ARCHIVUM MATHEMATICUM (BRNO)
Tomus 60 (2024), 135–144

COMPLETE SOLUTIONS OF A LEBESGUE-RAMANUJAN-NAGELL TYPE EQUATION

Priyanka Baruah, Anup Das, and Azizul Hoque

Abstract. We consider the Lebesgue-Ramanujan-Nagell type equation $x^2 + 5^a 13^b 17^c = 2^m y^n$, where $a, b, c, m \geq 0, n \geq 3$ and $x, y \geq 1$ are unknown integers with $\gcd(x, y) = 1$. We determine all integer solutions to the above equation. The proof depends on the classical results of Bilu, Hanrot and Voutier on primitive divisors in Lehmer sequences, and finding all S-integral points on a class of elliptic curves.

1. Introduction

The Lebesgue-Ramanujan-Nagell type equation

$$x^2 + D^m = \lambda y^n, \quad \lambda = 1, 2, 4\,, \tag{1.1}$$

in integer unknowns x, y, $m \geq 1$ and $n \geq 3$, has a long and distinguished history. The first result concerning the solutions of (1.1) was due to Lebesgue [21], who proved that (1.1) has no solutions when $\lambda = D = 1$ and $y > 1$. Later, many authors become interested in this equation and thus there are good amount of research concerning the solutions of (1.1). We direct the readers to the beautiful survey [20] for further information. Several authors studied some generalizations of (1.1) in [5, 9, 10, 13, 16, 19].

Recently, many authors become interested to find the integer solutions of the Lebesgue-Ramanujan-Nagell type equation

$$x^2 + p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} = y^n,\ x, y \geq 1,\ \gcd(x, y) = 1,\ a_1, a_2, \dots, a_k \geq 0,\ n \geq 3\,,$$

where $p_1, p_2, \dots, p_k$ are distinct primes with $k \geq 2$. There are many results concerning the integer solutions of this equation, but we refer to the very recent papers [2, 4, 6, 11, 12, 15, 17, 27]. From existing results, it is quite natural to consider Diophantine equations similar to the above one, where the right side is a product of an unknown integer with unknown exponent and a known prime with unknown exponent.

2020 *Mathematics Subject Classification*: primary 11D61; secondary 11D41, 11Y50.

Key words and phrases: Diophantine equation, Lehmer sequence, elliptic curve, quartic curve, S-integral points.

Received December 27, 2023, revised February 2024. Editor C. Greither.

DOI: 10.5817/AM2024-3-135

Here, we consider the Diophantine equation

$$(1.2)\quad x^2+5^a13^b17^c=2^my^n,\ x\geq 1,\ y>1,\ \gcd(x,y)=1,\ a,b,c,m\geq 0,\ n\geq 3\,,$$

and we find all its the integer solutions (x,y,a,b,c,m,n). It is noted that by reading (1.2) modulo 4, we see that $x^2+1\equiv 0 \pmod 4$ which does not satisfy for any integer x. Therefore (1.2) has no solution when $m\geq 2$, and thus we consider (1.2) for $m=0,1$. More precisely, we prove the following:

Theorem 1.1. *If $n\neq 3,4,6,12$, then (1.2) has no integer solutions. In case of $n=3,4,6,12$, the integer solutions (x,y,a,b,c,m) are given below.*

(i) *For $n=3$, (x,y,a,b,c,m) are given in Table 2.*

(ii) *For $n=4$, (x,y,a,b,c,m) are given in Table 1.*

(iii) *For $n=6$, $(x,y,a,b,c,m)=(716,9,1,1,2,0)$.*

(iv) *For $n=12$, $(x,y,a,b,c,m)=(716,3,1,1,2,0)$.*

Remarks. We mention some earlier results which can be retrieved from Theorem 1.1.

(i) For $m\geq 2$, reducing (1.2) modulo 4, one can see that it has no integer solutions.

(ii) Abu Muriefah and Arif proved that the Diophantine equation $x^2+5^a=y^n$, $n\geq 3$, $x\geq 1$, $y>1$, $\gcd(x,y)=1$, has no integer solutions when a is odd (see, [1, Theorem]). Later, Tao completely solved it in [25] and proved that it has no integer solutions. We can get these results from our theorem. In fact, our theorem shows that $(x,y,a,m,n)=(239,13,0,1,4),(7,3,1,1,3),(99,17,2,1,3)$ are the only integer solutions of the Diophantine equation

$$x^2+5^a=2^my^n, n\geq 3, x\geq 1, y>1, \gcd(x,y)=1.$$

(iii) It follows from [23, Theorem 1.1] that $(x,y,b,n)=(70,17,1,3)$ is the only integer solution of the Diophantine equation $x^2+13^b=y^n$ $(b,x,y\geq 1,\gcd(x,y)=1,n\geq 3)$. A consequence of Theorem 1.1 extends this result to the Diophantine equation $x^2+13^b=2^my^n$ $(x,y\geq 1,\gcd(x,y)=1,b,m\geq 0,n\geq 3)$. In this case, the only integer solutions are $(x,y,b,m,n)=(70,17,1,0,3),(9,5,2,1,3),(239,13,0,1,4)$.

(iv) In [3], Abu Muriefah et al. proved that the Diophantine equation $x^2+5^a13^b=y^n, n\geq 3, x\geq 1, y>1, \gcd(x,y)=1$, has no integer solutions, except $(x,y,a,b)=(70,17,0,1),(142,29,2,2),(4,3,1,1)$. We can get this result from our theorem. Our theorem also confirms that the integer solution of the Diophantine equation $x^2+5^a13^b=2y^n$ $(n\geq 3,x\geq 1,y>1,\gcd(x,y)=1)$, are $(x,y,a,b,n)=(239,13,0,0,4),(9,5,0,2,3),(7,3,1,0,3),(99,17,2,0,3),$ $(19,7,2,1,3),(253,73,2,4,3),$ $(79137,1463,2,3,3),(188000497,260473,8,4,3)$.

(v) Pink and Rábai completely solved the Diophantine equation $x^2+5^a17^c=y^n$ $(x,y\geq 1,\gcd(x,y)=1,a,c\geq 0,n\geq 3)$ in [24]. However, our theorem gives all the integer solutions of an extension of this equation, namely $x^2+5^a17^c=2^my^n$ $(x,y\geq 1,\gcd(x,y)=1,a,c\geq 0,n\geq 3)$.

Theorem 1.1 yields the following straightforward corollary. In case of $m=0$, this corollary follows from the work of Gou and Wang [18].

Corollary 1.1. *The Diophantine equation*

$$x^2 + 17^k = 2^m y^n, \ x \geq 1, y > 1, \gcd(x, y) = 1, k, m \geq 0, n \geq 3,$$

has no integer solutions, except $(x, y, k, m, n) = (8, 3, 1, 0, 4), (31, 5, 2, 1, 4), (239, 13, 0, 1, 4)$.

The next corollary immediately follows from Theorem 1.1.

Corollary 1.2. *The only integer solutions of the Diophantine equation*

$$x^2 + 13^k 17^\ell = 2^m y^n, \ x \geq 1, y > 1, \gcd(x, y) = 1, k, \ell, m \geq 0, n \geq 3$$

are $(x, y, k, \ell, m, n) = (70, 17, 1, 0, 0, 3)$, $(9, 5, 2, 0, 1, 3)$, $(8, 3, 0, 1, 0, 4), (31, 5, 0, 2, 1, 4), (239, 13, 0, 0, 1, 4)$.

We organize this article as follows. In §2, we deal with the exponent n satisfying $4 \mid n$. In this case, we transform (1.2) into quartic curves, and thus the problem is reduced to finding all $\{5, 13, 17\}$-integral points on these curves. Recall that for a finite set of prime numbers S, an S-integer is a rational number r/s with coprime integers r and $s > 0$ such that any prime factor of s lies in S. We treat (1.2) in §3 for prime exponent $n \geq 3$. For $n \geq 5$ with $n \neq 7$, we apply the result of Bilu, Hanrot and Voutier [7] concerning the existence of primitive divisors in Lehmer sequences. In case of $n = 7$, we first use some criteria for the existence of primitive divisors in Lehmer sequences to handle some cases of (1.2). For the remaining cases, we somehow transform them into elliptic curves. Analogously, we transform (1.2) into elliptic curves for $n = 3$. Then we solve the problem by finding all $\{5, 13, 17\}$-integral points on these elliptic curves. In §4, we summarize the proof of Theorem 1.1. All the computations are done using MAGMA [8].

2. The case: $4 \mid n$

Here, we prove the following proposition.

Proposition 2.1. *If n is a multiple of* 4*, then all integer solutions of* (1.2) *are given in Table 1.*

Tab. 1: All the solutions of (1.2) when $4 \mid n$

x	y	a	b	c	m	n	x	y	a	b	c	m	n
8	3	0	0	1	0	4	4	3	1	1	0	0	4
26556	163	5	1	1	0	4	36	7	1	1	1	0	4
716	27	1	1	2	0	4	716	3	1	1	2	0	12
239	13	0	0	0	1	4	31	5	0	0	2	1	4

Proof. Assume that $n = 4t$, where $t \geq 1$ is an integer. Then (1.2) can be written as

$$x^2 + 5^a 13^b 17^c = 2^m \left(y^t\right)^4, \ x \geq 1, y > 1, \gcd(x, y) = 1, a, b, c, m \geq 0, t \geq 1. \tag{2.1}$$

Recall that (2.1) has no integer solution when $m \geq 2$. Let $a \equiv a_1 \pmod 4, b \equiv b_1$

(mod 4) and $c \equiv c_1 \pmod 4$. Then (2.1) can be written as

$$x^2 + 5^{a_1} 13^{b_1} 17^{c_1} z^4 = 2^m \left(y^t\right)^4,$$

where $5^a 13^b 17^c = 5^{a_1} 13^{b_1} 17^{c_1} z^4$. This can be written as

$$X^2 = 2^m Y^4 - 5^{a_1} 13^{b_1} 17^{c_1}, \tag{2.2}$$

where $X = x/z^2$ and $Y = y^t/z$. Now the problem of finding integer solutions of (2.1) is transformed to finding all $\{5, 13, 17\}$-integer points on the 128 quartic curves defined by (2.2). Here, we use MAGMA [8] subroutine `SIntegralLjunggrenPoints` to determine all $\{5, 13, 17\}$-integer points on these elliptic curves. Note that we avoid $\{5, 13, 17\}$-integer points with $XY = 0$ as they yield to $xy = 0$. Also taking into account that $\gcd(x, y) = 1$, we don't consider $\{5, 13, 17\}$-integer points such that the numerators of X and Y are not coprime. We finally get only 8 integer points (X, Y) with $XY \neq 0$ and the numerators of X and Y are coprime. We then use the relations

$$X = \frac{x}{z^2}, Y = \frac{y^t}{z} \text{ and } 5^a 13^b 17^c = 5^{a_1} 13^{b_1} 17^{c_1} z^4,$$

to find the integer solutions (x, y, a, b, c, m, n), which are listed in Table 1. □

3. The case: $n \geq 3$ is prime

We rewrite (1.2) by changing n to p to emphasize that the exponent is prime:

$$x^2 + 5^a 13^b 17^c = 2^m y^p,\ x, \geq 1, y > 1, \gcd(x, y) = 1, a, b, c \geq 0, p \geq 2, m = 0, 1. \tag{3.1}$$

Proposition 3.1. *The equation* (3.1) *has no integer solutions for* $p > 3$*. When* $p = 3$*, its integer solutions are given by* $(x, y, a, b, c, m) \in \mathfrak{S}$*, where*

$$\begin{aligned} \mathfrak{S} := \{&(70, 17, 0, 1, 0, 0), (716, 81, 1, 1, 2, 0), (94, 21, 2, 0, 1, 0), (142, 29, 2, 2, 0, 0),\\ &(2034, 161, 3, 0, 2, 0), (9, 5, 0, 2, 0, 1), (7, 3, 1, 0, 0, 1), (99, 17, 2, 0, 0, 1),\\ &(63, 13, 2, 0, 1, 1), (19, 7, 2, 1, 0, 1), (33, 7, 2, 2, 1, 1), (118699, 1917, 2, 2, 1, 1),\\ &(79137, 1463, 2, 3, 0, 1), (253, 73, 2, 4, 0, 1), (188000497, 260473, 8, 4, 0, 1),\\ &(267689, 3297, 2, 2, 3, 1), (336049, 4317, 10, 0, 3, 1), (17127, 553, 6, 2, 1, 1)\}. \end{aligned}$$

Before proceeding further, we need to recall some results and to fix some notations. The following lemma follows from [26, Corollary 3.1]; however, the idea goes back to the work of Ljunggren [22].

Lemma 3.1. *Let* d $(\neq 3)$ *be square-free positive integer. If* $n \geq 3$ *is an odd integer coprime to* $h(-d)$*, the class number of* $\mathbb{Q}(\sqrt{-d})$*, then all integer solutions* (X, Y, Z) *of the Diophantine equation*

$$X^2 + dY^2 = 2^m Z^n,\ X, Y \geq 1, \gcd(X, dY) = 1, m = 0, 1,$$

can be expressed as

$$\frac{X + Y\sqrt{-d}}{\sqrt{2^m}} = \varepsilon_1 \left(\frac{u + \varepsilon_2 v\sqrt{-d}}{\sqrt{2^m}}\right)^n,$$

where u and v are positive integers satisfying $2^m Z = u^2 + dv^2$ and $\gcd(u, dv) = 1$, *and $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.*

Assume that F_k (resp. L_k) denote the k-th Fibonacci (resp. Lucas) number defined by $F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}$ (resp. $L_0 = 2, L_1 = 1, L_k = L_{k-1} + L_{k-2}$) for $k \geq 2$. Then the following lemma follows from Theorems 1,2,3 and 4 in [14].

Lemma 3.2. *Let F_k and L_k be the k-th Fibonacci and Lucas numbers, respectively. Then*

(i) *if $F_k = x^2$, then $(k, x) = (0, 0), (1, \pm 1), (2, \pm 1), (12, \pm 12)$;*

(iii) *if $F_k = 2x^2$, then $(k, x) = (0, 0), (3, \pm 1), (6, \pm 2)$;*

(iii) *if $L_k = x^2$, then $(k, x) = (1, \pm 1), (3, \pm 2)$;*

(iv) *if $L_k = 2x^2$, then $(k, x) = (0, \pm 1), (6, \pm 3)$.*

A pair (α_1, α_2) of algebraic integers is said to be a Lehmer pair if $(\alpha_1 + \alpha_2)^2$ and $\alpha_1\alpha_2$ are two non-zero coprime rational integers, and α_1/α_2 is not a root of unity. Also for a positive integer n, the Lehmer number corresponds to the pair (α_1, α_2) is defined as

$$\mathfrak{L}_n(\alpha_1, \alpha_2) = \begin{cases} \dfrac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} & \text{if } \ n \text{ is odd,} \\ \dfrac{\alpha_1^n - \alpha_2^n}{\alpha_1^2 - \alpha_2^2} & \text{if } \ n \text{ is even.} \end{cases}$$

Note that all Lehmer numbers are non-zero rational integers. A prime divisor p of $\mathfrak{L}_n(\alpha_1, \alpha_2)$ is primitive if $p \nmid (\alpha_1^2 - \alpha_2^2)^2 \mathfrak{L}_1(\alpha_1, \alpha_2)\mathfrak{L}_2(\alpha_2, \alpha_2) \cdots \mathfrak{L}_{n-1}(\alpha_1, \alpha_2)$. Further, $\big((\alpha_1 + \alpha_2)^2, (\alpha_1 - \alpha_2)^2\big)$ is known as the parameters of the Lehmer pair (α_1, α_2).

Proof of Proposition 3.1. We first rewrite (3.1) as follows:

$$x^2 + dz^2 = 2^m y^p, \ x, y \geq 1, \gcd(x, y) = 1, a, b, c \geq 0, p \geq 2, m = 0, 1, \tag{3.2}$$

where $d \in \{1, 17, 13, 221, 5, 85, 65, 1105\}$ and $z = 5^{a_1} 13^{b_1} 17^{c_1}$ for some integers $a_1, b_1, c_1 \geq 0$. Using MAGMA, we see that $h(-d) \in \{1, 2, 4, 8, 16\}$, and thus $p \nmid h(-d)$. As $\gcd(x, y) = 1$, so that $\gcd(x, dz) = 1$. Therefore by Lemma 3.1, we have (from (3.2))

$$\frac{x + z\sqrt{-d}}{\sqrt{2^m}} = \varepsilon_1 \left(\frac{u + \varepsilon_2 v\sqrt{-d}}{\sqrt{2^m}} \right)^p, \tag{3.3}$$

where u and v are positive integers satisfying $2^m y = u^2 + dv^2$ and $\gcd(u, dv) = 1$, and $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Note that $2 \nmid duvy$.

We define,

$$\alpha := \frac{u + \varepsilon_2 v\sqrt{-d}}{\varepsilon_1\sqrt{2^m}}.$$

Then α and its conjugate $\bar{\alpha}$ are algebraic integers such that $\gcd((\alpha + \bar{\alpha})^2, \alpha\bar{\alpha}) = 1$. It is easy to see that $\alpha/\bar{\alpha}$ satisfies

$$yZ^2 - 2^{1-m}(u^2 - v^2 d)Z + y = 0\,.$$

This shows that $\alpha/\bar{\alpha}$ is not a root of unity as $\gcd(2^{m-1}(u^2 - v^2d), y) = 1$. Thus $(\alpha, \bar{\alpha})$ is a Lehmer pair and $(2^{2-m}u^2, -2^{2-m}v^2d)$ is the corresponding parameter.

Let $\mathfrak{L}_n$ be the Lehmer number corresponding to the Lehmer pair $(\alpha, \bar{\alpha})$. Then

$$|\mathfrak{L}_p(\alpha, \bar{\alpha})| = \frac{z}{v} = \frac{5^{a_1}13^{b_1}17^{c_1}}{v}\,. \tag{3.4}$$

We divide the remaining part of the proof into several parts depending of the values of p.

Case I: When $p > 7$. Assume that q is a primitive divisor of $\mathfrak{L}_p(\alpha, \bar{\alpha})$. Then by (3.4), $q \in \{5, 13, 17\}$. We now utilize the fact that any primitive divisor of $\mathfrak{L}_p(\alpha, \bar{\alpha})$ is congruent to ± 1 modulo p, to conclude that none of these values of q is primitive divisor of $\mathfrak{L}_p(\alpha, \bar{\alpha})$. This contradicts to a consequence of the Primitive Divisor Theorem for Lehmer sequences which states that, if $p \geq 3$, then $\mathfrak{L}_p(\alpha, \bar{\alpha})$ has a primitive prime divisor except for finitely many pairs $(\alpha, \bar{\alpha})$. The Lehmer sequences correspond to these exceptional pairs $(\alpha, \bar{\alpha})$ are given in [7, Tables 2 and 4] in terms of their parameters $((\alpha + \bar{\alpha})^2, (\alpha - \bar{\alpha})^2)$. Since $(2^{2-m}u^2, -2^{2-m}v^2d)$ are the parameters, so that $p = 13$ and $(2^{2-m}u^2, 2^{2-m}v^2d) = (1, 7)$, which is not possible as $m = 0, 1$. This concludes that (3.1) has no integer solutions when $p > 7$.

Case II: When $p = 7$. As in Case I, if q is a primitive divisor of $\mathfrak{L}_7(\alpha, \bar{\alpha})$, then only possibility is $q = 13$. If $b_1 = 0$, then $\mathfrak{L}_7(\alpha, \bar{\alpha})$ has no primitive divisors, and hence as in Case I by [7, Table 2], we have $(2^{2-m}u^2, 2^{2-m}v^2d) = (1, 7), (1, 19), (3, 5), (5, 7), (13, 3), (14, 22)$. These are not possible as $m = 0, 1$. Therefore (3.1) has no solutions. Analogously, we can conclude that (3.1) has no solutions if $a_1 = b_1 = c_1 = 0$.

We now consider $b_1 \geq 1$. Note that

$$(\alpha^2 - \bar{\alpha}^2)^2 = -2^{4-2m}u^2v^2d. \tag{3.5}$$

Thus if $13 \mid vd$ then 13 is not a primitive divisor, and hence as before (3.1) has no solutions. Therefore $d = 1, 5, 17, 85$ and $13 \nmid v$. Also using the fact that for a primitive divisor q, the sign of $q \equiv \pm 1 \pmod{p}$ coincides with that of the Legendre symbol $\left(\frac{-4d}{q}\right)$, we get $d = 5, 85$. Equating imaginary parts in (3.3), we get

$$2^{3m}5^{a_1}13^{b_1}17^{c_1} = \varepsilon v(7u^6 - 35u^4v^2d + 21u^2v^4d^2 - v^6d^3)\,, \tag{3.6}$$

where $\varepsilon = \varepsilon_1\varepsilon_2 = \pm 1$. Since $\gcd(u, v) = 1$ and $2, 13 \nmid v$, so that (3.6) gives

$$v = 1, 5^{a_1}, 17^{c_1}, 5^{a_1}17^{c_1}\,.$$

For $v = 1$, (3.6) becomes

$$2^{3m}5^{a_1}13^{b_1}17^{c_1} = \varepsilon(7u^6 - 35u^4d + 21u^2d^2 - d^3)\,.$$

We can rewrite this equation as

$$DY^2 = 7X^3 - 35dX^2 + 21d^2X - d^3, \tag{3.7}$$

where $X = u^2, Y = 2^m 5^{a_2} 13^{b_2} 17^{c_2}, a_2 = \lfloor a_1/2 \rfloor, b_2 = \lfloor b_1/2 \rfloor, c_2 = \lfloor c_1/2 \rfloor$ and $D = \varepsilon 2^m 5^i 13^j 17^k$ with $i, j, k \in \{0, 1\}$. We now multiply both sides of (3.7) by $7^2 D^3$ and then rewrite as follows:

$$V^2 = U^3 - 35dDU + 147d^2D^2U - 49d^3D^3\,, \tag{3.8}$$

where $U = 7DX$ and $V = 7D^2Y$. Here, we use `IntegralPoints` subroutine of MAGMA [8] to compute all integral points on the 64 elliptic curves defined by (3.8). We then apply the relations $U = 7DX, V = 7D^2Y, X = u^2, Y = 2^m 5^{a_2} 13^{b_2} 17^{c_2}, a_2 = \lfloor a_1/2 \rfloor, b_2 = \lfloor b_1/2 \rfloor, c_2 = \lfloor c_1/2 \rfloor$ and $D = \varepsilon 2^m 5^i 13^j 17^k$ with $i, j, k \in \{0, 1\}$. We check that none of these integral points leads to an integer solution of (3.1).

We now consider $v = 5^{a_1}$ with $a_1 \geq 1$. Then (3.6) becomes

$$2^{3m} 13^{b_1} 17^{c_1} = \varepsilon(7u^6 - 35u^4v^2d + 21u^2v^4d^2 - v^6d^3)\,.$$

Dividing both sides of this equation by v^6, we obtain the following elliptic curves

$$DY^2 = 7X^3 - 35dX^2 + 21d^2X - d^3, \tag{3.9}$$

where $X = u^2/v^2, Y = 2^m 13^{b_2} 17^{c_2}/v^3, b_2 = \lfloor b_1/2 \rfloor, c_2 = \lfloor c_1/2 \rfloor$ and $D = \varepsilon 2^m 13^j 17^k$ with $j, k \in \{0, 1\}$. As before, multiplying both sides of (3.9) by $7^2 D^3$, we get

$$V^2 = U^3 - 35dDU + 147d^2D^2U - 49d^3D^3, \tag{3.10}$$

where $U = 7DX$ and $V = 7D^2Y$. Here, we use `SIntegralPoints` subroutine of MAGMA to compute all $\{5\}$-integral points on the 32 elliptic curves defined by (3.9). As in the previous case, we apply the relations $U = 7DX, V = 7D^2Y, X = u^2/5^{2a_1}, Y = 2^m 13^{b_2} 17^{c_2}/5^{3a_1}, a_2 = \lfloor a_1/2 \rfloor, b_2 = \lfloor b_1/2 \rfloor, c_2 = \lfloor c_1/2 \rfloor$ and $D = \varepsilon 2^m 13^j 17^k$ with $j, k \in \{0, 1\}$, to find the corresponding integer solutions of (3.1). However, none of these integral points leads to an integer solution of (3.1).

Assume that $v = 17^{c_1}$ with $c_1 \geq 1$. Then (3.6) becomes

$$2^{3m} 5^{a_1} 13^{b_1} = \varepsilon(7u^6 - 35u^4v^2d + 21u^2v^4d^2 - v^6d^3)\,.$$

As before, we first divide both sides of this equation by v^6 and then multiply by $7^2 D^3$ to arrive at

$$V^2 = U^3 - 35dDU + 147d^2D^2U - 49d^3D^3, \tag{3.11}$$

where $U = 7DX = 7Du^2/v^2, V = 7D^2 2^m 5^{a_2} 13^{b_2}/v^3, a_2 = \lfloor a_1/2 \rfloor, b_2 = \lfloor b_1/2 \rfloor$ and $D = \varepsilon 2^m 5^i 13^j$ with $i, j \in \{0, 1\}$. As in the previous case, we compute all $\{17\}$-integral points on the elliptic curves defined by (3.10), but none of these integral points leads to an integral solution of (3.1).

Finally let $v = 5^{a_1} 17^{c_1}$ with $a_1, c_1 \geq 1$. Then (3.6) becomes

$$2^{3m} 13^{b_1} = \varepsilon(7u^6 - 35u^4v^2d + 21u^2v^4d^2 - v^6d^3)\,.$$

In the same way as before, we divide both sides of this equation by v^6 and then multiply by $7^2 D^3$ to get

$$V^2 = U^3 - 35dDU + 147d^2D^2U - 49d^3D^3, \tag{3.12}$$

where $U = 7DX = 7Du^2/v^2, V = 7D^2 2^m 13^{b_2}/v^3, b_2 = \lfloor b_1/2 \rfloor$ and $D = \varepsilon 2^m 13^j$ with $j \in \{0, 1\}$. We compute all $\{5, 17\}$-integral points on the 16 elliptic curves defined by (3.11), but none of these integral points leads to an integer solution of (3.1).

Case III: When $p = 5$**.** Let q be a primitive divisor of $\mathfrak{L}_5(\alpha,\bar{\alpha})$. Then by (3.4) only possibility $q = 5$ or $q = 13$ or 17. Again the fact that any primitive divisor of $\mathfrak{L}_p(\alpha,\bar{\alpha})$ is congruent to ± 1 modulo p confirms that none of $5, 13$ and 17 is a primitive divisor of $\mathfrak{L}_5(\alpha,\bar{\alpha})$. Therefore as in Case I, using Primitive Divisor Theorem for Lehmer sequences and [7, Table 4], we get

$$(2^{2-m}u^2, 2^{2-m}v^2d) = \begin{cases} (F_{k-2\varepsilon}, 4F_k - F_{k-2\varepsilon}) & \text{with } k \geq 3\,, \\ (L_{k-2\varepsilon}, 4L_k - L_{k-2\varepsilon}) & \text{with } k \neq 1\,, \end{cases} \tag{3.13}$$

where F_k (resp. L_k) denotes the k-th Fibonacci (resp. Lucas) number. Utilizing Lemma 3.2 in (3.13), we conclude the following:

- $(k-2\varepsilon, m, 2u) = (1,0,1), (2,0,1), (12,0,12)$; none of these are possible as u is odd.
- $(k-2\varepsilon, m, u) = (3,1,1), (6,1,2)$, but the only possibility $(k-2\varepsilon, u) = (3,1,1)$. This leads to $(k,m,u,v,d) = (5,1,1,3,1)$, and thus $y = (u^2+dv^2)/2 = 5$. Therefore (3.1) becomes $x^2 + 1 = 2 \times 5^5$, which has no integer solutions.
- $(k-2\varepsilon, m, 2u) = (1,0,1), (3,0,2)$, but the only possibility is $(k-2\varepsilon, m, u) = (3,0,1)$. This leads to $v^2d = 8$, which is not possible as vd is odd.
- $(k-2\varepsilon, m, u) = (6,1,3)$, which gives $(k,m,u,vd) = (4,1,3,5), (8,1,3,89)$. The only possibility is $(k,m,u,vd) = (4,1,3,5)$, which yields $y = 7$. Thus (3.1) becomes $x^2 + 5 = 2 \times 7^5$, which has no integer solutions.

Case IV: $p = 3$**.** In this case, the facts of primitive divisors of $\mathfrak{L}_3(\alpha,\bar{\alpha})$ does not provide any fruitful conclusion. Thus, we transfer the problem of finding integer solutions of (3.1) into the problem of finding $\{5, 13, 17\}$-integral points on the corresponding elliptic curves. For $p = 3$, (3.1) becomes

$$x^2 + 5^a 13^b 17^c = 2^m y^3,\ x, y \geq 1, \gcd(x,y) = 1, a, b, c \geq 0, p \geq 2, m = 0, 1\,. \tag{3.14}$$

We write

$$2^{2m} 5^a 13^b 17^c = Az^6, \tag{3.15}$$

where $(A, z) = \left(2^{2m}5^{a_1}13^{b_1}17^{c_1}, 5^{a_2}3^{b_2}7^{c_2}\right)$ with $a = 6a_2 + a_1, b = 6b_2 + b_1, c = 6c_2 + c_1$, $a_1, b_1, c_1 \in \{0,1,2,3,4,5\}$ and $a_2, b_2, c_2 \geq 0$.

We now multiply the both sides of (3.14) by 2^{2m} and then subsequently divide both sides of z^6, and put $X := 2^m x/z^3$ and $2^m y/z^2$ to reduce to

$$X^2 = Y^3 - A\,. \tag{3.16}$$

Here, we again use `SIntegralPoints` subroutine of MAGMA to determine all $\{5, 13, 17\}$-integral points on the 432 elliptic curves defined by (3.16). Taking into account that $xy \neq 0$ and $\gcd(x,y) = 1$, we don't consider the $\{5, 13, 17\}$-integer points such that $XY = 0$ or the numerators of $X/2^m$ and $Y/2^m$ are not coprime. Finally, we utilize the relation (3.15) along with the conditions on A and z, and on their exponents to find the corresponding integer solutions of (3.14). These solutions are given by $(x, y, a, b, c, m) \in \mathfrak{S}$. $\square$

4. Proof of Theorem 1.1

We first assume that n is a multiple of 4. Then all integer solution of (1.2) are given by Proposition 2.1 (see, Table 1).

Let $n \geq 3$ be an integer such that $4 \nmid n$. Then we can write $n = p\ell$ for some odd prime p and some odd integer $\ell \geq 1$. Thus, (1.2) can rewritten as

$$x^2+5^a13^b17^c = 2^m(y^{n/p})^p,\ x \geq 1, y > 1, \gcd(x,y) = 1, a,b,c,m \geq 0, p \geq 3\,. \tag{4.1}$$

By Proposition 3.1, the only integer solutions of (4.1) are given by

$$(x, y^{n/p}, a, b, c, m) \in \mathfrak{S}.$$

These solutions lead to the integer solutions of (1.2), which are listed in Table 2.

Tab. 2: All the solutions of (1.2) when $3 \mid n$

x	y	a	b	c	m	n	x	y	a	b	c	m	n
70	17	0	1	0	0	3	716	81	1	1	2	0	3
716	9	1	1	2	0	6	716	3	1	1	2	0	12
94	21	2	0	1	0	3	142	29	2	2	0	0	3
2034	161	3	0	2	0	3	9	5	0	2	0	1	3
7	3	1	0	0	1	3	99	17	2	0	0	1	3
63	13	2	0	1	1	3	19	7	2	1	0	1	3
33	7	2	2	1	1	3	118699	1917	2	2	1	1	3
79137	1463	2	3	0	1	3	253	73	2	4	0	1	3
188000497	260473	8	4	0	1	3	267689	3297	2	2	3	1	3
336049	4317	10	0	3	1	3	17127	553	6	2	1	1	3

Acknowledgement. The authors are grateful to Professor Kalyan Chakraborty for carefully reading and valuable comments which have improved the presentation of this manuscript. The authors are thankful to the anonymous referee(s) for valuable comments/suggestions. This work is supported by SERB MATRICS Project (No. MTR/2021/000762) and SERB CRG Grant (No. CRG/2023/007323), Govt. of India.

References

[1] Abu Muriefah, F.S., Arif, S.A., *The Diophantine equation* $x^2 + 5^{2k+1} = y^n$, Indian J. Pure Appl. Math. **30** (3) (1999), 229–231.

[2] Abu Muriefah, F.S., Luca, F., Siksek, S., Tengely, S., *On the Diophantine equation* $x^2 + C = 2y^n$, Int. J. Number Theory **5** (6) (2009), 1117–1128.

[3] Abu Muriefah, F.S., Luca, F., Togbé, A., *On the Diophantine equation* $x^2 + 5^a13^b = y^n$, Glasgow Math. J. **50** (1) (2008), 175–181.

[4] Alan, M., Zengin, U., *On the Diophantine equation* $x^2 + 3^a41^b = y^n$, Period. Math. Hungar. **81** (2020), 284–291.

[5] Arif, S.A., Al-Ali, A.S., *On the Diophantine equation* $ax^2 + b^m = 4y^n$, Acta Arith. **103** (4) (2002), 343–346.

[6] Bhatter, S., Hoque, A., Sharma, R., *On the solutions of a Lebesgue-Nagell type equation*, Acta Math. Hungar. **158** (2019), 17–26.

[7] Bilu, Y., Hanrot, G., Voutier, P.M., *Existence of primitive divisors of Lucas and Lehmer numbers (with an appendix by M. Mignotte)*, J. Reine Angew. Math. **539** (2001), 75–122.

[8] Bosma, W., Cannon, J., Playoust, C., *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (3) (1997), 235–265.

[9] Bugeaud, Y., *On some exponential Diophantine equations*, Monatsh. Math. **132** (2001), 93–97.

[10] Chakraborty, K., Hoque, A., *On the Diophantine equation $dx^2 + p^{2a}q^{2b} = 4y^p$*, Results Math. **77** (1) (2022), 11 pp., article no. 18.

[11] Chakraborty, K., Hoque, A., Sharma, R., *Complete solutions of certain Lebesgue-Ramanujan-Nagell equations*, Publ. Math. Debrecen **97** (3/4) (2020), 339–352.

[12] Chakraborty, K., Hoque, A., Sharma, R., *On the solutions of certain Lebesgue-Ramanujan-Nagell equations*, Rocky Mountain J. Math. **51** (2) (2021), 459–471.

[13] Chakraborty, K., Hoque, A., Srinivas, K., *On the Diophantine equation $cx^2 + p^{2m} = 4y^n$*, Results Math. **76** (2021), 12 pp., article no. 57.

[14] Cohn, J.H.E., *Square Fibonacci numbers, etc.*, Fibonacci Quart. **2** (2) (1964), 109–113.

[15] Dabrowski, A., *On the Lebesgue-Nagell equation*, Colloq. Math. **125** (2) (2011), 245–253.

[16] Dabrowski, A., Günhan, N., Soydan, G., *On a class of Lebesgue-Ljunggren-Nagell type equations*, J. Number Theory **215** (2020), 149–159.

[17] Demirci, M., *On the Diophantine equation $x^2+5^ap^b = y^n$*, Filomat **31** (16) (2017), 5263–5269.

[18] Gou, S., Wang, T.T., *The Diophantine equation $x^2 + 2^a.17^b = y^n$*, Czechoslovak Math. J. **62** (2012), 645–654.

[19] Hoque, A., *On a class of Lebesgue-Ramanujan-Nagell equation*, Period. Math. Hungar. (2023), `https://doi.org/10.1007/s10998-023-00564-z`.

[20] Le, M., Soydan, G., *A brief survey on the generalized Lebesgue-Ramanujan-Nagell equation*, Surv. Math. Appl. **15** (2020), 473–523.

[21] Lebesgue, V.A., *Sur l'impossibilité, en nombres entiers, de l'équation $x^m = y^2 + 1$*, Nouvelles Annales des Math. **9** (1850), 178 pp.

[22] Ljunggren, W., *On the Diophantine equation $Cx^2 + D = 2y^n$*, Math. Scand. **18** (1966), 69–86.

[23] Luca, F., Togbé, A., *On the equation $x^2 + 2^\alpha 13^\beta = y^n$*, Colloq. Math. **116** (1) (2009), 139–146.

[24] Pink, I., Rábai, Z., *On the Diophantine equation $x^2 + 5^k 17^l = y^n$*, Commun. Math. **19** (2011), 1–9.

[25] Tao, L., *On the Diophantine equation $x^2 + 5^m = y^n$*, Ramanujan J. **19** (2009), 325–338.

[26] Yuan, P., *On the Diophantine equation $ax^2 + by^2 = ck^n$*, Indag. Math. (N.S.) **16** (2) (2005), 301–320.

[27] Zhu, H., Le, M., Soydan, G., Togbé, A., *On the exponential Diophantine equation $x + 2^ap^b = y^n$*, Period. Math. Hungar. **70** (2015), 233–247.

Department of Mathematics, Gauhati University,
Guwahati-781014, Assam, India
E-mail: `baruah067@gmail.com` `anup@gauhati.ac.in`

Department of Mathematics, Faculty of Science,
Rangapara College, Rangapara,
Sonitpur-784505, Assam, India
E-mail: `ahoque.ms@gmail.com`