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ON THE COMPRESSED DESCARTES-PLANE AND ITS APPLICATIONS

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Dedicated to Professor Á. Varecza on his 60th birthday

Abstract. The open interval $\underbracket{R} = (-1, 1)$ with the sub-addition $\oplus$ and sub-multiplication $\odot$ is considered as a compressed model of the set of real numbers R. The paper contains the discussion of sub-linear function $y = (m \otimes x) \oplus b$ and shows its graph in the compressed Descartes-plane $\underbracket{R}^2 = \{(x, y) \in R^2 : -1 < x < 1 \text{ and } -1 < y < 1\}$.

Introduction

The set of compressed real numbers $\underbracket{R}$ can be introduced such that the set of real numbers is isomorphic with it. The isomorphism is given by the compression function defined by the equation

$$x = \operatorname{th} u, \quad u \in R. \tag{0.1}$$

Moreover, the operations working on $\underbracket{R}$ are: the sub-addition

$$x \oplus y = \frac{x + y}{1 + x \cdot y}, \quad x, y \in \underbracket{R} \tag{0.2}$$

and the sub-multiplication

$$x \odot y = \operatorname{th}((\operatorname{area}\operatorname{th} x)(\operatorname{area}\operatorname{th} y)), \quad x, y \in \underbracket{R}. \tag{0.3}$$

Of course, we have the identities

$$\underbracket{u + v} = \underbracket{u} \oplus \underbracket{v}, \quad u, v \in R \tag{0.4}$$

and

$$\underbracket{u \cdot v} = \underbracket{u} \odot \underbracket{v}, \quad u, v \in R. \tag{0.5}$$

The number

$$\underbracket{u} = \operatorname{th} u, \quad u \in R$$

is called the compressed of u. On the other hand, the number

$$\overbracket{x} = \operatorname{area}\operatorname{th} x, \quad x \in \underbracket{R}$$

is called the exploded of x.

The inverse of sub-addition is sub-subtraction

$$x \ominus y = \frac{x - y}{1 - x \cdot y}, \quad x, y \in \underbracket{R} \tag{0.6}$$

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and the inverse of sub-multiplication is sub-division

$$x \odot y = \underbracket{\left(\frac{\overbracket{x}}{\overbracket{y}}\right)}, \quad x, y \in \underbracket{R}, \quad y \neq 0. \tag{0.7}$$

The arrangement on R gives the arrangement on $\underbracket{R}$ too. Namely, for any pair $x, y \in \underbracket{R}$, $x < y$ if and only if $\overbracket{x} < \overbracket{y}$.

If a real function f is given by the equation

$$v = f(u), \quad u, v \in R \tag{0.8}$$

then we define

$$\operatorname{sub} f(x) = \underbracket{f(\overbracket{x})}, \quad x \in \underbracket{R} \tag{0.9}$$

and $\operatorname{sub} f$ is called the sub-function of the function f. Of course, for a compressed real number x the function $\operatorname{sub} f$ is determined if and only if $\overbracket{x}$ belongs to the definition-domain of function f. Now we have the following obvious fact as a

Lemma 0.10. *If function f is monotonic on the interval $(\alpha, \beta) \subset R$, where $\alpha = -\infty$ and $\beta = \infty$ are allowed then the sub-function* $\operatorname{sub} f$ *is monotonic in the same sense on the interval $(\underbracket{\alpha}, \underbracket{\beta})$.*

By Lemma 0.10. we obtain the next corollaries.

Corollary 0.11. *For any $x, y, z \in \underbracket{R}$, if $x < y$ then*

$$x \oplus z < y \oplus z$$

holds.

Corollary 0.12. *For any $x, y, z \in \underbracket{R}$, if $x < y$ and $z > 0$ then*

$$x \odot z < y \odot z$$

holds.

The compressed Descartes-plane is

$$\underbracket{R}^2 = \{(x, y) : x, y \in \underbracket{R}\}$$

that is, the set of arranged pairs of compressed real numbers.

1. The sub-linear function

Considering the linear function, given by the equation

$$v = \mu u + \beta, \quad u, v \in R \tag{1.1}$$

where μ and β are fixed real numbers, we have that its sub-function

$$y = (m \odot x) \oplus b, \quad x = \underbracket{u}, y = \underbracket{v} \tag{1.2}$$

with

$$m = \underbracket{\mu} \tag{1.3}$$

and

$$b = \underbracket{\beta} \tag{1.4}$$

because by (1.1), (0.9), (0.1), (0.4), (0.5), (1.4) and (1.3)

$$\underbracket{\mu \overbracket{x} + \beta} = \underbracket{\mu \overbracket{x}} \oplus \underbracket{\beta} = (\underbracket{\mu} \odot \underbracket{(\overbracket{x})}) \oplus b = (m \odot x) \oplus b$$

is obtained. So, the sub-function defined by the equation (1.2) is called sub-linear function.

Introducing the compression-transformation

$$(u, v) \rightarrowtail (\underbracket{u}, \underbracket{v}), \quad (u, v) \in R^2$$

and the explosion-transformation

$$(x,y) \rightarrowtail (\overset{\sqcup}{x}, \overset{\sqcup}{y}), \quad (x,y) \in \underset{\sqcup}{R}^2$$

we say that the sub-linear function is the compressed of the linear function. Conversely, the linear function is the exploded of the sub-linear function.

Our aim is to discuss the graph of sub-linear function under (1.2). First of all we remark, that the *sub-linear function is defined on the whole* $\underset{\sqcup}{R}$. Moreover, Lemma 0.10 says that *it is monotonic increasing if* $m > 0$ *and decreasing if* $m < 0$. *If* $m = 0$ *then the sub-linear function is constant.*

Using (0.2), (0.3) and (1.3) we have that the equation (1.2) is equivalent with the equation

$$y(x) = \frac{b + \operatorname{th}(\mu \operatorname{area\,th} x)}{1 + b \operatorname{th}(\mu \operatorname{area\,th} x)}, \quad -1 < x < 1. \tag{1.5}$$

Hence,

$$\lim_{\substack{x \to 1 \\ x < 1}} ((m \odot x) \oplus b) = \begin{cases} 1 & \text{if} \quad \mu > 0 \\ -1 & \text{if} \quad \mu < 1 \end{cases}, \quad (\text{see}(1.3)) \tag{1.6}$$

and

$$\lim_{\substack{x \to -1 \\ x > -1}} ((m \odot x) \oplus b) = \begin{cases} -1 & \text{if} \quad \mu > 0 \\ 1 & \text{if} \quad \mu < 1 \end{cases}, \quad (\text{see}(1.3)) \tag{1.7}$$

hold. By the equation

$$y'(x) = \frac{\mu(1 - b^2)}{(1 - x^2)(\operatorname{ch}(\mu \operatorname{area\,th} x) + b \operatorname{sh}(\mu \operatorname{area\,th} x))^2}, \quad -1 < x < 1, \tag{1.8}$$

we can compute that

$$\lim_{\substack{x \to 1 \\ x < 1}} ((m \odot x) \oplus b)' = \begin{cases} 0 & \text{if} \quad \mu > 1, \\ \infty & \text{if} \quad 0 < \mu < 1 \\ -\infty & if \quad -1 < \mu < 0, \\ 0 & \text{if} \quad \mu < -1 \end{cases} \quad (\text{see}(1.3)) \tag{1.9}$$

and

$$\lim_{\substack{x \to -1 \\ x > -1}} ((m \odot x) \oplus b)' = \begin{cases} 0 & \text{if} \quad \mu > 1, \\ \infty & \text{if} \quad 0 < \mu < 1 \\ -\infty & \text{if} \quad -1 < \mu < 0, \\ 0 & \text{if} \quad \mu < -1 \end{cases}, \quad (\text{see}(1.3)) \tag{1.10}$$

Now we will consider the special case $b = 0$, that is

$$y = m \odot x, \quad x \in \underset{\sqcup}{R}. \tag{1.11}$$

This sub-linear function is the compressed of linear function

$$v = \mu \cdot u, \quad u \in R. \tag{1.12}$$

On the other hand (1.8) shows, that

$$(m \odot x)'\Big|_{x=0} = \mu.$$

This means that the tangent of the curve (1.11) at the point $O = (0,0)$ is the line with the equation

$$y = \mu x. \tag{1.13}$$

So, the exploded of (1.11) is the tangent of (1.11), too.

The cases $m = \underset{\sqcup}{1}$, $m = 0$ and $m = -\underset{\sqcup}{1}$ are trivial, because by (0.3) we have

$$m \odot x = \begin{cases} x & \text{if} \quad m = \underset{\sqcup}{1}, \quad (\mu = 1) \\ 0 & \text{if} \quad m = 0, \quad (\mu = 0) \\ -x & if \quad m = -\underset{\sqcup}{1}, \ (\mu = -1) \end{cases}, \quad \text{(see (1.3))}.$$

For the other cases we have

Theorem 1.14. *If $m \in \underset{\sqcup}{R}$ such that $m \neq \underset{\sqcup}{1}, 0, -\underset{\sqcup}{1}$ then the curve of sub-linear function defined by the equation (1.11) has a unique inflexion-point at $x = 0$.*

Proof. Considering (1.5) and (1.8) in the special case $b = 0$, we have

$$y(x) = \operatorname{th}(\mu \operatorname{area th} x), \quad \mu \in R, \quad \mu \neq 1, 0, -0 \tag{1.15}$$

and

$$y'(x) = \frac{\mu}{(1 - x^2)\operatorname{ch}^2(\mu(\operatorname{area th} x))}, \quad -1 < x < 1, \tag{1.16}$$

respectively. By (1.16) we get

$$y''(x) = -\frac{\mu((1 - x^2)\operatorname{ch}^2(\mu(\operatorname{area th} x)))'}{((1 - x^2)\operatorname{ch}^2(\mu(\operatorname{area th} x)))^2}, \quad \mu \neq 1, 0, -0. \tag{1.17}$$

A usual computation gives that $y''(x) = 0$ if and only if

$$\operatorname{th}(\mu \operatorname{area th} x) = \frac{x}{\mu}, \quad \mu \neq 1, 0, -1. \tag{1.18}$$

Clearly, if $x = 0$, then (1.18) is fulfilled. We will show that this is the unique case. Considering that the functions standing on both sides of the equation (1.18) are odd we can assume that

$$0 < x < 1. \tag{1.19}$$

Now, let us consider the following four cases:

(i) Case $\mu > 1$. By (1.19) we have

$$\frac{x}{\mu} < x = \operatorname{th}(\operatorname{area th} x) < \operatorname{th}(\mu(\operatorname{area th} x).$$

(ii) Case $0 < \mu < 1$. By (1.19) we have

$$\frac{x}{\mu} > x = \operatorname{th}(\operatorname{area th} x) > \operatorname{th}(\mu(\operatorname{area th} x)).$$

(iii) Case $-1 < \mu < 0$. By (1.19) we have

$$\frac{x}{\mu} < -x = \operatorname{th}(-\operatorname{area th} x) < \operatorname{th}(\mu \operatorname{area th} x).$$

(iv) Case $\mu < -1$. By (1.19) we have

$$\frac{x}{\mu} > -x = \operatorname{th}(-\operatorname{area th} x) > \operatorname{th}(\mu \operatorname{area th} x.$$

Our proof is complete. □

Collecting the results mentioned above we have Figure 1.20.

After Theorem 1.14, the graph of the special sub-linear function (1.11) is already known. (See Fig. 1.20.) The hyperbola

$$y = \frac{x - b\mu}{\mu - bx}, \quad \mu \in R, \ b \in \underset{\sqcup}{R}, \ \mu, b \neq 0 \tag{1.21}$$

has the definition-domain

$$R \backslash \{\frac{\mu}{b}\}. \tag{1.22}$$

We need

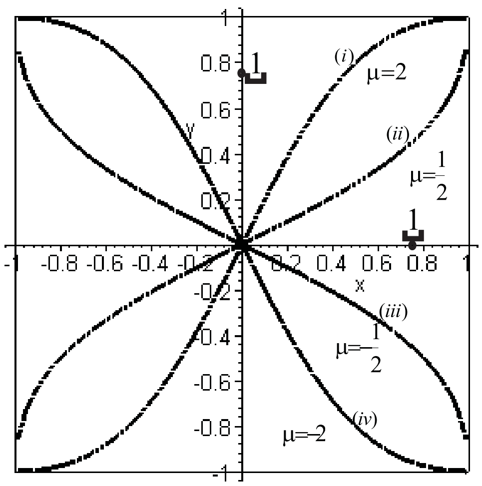

FIG. 1.20.

Lemma 1.23. *If $\mu \in R$ such that*

$$\mu > 1 \tag{1.24}$$

and $b \in \underline{R}$ such that

$$b > 0 \tag{1.25}$$

then the graphs of the sub-linear function defined under (1.11) and the hyperbola (1.21) have a unique common point.

Proof. Considering (1.6) and (1.7) we have that the y-domain of the sub-linear function defined under (1.11) is the interval (-1, 1). On the other hand, in the case of hyperbola (1.21), by (1.24) and (1.25)

$$y(-1) = \frac{-1 - b\mu}{\mu + b} > -1 \quad \text{and} \quad y(1) = \frac{1 - b\mu}{\mu - b} < 1.$$

Moreover, (1.21), (1.22), (1.24) and (1.25) show that $\underline{R}$ is a subset of the definition domain of the hyperbola which is strictly monotonic in it. So, the graphs intersect each other by the Bolzano–Darboux property. Our task is to sow that their intersection has only one point in $\underline{R}^2$. Clearly, the abscissas of common points belong to $\underline{R}$.

By (1.24) we have that for any $x \in \underline{R}$ the number $\frac{x}{\mu} \in \underline{R}$, too. So, by (0.6) we can write for any $x \in \underline{R}$, that

$$\frac{x - b\mu}{\mu - bx} = \frac{\frac{x}{\mu} - b}{1 - b\frac{x}{\mu}} = \frac{x}{\mu} \ominus b, \quad (x \in \underline{R}).$$

Hence, the hyperbola graph above the interval (-1, 1) has the equation

$$y = \frac{x}{\mu} \ominus b, \quad x \in \underline{R}$$

Casting a glance at (1.11) for the abscissas of the common points we have the equation:

$$\frac{x}{\mu} \ominus b = m \odot x. \tag{1.26}$$

Hence,

$$\frac{x}{\mu} = (m \odot x) \oplus b. \tag{1.27}$$

Exploding both sides of equation (1.27), by (0.4), (0.5), (1.3) and (1.4) we have

$$\operatorname{area\,th} \frac{x}{\mu} = \mu \cdot \operatorname{area\,th} + \beta. \tag{1.28}$$

It is sufficient to show, that the equation (1.28) which is equivalent to (1.27) and (1.26) has only one solution. To prove this fact we consider the function

$$d(x) = (\mu \operatorname{area\,th} x + \beta) - \operatorname{area\,th} \frac{x}{\mu}, \quad -1 < x < 1. \tag{1.29}$$

Hence,

$$d'(x) = \mu\Big(\frac{1}{1-x^2} - \frac{1}{\mu^2 - x^2}\Big), \quad -1 < x < 1.$$

By (1.24)we have that for any $x \in (-1,1), d'(x) > 0$. So, the function (1.29) is strictly increasing on the interval (-1, 1). This means that the equation

$$(\mu \operatorname{area\,th} x + \beta) - \operatorname{area\,th} \frac{x}{\mu} = 0$$

together with the equation (1.28) has only one solution. □

By Lemma 1.23. we have

Definition 1.30. Let $\mu \in R$, $b \in \underline{R}$, $\mu \neq 1, 0, -1$; $b \neq 0$. The solution of equation

$$\frac{x - \mu b}{\mu - bx} = m \odot x, \quad (m = \underline{\mu})$$

is denoted by $x_{m,b}$ and called inflexion-abscissa.

Returning to the general sub-linear function (1.2) the cases $m = 0$ and $b \neq 0$ are trivial because in these cases the sub-linear functions are constant:

$$y = b, \quad (b \in \underline{R}).$$

Now we are turning to the interesting special cases when

$$|m| = \underline{1} \quad \text{and} \quad b \neq 0$$

are fulfilled.

In the case

$$m = \underline{1} \quad \text{and} \quad 0 < b < 1 \tag{1.31}$$

the equation (1.5) gives the special case

$$y = \frac{b+x}{1+bx}, \quad -1 < x < 1.$$

It is a piece of the hyperbola which is symmetrical for the line $y = -x$ and has the assimptots $x = -\frac{1}{b}(< -1)$ and $y = \frac{1}{b}(> 1)$. In this case

$$\lim_{\substack{x\to 1\\ x<1}} y'(x) = \frac{1-b}{1+b} \quad \text{and} \quad \lim_{\substack{x\to -1\\ x>-1}} y'(x) = \frac{1+b}{1-b}.$$

In the case

$$m = -\underline{1} \quad \text{and} \quad 0 < b < 1 \tag{1.32}$$

the equation (1.5) gives the special case

$$y = \frac{b-x}{1-bx}, \quad -1 < x < 1.$$

It is a piece of the hyperbola which is symmetrical for the line $y = x$ and has the assimptots $x = \frac{1}{b}(> 1)$ and $y = \frac{1}{b}(> 1)$. In this case

$$\lim_{\substack{x\to 1\\ x<1}} y'(x) = \frac{b+1}{b-1} \quad \text{and} \quad \lim_{\substack{x\to -1\\ x>-1}} y'(x) = \frac{b-1}{b+1}.$$

In the case

$$m = \underline{1} \quad \text{and} \quad -1 < b < 0 \tag{1.33}$$

the equation $y = x \oplus b$ has the inverse $y = x \oplus |b|$ which is the case (1.31) with $0 < |b| < 1$.

In the case

$$m = -\underline{1} \quad \text{and} \quad -1 < b < 0 \tag{1.34}$$

the equation (1.2) has the form $y = b \ominus x = -(x \oplus |b|)$. Considering the graph of case (1.31) with $0 < |b| < 1$ and reflecting it for the "x" axis we have the graph of this case.

Collecting the cases (1.31), (1.32), (1.33) and (1.34) with $|b| = \frac{1}{2}$ we have the next figure:

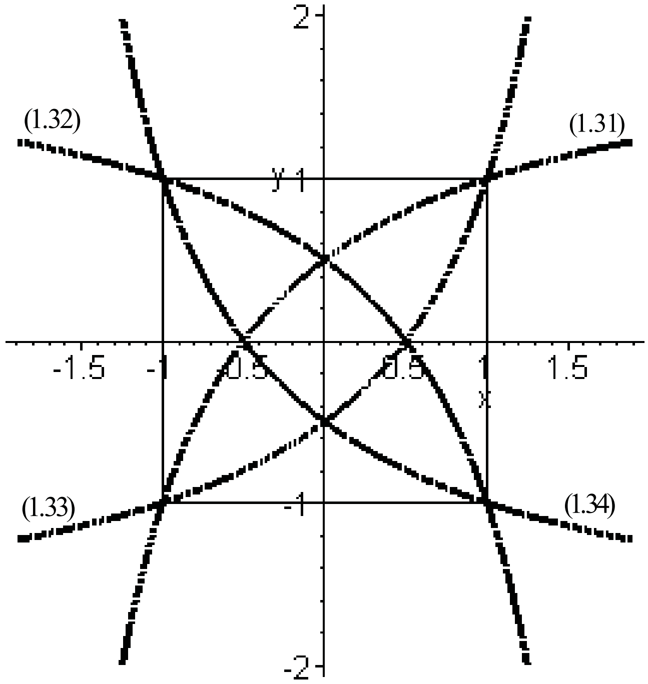

FIG. 1.35.

In the last part of the discussion we consider the cases $m, b \in \underline{R}$ such that $|m| \neq \underline{1}$, and $m, b \neq 0$. We need

Lemma 1.36. *Let $m, b \in \underline{R}$. If the point (x_0, y_0) lies on the graph of the curve having the equation (1.2) then:*

a) point $(-x_0, -y_0)$ lies on the graph of the curve having the equation

$$y = (m \odot x) \oplus (-b), \tag{1.37}$$

b) point $(-x_0, y_0)$ lies on the graph of the curve having the equation

$$y = ((-m) \odot x) \oplus b, \tag{1.38}$$

c) point (y_0, x_0) lies on the graph of the curve having the equation

$$y = (x \bigodot m) \oplus ((-b) \bigodot m), \tag{1.39}$$

where $m \neq 0$ is assumed.

Proof. Case a)

Using (1.2), by (0.2) and (0.3) we obtain

$$\begin{aligned} y_0 &= (m \odot x_0) \oplus b = (m \odot (-(-x_0))) \oplus b = \\ &= (m \odot (\underbracket{-1} \odot (-x_0))) \oplus (\underbracket{-1} \odot (-b)) = \underbracket{-1} \odot ((m \odot (-x_0)) \oplus (-b)) = \\ &= -((m \odot (-x)) + (-b)). \end{aligned}$$

So, $(-x_0, -y_0)$ satisfies (1.37)

Case b)

Using (1.2), by (0.2) and (0.3), we obtain

$$\begin{aligned} y_0 &= (m \odot x_0) \oplus b = (m \odot (\underbracket{-1} \odot (-x_0)) \oplus b = \\ &= ((m \odot \underbracket{-1}) \odot (-x_0)) \oplus b = ((-m) \odot (-x_0)) \oplus b. \end{aligned}$$

So, $(-x_0, y_0)$ satisfies (1.38).

Case c)

Starting from (1.2), by (0.6), (0.2), (0.3) and (0.7), we can write

$$\begin{aligned} y_0 &= (m \odot x_0) \oplus b \\ y_0 \ominus b &= m \odot x_0 \\ y_0 \oplus (-b) &= x_0 \odot m \\ (y \oplus (-b)) \bigodot m &= x_0 \\ (y_0 \bigodot m) \oplus ((-b) \bigodot m) &= x_0, \end{aligned}$$

that is (y_0, x_0) satisfies (1.39). □

Finally, we have

Theorem 1.40. *If $m, b \in \underbracket{R}$ such that $m \neq \underbracket{1}, 0, \underbracket{-1}$ and $b \neq 0$ then the curve of the sub-linear function defined by the equation (1.2) has the unique inflexion-point*

$$\left(x_{m,b}, \frac{x_{m,b}}{\mu}\right), \quad (\mu = \overbracket{m}) \tag{1.41}$$

where $x_{m,b}$ is the inflexion-abscissa.

Proof. We have already seen that the equation (1.2) is equivalent to the equation (1.5) which has the derivative (1.8). Hence,

$$y''(x) = \mu(1 - b^2)\Big(\frac{1}{(1 - x^2)(\operatorname{ch}(\mu \operatorname{area\,th} x) + b \operatorname{sh}(\mu \operatorname{area\,th} x))^2}\Big)', \quad -1 < x < 1.$$

We have to solve the equation

$$y''(x) = 0, \quad -1 < x < 1 \tag{1.42}$$

which is equivalent to the equation

$$((1 - x^2)(\operatorname{ch}(\mu \operatorname{area\,th} x) + b \operatorname{sh}(\mu \operatorname{area\,th} x))^2)' = 0, \quad -1 < x < 1.$$

Hence, a usual computation gives that (1.42) is equivalent to the equation

$$(x - b\mu) \operatorname{ch}(\mu \operatorname{area\,th} x) = (\mu - bx) \operatorname{sh}(\mu \operatorname{area\,th} x), \quad -1 < x < 1. \tag{1.43}$$

Now we distinguish the following eight cases

(i)/1: $\mu > 1$ and $b > 0$,
(i)/2: $\mu > 1$ and $b < 0$,
(ii)/1: $0 < \mu < 1$ and $b > 0$,
(ii)/2: $0 < \mu < 1$ and $b < 0$,
(iii)/1: $-1 < \mu < 0$ and $b > 0$,
(iii)/2: $-1 < \mu < 0$ and $b < 0$,
(iv)/1: $\mu < -1$ and $b > 0$,
(iv)/2: $\mu < -1$ and $b < 0$.

Let is begin with the case (i)/1. Writing (1.43) in the form

$$\frac{x - b\mu}{\mu - bx} = m \odot x, \quad \text{(see (1.3) and (0.3))}$$

by Lemma 1.23 and Definition 1.30 we have that the unique solution of (1.42) is the inflection-abscissa $x_{m,b}$. Writing the left-hand side in the form $\frac{x}{\mu} \ominus b$ the equalities (1.26) and (1.27) show that the point (1.41) lies on the curve of the sub-linear function (1.2). Moreover, (1.7) and (1.10) show that the curve starts from the point (-1, -1) in a convex way. So, we have that it is convex the interval $(-1, x_{m,b})$ and concave in the interval $(x_{m,b}, 1)$. Thus our statement is proved for the case (i)/1.

In the case (i)/2 we apply the case a) of Lemma 1.36 which says that the graph is a reflection with respect to the point $O = (0, 0)$ of the graph given in the case (i)/1.

In the case (ii)/2 we apply the case c) of Lemma 1.36 which says that the graph is a reflection with respect to line $y = x$ of a graph belonging to the case (i)/2.

In the case (ii)/1 we apply the case a) of Lemma 1.36 which says that the graph is a reflection with respect to the point $O = (0, 0)$ of the graph given in the case (ii)/2.

In the case (iv)/1 we apply the case b) of Lemma 1.36 which says that the graph is a reflection with respect to the "y" axis of the graph given in the case (i)/1.

In the case (iv)/2 we apply the case a) of Lemma 1.36 which says that the graph is a reflection with respect to the point $O = (0, 0)$ of the graph given in the case (iv)/1.

In the case (iii)/1 we apply the case a) of Lemma 1.36 which says that the graph is a reflection with respect to the point $O = (0, 0)$ of the graph given in the case (ii)/1.

In the case (iii)/2 we apply the case a) of Lemma 1.36 which says that the graph is a reflection with respect to the point $O = (0, 0)$ of the graph given in the case (iii)/1.

The proof of Theorem 1.40 is complete. □

Finally, using Theorem 1.40 by (1.6), (1.7), (1.9), (1.10). Fig. 1.20, (1.21), Definition (1.30) and (1.41) we give the graph of the sub-linear function $y = (\underline{2} \odot x) \oplus \frac{1}{2}$:

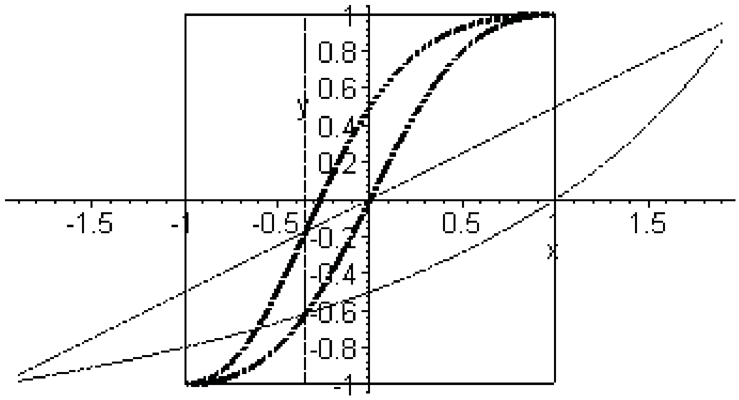

Fig. 1.41.

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