{S}_{\text{box}}$ is a surface in R^3 with the equation $w = \underbracket{\text{area}}\operatorname{th}(\operatorname{th} u + \operatorname{th} v)$. By (3.13) and (2.31) we have that $(u, v, w) \in \overbracket{\ell}_{\text{box}}$ if $|\tau| < (\sqrt{\frac{3}{2}})$. (We can see that the parameter τ has to be an exploded real number.) Of course, $\overbracket{\ell}_{\text{box}} \subset \overbracket{S}_{\text{box}}$. If $(u, v) \in R^2 \backslash T$ then the points of super-plane $\overbracket{S}$ are invisible. Clearly, $\overbracket{e}_{\text{window}} = e$ and $e = S \cap \overbracket{S}_{\text{box}}$, so $e = S \cap \overbracket{S}$. Moreover, $\underbracket{e} = \underbracket{S} \cap S \cap \overbracket{S}$. All relationship will be introduced in the Figure 3.14.

We remark that an open hexagon is the sub-box phenomenon of the plane S.

Example 3.15. The form of a traditional sphere is characterized by its radius, merely. It is not true for the super-sphere introduced in Example 2.81. To show that we consider super-spheres $S_O(1)$ and $S_{U_0}(1)$ where $O = (0, 0, 0)$ and $U_0 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ having the equations $d_{\overbracket{R^3}}(U, O) = 1$ and $d_{\overbracket{R^3}}(U, U_0) = 1$, respectively. Applying (2.84), (2.66) we obtain

$$S_O(1) = \{U = (u, v, w) \in \overbracket{R^3} : \overbracket{\sqrt{(\underbracket{u})^2 + (\underbracket{v})^2 + (\underbracket{w})^2}} = 1\} \tag{3.16}$$

and

$$S_{U_0}(1) = \Big\{U = (u, v, w) \in \overbracket{R^3} : \overbracket{\sqrt{(\underbracket{u} - \operatorname{th}\frac{1}{2})^2 + (\underbracket{v} - \operatorname{th}\frac{1}{2})^2 + (\underbracket{w} - \operatorname{th}\frac{1}{2})^2}} = 1\Big\}, \tag{3.17}$$

respectively.

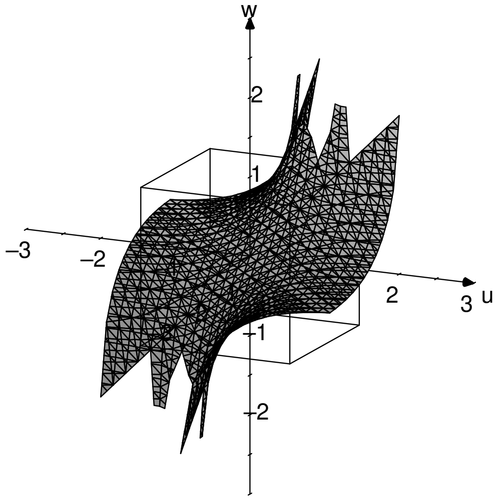


FIGURE 3.14.

Considering (3.16) by (1.2) and (1.17) we have that if $(u, v, w) \in S_O\ (1)$ then $0 \leq \max(|\underline{u}|, |\underline{v}|, |\underline{w}|) \leq \underline{1}$, so definition 2.13 says that $S_O(1)$ is bounded in R^3. This means that

$$S_O(1)_{\text{box}} = S_O(1). \tag{3.18}$$

On the other hand, the point $P_0 = (\frac{1}{2}, \frac{1}{2}, \overline{\text{th}\,1 + \text{th}\,\frac{1}{2}}) \in S_{U_0}(1)$ but $P_0 \notin R^3$. Hence

$$S_{U_0}(1)_{\text{box}} \subset S_{U_0}(1).$$

By (3.16) and (3.18) we have that $S_O(1)_{\text{box}}$ is described by the equation

$$\text{th}^2\, u + \text{th}^2\, v + \text{th}^2\, w = \text{th}^2\, 1$$

while by (3.17) $S_{U_0}(1)_{\text{box}}$ is described by the equation

$$\Big(\text{th}\, u - \text{th}\,\frac{1}{2}\Big)^2 + \Big(\text{th}\, v - \text{th}\,\frac{1}{2}\Big)^2 + \Big(\text{th}\, w - \text{th}\,\frac{1}{2}\Big)^2 = \text{th}^2\, 1.$$

Both box phenomena are shown in Figures 3.19. and Figure 3.20.

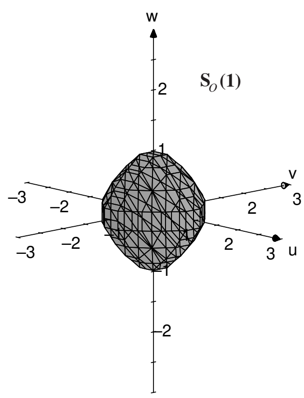


FIGURE 3.19.

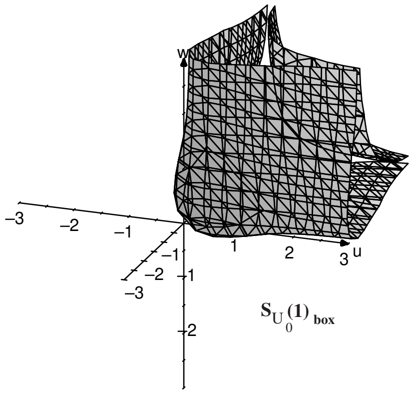


FIGURE 3.20.

If the radius is growing, the super-sphere $S_O(r)$ has newer and newer forms, which do not show a similarity in the usual sense. If the radius is less then $\overbracket{1}$ then $S_O(r)_{\text{box}} = S_O(r)$. The super-sphere $S_O(\overbracket{1})$ has six invisible points, namely $(\overbracket{1},0,0)$, $(0,\overbracket{1},0)$, $(\overbracket{-1},0,0)$, $(0,\overbracket{-1},0)$, $(0,0,\overbracket{1})$ and $(0,0,\overbracket{-1})$. If $r > \overbracket{1}$ then $S_O(r)$ has an infinite number of invisible points. Some examples are shown in figures 3.21–3.24.

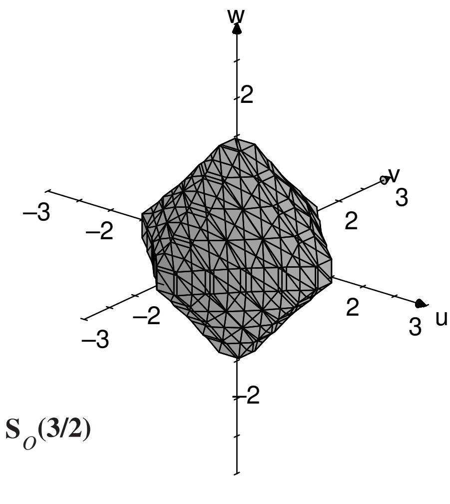


FIGURE 3.21.

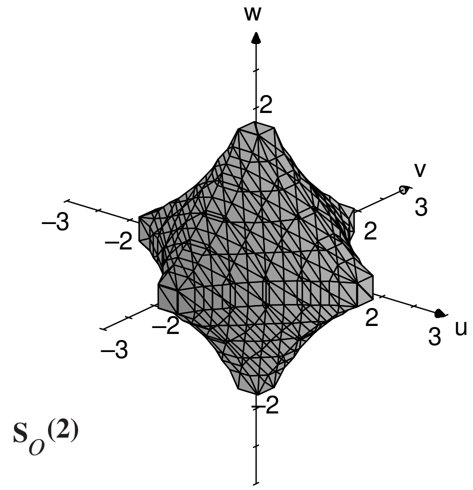


FIGURE 3.22.

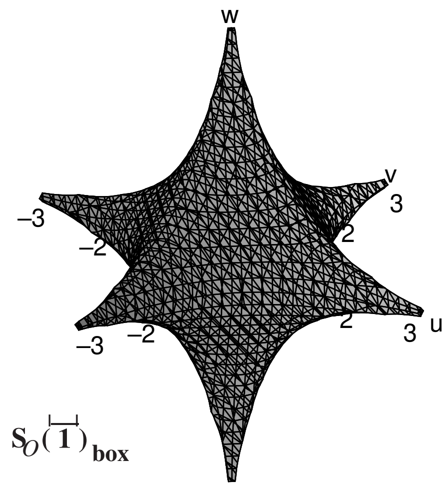


FIGURE 3.23.

4. THE LIMIT AND CONTINUITY OF FUNCTIONS WITH SEVERAL VARIABLES

First of all we consider a sequence $\{U_n\}_{n=1}^{\infty}$ such that $U_n = (u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(k)}) \in \overline{R^k}$, $n = 1, 2, 3, \dots$ where $u_n^{(\ell)} \in \overline{R}$, $\ell = 1, 2, 3, \dots, k$.

Definition 4.1. Having a $U_0 = (u_0^{(1)}, u_0^{(2)}, \dots, u_0^{(k)}) \in \overline{R^k}$ we say that

$$\lim_{n\to\infty} U_n = U_0$$

if

$$\lim_{n\to\infty} d_{\overline{R^k}}(U_n, U_0) = 0. \tag{4.2}$$

Definition 2.61, (2.64) with $k = 1$, (1.10) and Theorem 2.46 in [1] show that Definition 4.1 is a generalization of Definition 1.17 in [1].

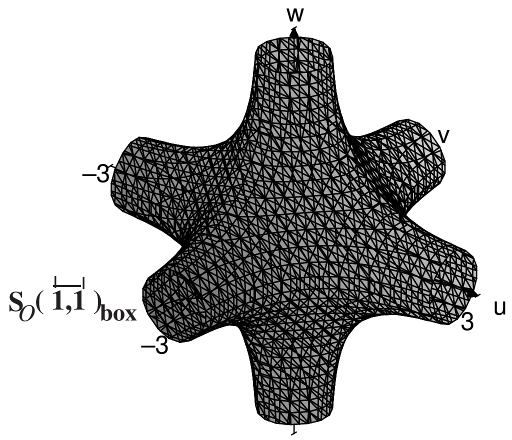


FIGURE 3.24.

Theorem 4.3. *Let us assume that U_n, $U_0 \in \overbracket{R^k}$, $n = 1, 2, \ldots$. These sequence $\{U_n\}_{n=1}^{\infty}$ converges to U_0 if and only if*

$$\lim_{n\to\infty} u_n^{(\ell)} = u_0^{(\ell)}, \quad \ell = 1, 2, 3, \ldots, k, \tag{4.4}$$

holds.

Proof. Applying (2.66) the condition (4.2) is equivalent to the condition

$$\lim_{n\to\infty} \overbracket{d_{R^k}(\underbracket{U_n}, \underbracket{U_0})} = 0. \tag{4.5}$$

Applying Theorem 1.19 in [1] and using (1.3) we get that (4.5) is equivalent to

$$\lim_{n\to\infty} d_{R^k}(\underbracket{U_n}, \underbracket{U_0}) = 0. \tag{4.6}$$

Moreover, it is known that (4.6) is equivalent to

$$\lim_{n\to\infty} \underbracket{u_n^{(\ell)}} = \underbracket{u_0^{(\ell)}}, \quad \ell = 1, 2, 3, \ldots, k. \tag{4.7}$$

Using (1.2) and applying Theorem 1.19 in [1] again we obtain that (4.7) is equivalent to (4.4). ☐

Extending Definition 4.32 in [1] we give

Definition 4.8. Let F be a given function with $D_F \subset \overbracket{R^k}$ and let U_0 be a given element of $\overbracket{R^k}$. Let us assume that there exists a sequence $\{U_n\}_{n=1}^{\infty}$ such that $U_n \neq U_0$, $U_n \in D_F$ and $\lim_{n\to\infty} U_n = U_0$. If there exists an exploded real number v_0 such that for any $\{U_n\}_{n=1}^{\infty}$ mentioned above

$$\lim_{n\to\infty} F(U_n) = v_0 \tag{4.9}$$

holds, then we say that $\lim_{U\to U_0} F(U) = v_0$.

Theorem 4.10. *The function F has the limit v_0 at the point $U_0 \in \overbracket{R^k}$ if and only if*

$$\lim_{X\to\underbracket{U_0}} \operatorname{sub} F(X) = \underbracket{v_0}.$$

Proof. Denoting by $X_n = \underbracket{U_n}$ and $X_0 = \underbracket{U_0}$ we repeat that $\lim_{n\to\infty} U_n = U_0$ is equivalent to $\lim_{n\to\infty} X_n = X_0$. (See Definition 4.1 with (4.2), (4.5) and (4.6).) By Theorem 1.19 in [1] the condition (4.9) is equivalent to the condition

$$\lim_{n\to\infty} \underbracket{F(\overbracket{X_n})} = \underbracket{v_0}.$$

By (2.21) we have that $\operatorname{sub} F(x_n) = \underbracket{F(\overbracket{X_n})}$, so our proof is complete. $\square$

With respect to identities (2.66) and (1.3), Theorem 4.10 yields

Theorem 4.11. *The function F has the limit v_0 at the point $U_0 \in \overbracket{R^k}$ if and only if for any $\varepsilon(>0)$ there exists $\delta(>0)$ such that the conditions $U(\neq U_0) \in D_F$ and $\underset{R^k}{d\!\!-\!\!-}(U, U_0) < \delta$ imply $|F(U) \ominus v_0| < \varepsilon$.*

Definition 4.12. We say that F is continuous at the point U_0 if $U_0 \in D_F$ and
$$\lim_{U \to U_0} F(U) = F(U_0).$$

Example 4.13. The functions given by

(4.14)
$$f(u,v) = \frac{1}{2} \oplus \operatorname{spr} p_{\frac{1}{2}}(\operatorname{spr} p_2(1) \ominus \operatorname{spr} p_2(u \ominus \frac{1}{2}) \ominus \operatorname{spr} p_2(v \ominus \frac{1}{2}))$$

and

(4.15)
$$g(u,v) = \frac{1}{2} \ominus \operatorname{spr} p_{\frac{1}{2}}(\operatorname{spr} p_2(1) \ominus \operatorname{spr} p_2(u \ominus \frac{1}{2}) \ominus \operatorname{spr} p_2(v \ominus \frac{1}{2}))$$

describe the upper half and the lower half of super-sphere $S_{U_0}(1)$ with $U_0 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, respectively. Clearly,

$$D_f = D_g = \{(u,v) \in \overbracket{R^2} : (\underbracket{u} - \underbracket{(\tfrac{1}{2})})^2 + (\underbracket{v} - \underbracket{(\tfrac{1}{2})})^2 \leq (\underbracket{1})^2\}. \tag{4.16}$$

Moreover, the equations of box phenomena of half super-spheres are

$$w = \operatorname{area}\operatorname{th}(\operatorname{th}\frac{1}{2} + \sqrt{\operatorname{th}^2 1 - (\operatorname{th} u - \operatorname{th}\frac{1}{2})^2 - (\operatorname{th} v - \operatorname{th}\frac{1}{2})^2}) \tag{4.17}$$

and

$$w = \operatorname{area}\operatorname{th}(\operatorname{th}\frac{1}{2} - \sqrt{\operatorname{th}^2 1 - (\operatorname{th} u - \operatorname{th}\frac{1}{2})^2 - (\operatorname{th} v - \operatorname{th}\frac{1}{2})^2}), \tag{4.18}$$

respectively. (See Fig. 3.20.) The joint definition-domain is the window phenomenon of the super-disc D_f:

$$D_{f,\text{window}} = \{(u,v) \in R^2 : (\operatorname{th} u - \operatorname{th}\frac{1}{2})^2 + (\operatorname{th} v - \operatorname{th}\frac{1}{2})^2 \leq \operatorname{th}^2 1\}. \tag{4.19}$$

Especially interesting is the window phenomenon of the level-curve T given by $f(u,v) = \overbracket{1}$. The window phenomenon has the equation

$$(\operatorname{th} u - \operatorname{th}\frac{1}{2})^2 + (\operatorname{th} v - \operatorname{th}\frac{1}{2})^2 = \operatorname{th}^2 1 - (1 - \operatorname{th}\frac{1}{2})^2, \quad (u,v) \in R^2, \tag{4.20}$$

demonstrated by the following figure:

By (4.14), (2.21), (1.2), (1.5), (1.7), (1.3), (2.29) and (1.18) we obtain

$$\operatorname{sub} f(x,y) = \underbracket{(\tfrac{1}{2})} + \sqrt{(\underbracket{1})^2 - (x - \underbracket{(\tfrac{1}{2})})^2 - (y - \underbracket{(\tfrac{1}{2})})^2}, \quad (x,y) \in R^2 \tag{4.22}$$

which is a continuous function in its definition-domain given by the inequality
$$\left(x - \underbracket{(\tfrac{1}{2})}\right)^2 + \left(y - \underbracket{(\tfrac{1}{2})}\right)^2 = (\underbracket{1})^2.$$

(We can see that this definition-domain is $\underbracket{D_f}$ where D_f is given by (4.16) with respect to (2.23).) Applying (4.20) we have
$$\underbracket{T} = \{(x,y) \in R^2 : \left(x - \underbracket{(\tfrac{1}{2})}\right)^2 + \left(y - \underbracket{(\tfrac{1}{2})}\right)^2 = (\underbracket{1})^2 - \left(1 - \underbracket{(\tfrac{1}{2})}\right)^2.$$

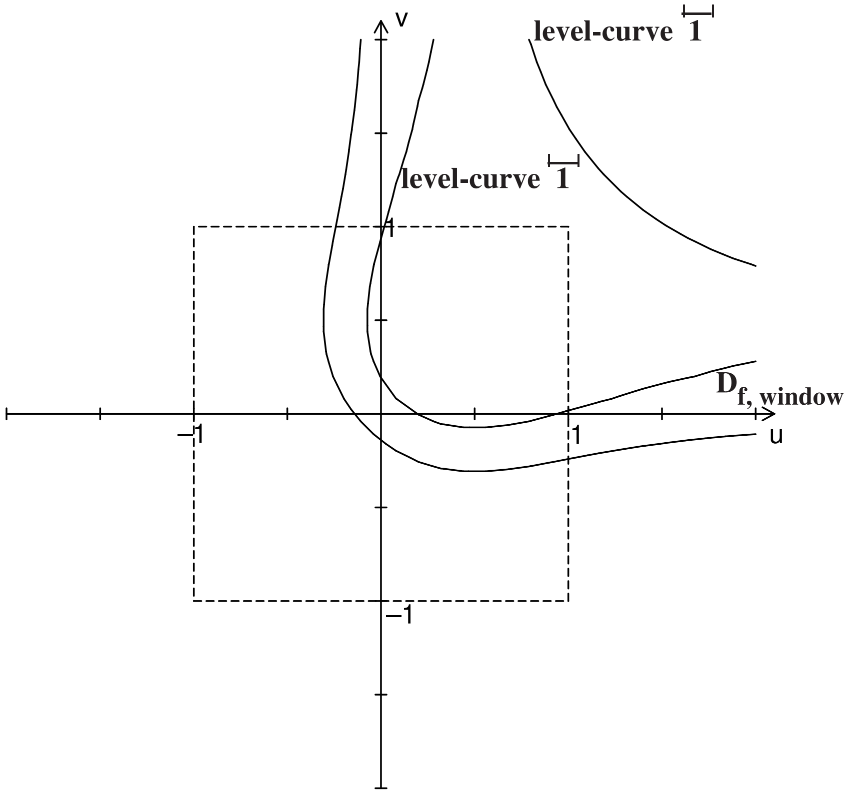


FIGURE 4.21.

Clearly, $\underset{\sqcup}{T} \subset \underset{\sqcup}{D_f}$ so sub f is continuous on $\underset{\sqcup}{T}$, too. Hence, Theorem 4.10 and Definition 4.12 give that f is continuous at any point $U = (u, v)$ of the level-curve T. Consequently, it is continuous at any point of the level-curve T_{window} demonstrated by Fig. 4.21. By (4.20) we can see that the point $U^* = (u^*, u^*)$ where

$$u^* = \operatorname{area\,th}\Big(\operatorname{th}\frac{1}{2} - \sqrt{\frac{\operatorname{th}^2 1 - (1 - \operatorname{th}\frac{1}{2})^2}{2}}\Big),$$

is an element of T_{window}, so

$$\lim_{U \to U^*} f(u, v) = \overset{\sqcup}{1}(= f(U^*)). \tag{4.23}$$

On the other hand, (4.17) shows

$$\lim_{\substack{U \to U^* \\ u, v < u^*}} f(u, v) = \infty. \tag{4.24}$$

We remark that (4.23) and (4.24) are not equivalent because if $u, v > u^*$ then $f(u, v) > \overset{\sqcup}{1}$. Another interesting problem is the behavior of the function g in the neighbourhood of the point $U^{**} = (\overset{\sqcup}{1}, \frac{1}{2})$. By (4.15), (1.18), (1.2), (1.7), (1.3), (2.29), (1.4) and (1.1) we obtain that $g(U^{**}) = \operatorname{area\,th}\big(\operatorname{th}\frac{1}{2} - \sqrt{\operatorname{th}^2 1 - (1 - \operatorname{th}\frac{1}{2})^2}\big)$. As g is continuous at the point U^{**}, we have that

$$\lim_{U \to U^{**}} g(u, v) = \operatorname{area\,th}\big(\operatorname{th}\frac{1}{2} - \sqrt{\operatorname{th}^2 1 - (1 - \operatorname{th}\frac{1}{2})^2}\big) \tag{4.25}$$

holds. On the other hand, (4.15) and (4.18) show that

$$\lim_{u\to\infty, v\to\frac{1}{2}} g(u,v) = \operatorname{area\,th}\Bigl(\operatorname{th}\frac{1}{2} - \sqrt{\operatorname{th}^2 1 - (1-\operatorname{th}\frac{1}{2})^2}\Bigr) \tag{4.26}$$

is also true. We remark that (4.25) and (4.26) are not equivalent because in the case of (4.26) the point $(u,v) \in R^2$, while in the case of (4.25) u may be greater than the positive discriminator. (Using (4.16) we can see that for $v = \frac{1}{2}$, the inequality $\overbracket{1} \le u < \overbracket{\underbracket{1} + \underbracket{(\tfrac{1}{2})}}$ is allowed.)

In general, we can say that if $U = (u_1, u_2, u_3, \dots, u_k) \in R^k$ and at least one of u_j $(j = 1,2,3,\dots,k)$ tends to ∞ (or $-\infty$) then

$$\lim_{u_1\to u_1^*, u_2\to u_2^*, \dots, a_k \to u_k^*} f(U) = v_0$$

where one ore more of $u_1^*, u_2^*, u_3^*, \dots, u_k^*, v_0$ may be ∞ or $-\infty$, is a restricted case of

$$\lim_{U\to \underbracket{U^*}} f(U) = v_0, \quad U^* = (u_1^*, u_2^*, u_3^*, \dots, u_k^*)$$

such that $U \in \overbracket{R^k}$ and if u_j^* is equal to ∞ (or $-\infty$) then we write $u_j^* = \overbracket{1}$ (or $u_j^* = \overbracket{-1}$) and $v_0 = \overbracket{1}$ (or $v_0 = \overbracket{-1}$), respectively.

5. On the geometry of space $\overset{\sqcup}{R}^3$.

The points of $\overbracket{R^3}$ were introduced under (2.1) while the super-lines and super-planes were defined in Definition 2.12. Considering the Euclidean geometry of space R^3 we can say that space $\overbracket{R^3}$ *has a super-Euclidean geometry* with the following (Hilbert-type) properties:

Property 5.1. *If U and V are distinct points of $\overbracket{R^3}$ then there exists a super-line L that contains both U and V.*

Property 5.2. *There is only one L such that $U \in L$ and $V \in L$.*

Property 5.3. *Any super-line has at least two points. There exists at least three points not all in one super-line.*

Property 5.4. *If U, V and W not are in the same super-line then there exists a super-plane S such that U, V and W are in S. Any super-plane has a point at least.*

Property 5.5. *If U, V and W are different non super-collinear points there is exactly one super-plane containing them.*

Property 5.6. *If two points lie in a super-line L and a super-plane S then every point of L lie in S.*

Property 5.7. *If two super-planes have a joint point then they have another joint point, too.*

Property 5.8. *There exist at least four points such that they are not on the same super-plane.*

We will say that the *point W is between the points U and V on super-line L if $\underbracket{W}$ is between $\underbracket{U}$ and $\underbracket{V}$ on line $\underbracket{L}$*. (See (2.1), (2.2), (2.7) and the first sentence of Definition 2.12.) The concept of "between" has the following properties:

Property 5.9. *If W is between U and V then U, V and W are three different points of a super-line and W is between V and U.*

Property 5.10. *For any arbitrary point U and V there exists at least one pont W lying on the super-line determined by U and V such that W is between U and V.*

Property 5.11. *For any three points of a super-line there is only one between the other two.*

Property 5.12 (Pasch-type property.)**.** *If U, V and W are not in the same super-line and L is a super-line of the super-plane determined by the points U, V and W such that L has not points U, V or W but is has a joint point with one of the super-segment UV of the super-line determined by U and V then L has a joint point with the super-segments UW or VW of the super-lines determined by U and W or V and W, respectively.* (The super-segment UV means the set of points which are between U and V on the super-line determined by U and V.)

We will say that *two sets of points are super-congruent if their compresseds are congruent.* (See (2.7)) Exploding a familiar convex angel $\sphericalangle XZY$ we have super-angle spr $\sphericalangle UWV$, where $U = \overbracket{X}$, $V = \overbracket{Y}$ and $W = \overbracket{Z}$. The point W is called the peak-point of super-angle. If the points X, Y and Z are not in the same super-line then super-angle is in the super-plane determined by U, V and W. The concept of "super-congruency" and super-angle have the following properties.

Property 5.13. *On a given super-half-line L there always exists at least one super-segment such that one of its end-points is the starting point of the super half-line L and this super-segment is super-congruent with an earlier given super-segment.*

Property 5.14. *If both super-segments p_1 and p_2 are super-congruent with the super-segment p_3 then p_1 and p_2 are super-congruent.*

Property 5.15. *If super-segment p_1 is super-congruent with super-segment q_1 and p_2 is super-congruent with q_2 then $p_1 \cup p_2$ is super-congruent with $q_1 \cup q_2$.*

Property 5.16. *On a given side of a super half-line there exists only one super-angle which is super-congruent with an earlier given super-angle. Each super-angle is super-congruent with itself.*

Property 5.17. *Let us consider two super-triangles. If two sides and the super-angles enclosed by these sides are super-congruent in the super-triangles mentioned above then they have another super-congruent super-angle.*

We say that the *super-lines L_1 and L_2 are super-parallel if their compresseds $\underbracket{L_1}$ and $\underbracket{L_2}$ are parallel.* Now we have

Property 5.18. *If super-line L_1 and point U are given such that U is off L_1 then there exists only one super-line L_2 through U that is super-parallel to L_1.*

Finally, we mention two properties for continuity.

Property 5.19 (Archimedes-type property.)**.** *If point U_1 is between points U and V on a super-line then there are points $U_2, U_3, \dots, U_n$ such that super-segments U_{j-1}, U_i, $(i = 2, 3, \dots, n)$ are super-congruent with he super-segment UU_1 and V is between points U and U_n.*

Property 5.20 (Cantor-type property.)**.** *If $\{U_nV_n\}_{n=1}^{\infty}$ is a sequence of super-segments lying on a super-line L such that for any $n = 1, 2, 3, \dots$ $U_{n+1}V_{n+1} \subset U_nV_n$ then there exists at least one point W of L such that W belongs to each U_nV_n.*

In the following we construct an extra-model for the familiar points of R^3.

Definition 5.21. A point $P \in \overbracket{R^3}$ is called an extra-point if $P \in R^3$. (Extra points are the visible points of $\overbracket{R^3}$.)

Definition 5.22. For any super-line L belonging to $\overbracket{R^3}$ the curve L_{box} is called an extra-line if $L_{\text{box}} \neq \emptyset$.

Definition 5.23. For any super-plane S belonging to $\overbracket{R^3}$, the surface S_{box} is called an extra-plane if $S_{\text{box}} \neq \emptyset$. Considering that the compressed of box-phenomenon of a super-plane may be an open triangle, quadrangle, pentagon or hexagon we speak *triangular, quadrangular, pentagonal and hexagonal extra-planes*, respectively.

Remark 5.24. Clearly,

- P is extra-point if and only if $\underbracket{P} \in \underbracket{R^3}$.
- L_{box} is an extra-line if and only if $\underbracket{L_{\text{box}}} \cap \underbracket{R^3} \neq \emptyset$.
- S_{box} is an extra-plane if and only if $\underbracket{S_{\text{box}}} \cap \underbracket{R^3} \neq \emptyset$.

For the sake of simplicity we do not introduce new symbols for the extra-lines and extra-planes instead of L_{box} and S_{box}, respectively. On the other hand, $(\underbracket{R^3})^0$ denotes the set of limit-points of the open cube $\underbracket{R^3}$.

Definition 5.25. We say that extra-lines $L^{(1)}_{\text{box}}$ and $L^{(2)}_{\text{box}}$ are extra-parallel if

$$\underbracket{L^{(1)}} \cap \underbracket{L^{(2)}} \in (\underbracket{R^3})^0$$

If the super-lines $L^{(1)}$ and $L^{(2)}$ are not identical, then by Property 5.2 they may have at most one joint point. So, if the extra-lines $L^{(1)}_{\text{box}}$ and $L^{(2)}_{\text{box}}$ - are extra-parallel then extra-parallelness point $P(L^{(1)}_{\text{box}}, L^{(2)}_{\text{box}}) = L^{(1)} \cap L^{(2)}$ is unambiguously determined. By this reason we say that *extra-lines $L^{(1)}_{\text{box}}$ and $L^{(2)}_{\text{box}}$ are extra-parallel with respect to the extra-parallelness point $P(L^{(1)}_{\text{box}}, L^{(2)}_{\text{box}})$*. It is important that the extra-parallelness point is not an extra-point.

The extra-parallelness of extra-lines, lying on the quadrangular extra plane $w = 0$ is demonstrated by the following figure:

Fig. 5.26. shows that $L^{(1)}_{\text{box}}$ and $L^{(2)}_{\text{box}}$ where $L^{(1)}$ and $L^{(2)}$ have the equations

$$v \ominus (\overbracket{3} \odot u) \oplus \overbracket{2.8} = 0$$

and

$$(\overbracket{2} \odot u) \oplus (\overbracket{3} \odot v) \ominus \overbracket{2.6} = 0,$$

respectively are extra-parallel with $P(L^{(1)}_{\text{box}}, L^{(2)}_{\text{box}}) = (\overbracket{1}, \overbracket{0.2}, 0)$, but super-lines $L^{(1)}$ and $L^{(2)}$ are not super-parallel. Moreover, considering the extra-point $P_0 = (\overbracket{0.4}, \overbracket{0.6}, 0)$ of the extra-line $L^{(2)}_{\text{box}}$ and the extra-line $L^{(3)}_{\text{box}}$, where the equation of $L^{(3)}$ is

$$(\overbracket{8} \odot u) \oplus v \ominus \overbracket{3.8} = 0$$

we have that $P_0 \in L^{(3)}_{\text{box}}$ as well as $L^{(1)}_{\text{box}}$ and $L^{(3)}_{\text{box}}$ are extra-parallel with

$$Q(L^{(1)}_{\text{box}}, L^{(2)}_{\text{box}}) = (\overbracket{0.6}, \overbracket{(-1)}, 0).$$

Definition 5.27. The extra-line L_{box} and extra-plane S_{box} are extra-parallel if

$$\underbracket{L} \cap \underbracket{S} \in (\underbracket{R^3})^0$$

and the point $P(L_{\text{box}}, S_{\text{box}}) = L \cap S$ is called extra-parallelness point of L_{box} and S_{box}.

Definition 5.28. The extra-planes $S^{(1)}_{\text{box}}$ and $S^{(2)}_{\text{box}}$ are called extra-parallel with respect to a super line $S^{(1)} \cap S^{(2)}$ if

$$\underbracket{S^{(1)}} \cap (\underbracket{R^3})^0 = \underbracket{S^{(2)}} \cap (\underbracket{R^3})^0 \neq \emptyset.$$

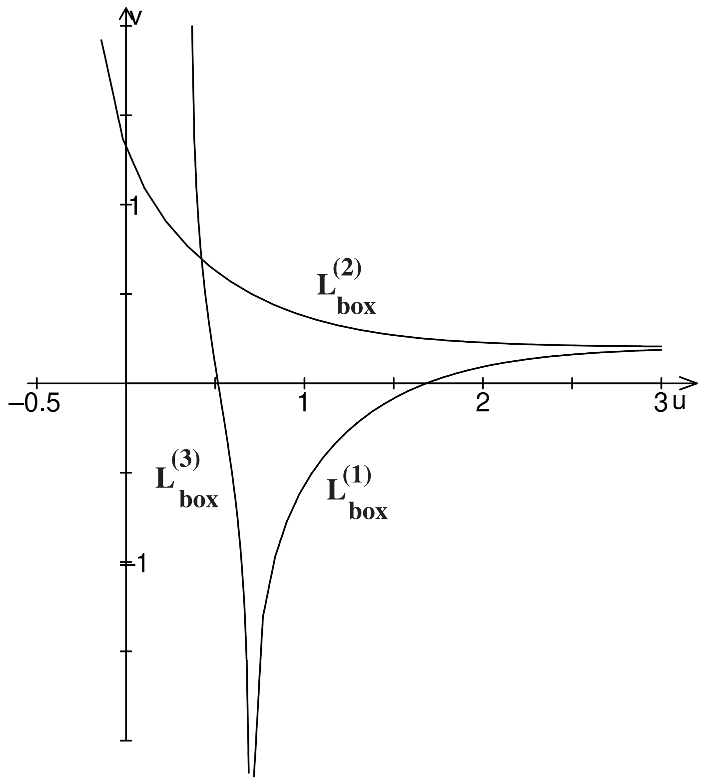


FIGURE 5.26.

Returning back to Fig. 3.14 we can check that considering the extra-point $Q = (\overbracket{0.5}, \overbracket{0.5}, 0)$ and the hexagonal extra-plane S_{box} having the equation $z = \operatorname{area\,th}(\operatorname{th} x + \operatorname{th} y)$, there are six extra-planes coinciding with Q such that they are extra-parallel with S_{box}.

Finally, we raise the problem: What kind of properties does the geometry for R^3 with extra-points, extra-lines and extra-planes have?

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