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NUMERICAL APPROACHES TO THE MODELLING OF QUASI-BRITTLE CRACK PROPAGATION

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Abstract. Computational analysis of quasi-brittle fracture in cement-based and similar composites, supplied by various types of rod, fibre, etc. reinforcement, is crucial for the prediction of their load bearing ability and durability, but rather difficult because of the risk of initiation of zones of microscopic defects, followed by formation and propagation of a large number of macroscopic cracks. A reasonable and complete deterministic description of relevant physical processes is rarely available. Thus, due to significance of such materials in the design and construction of buildings, semi-heuristic computational models must be taken into consideration. These models generate mathematical problems, whose solvability is not transparent frequently, which limits the credibility of all results of ad hoc designed numerical simulations. In this short paper such phenomena are demonstrated on a simple model problem, covering both micro- and macro-cracking, with references to needful generalizations and more realistic computational settings.

1. Introduction

Cement-based composites, supplied by various type of fibre, rod, etc. reinforcement, are the most frequently used materials in building structures thorough the world. Their load bearing ability and durability is conditioned by the minimization of the risk of initiation and propagation of fracture. Due its rather complicated structure, the so-called quasi-brittle fracture can be expected here, using the nomenclature of [27], unlike simple fracture models as the brittle or ductile ones. In the rough qualitative classification, under mechanical, thermal, etc. loads 4 deformation stages can be distinguished: i) reversible elastic deformation, ii) initiation of zones of microscopic defects, iii) formation and propagation of systems of macroscopic cracks, iv) destruction of material structure, from local to total one. The development of advanced materials, structure and technologies can rarely come from the experience with classical ones, moreover the significant size effect limits

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the replacement of the expensive results from long-time observations in situ by those from simplified laboratory experiment, thus some deeper both physical and mathematical analysis for the design of tools for computational modelling and simulation is required. This can be based on the principles of classical thermomechanics by [3], working with conservation of mass, (linear and angular) momentum and energy, supplied by appropriate constitutive relations. Unfortunately, their complete quantitative formulation, coming from exact microstructural considerations like [20], is not available because of a complicated material structure; consequently some semi-heuristic approach could be useful, as proposed by [13], although its well-possedness might be doubtful and further generalizations are needed.

Such time-delayed development of mathematical theory is typical for many algorithms needed in engineering problems: e.g. most engineering journals celebrated 70 years of the finite element method (FEM), connected with the design of wings of Boeing YB52 prototype by [30] (tested 15th April 1952, published 1956), whereas the history of the mathematical theory of FEM is about 15 years shorter, as evident from [36] and its references. Moreover, very detailed deterministic models suffer from the complicated (or quite impossible) identification of reasonable material characteristics for constitutive relations, generating non-trivial ill-posed inverse problems; thus all practical computational tools can be seen as certain compromises between the general physical theory, its simplified mathematical presentation and the design of effective and robust numerical algorithms, up to their software and hardware implementations, limited by the amount of time and money for such complete analysis, covering both the formal verification and the practical validation in the sense of [29]. Thus in this contributed conference paper we shall introduce a model problem based on a linear Neumann and Dirichlet boundary value and Cauchy initial value problem for one partial differential equation of evolution for i), coming from the conservation of momentum, modified by certain nonlinear terms covering ii) and iii) (Section 2), followed by its existence and convergence analysis (Section 3), with some references to useful generalizations (Section 4); this approach is not able to handle later stages of both ii) and iii) tending to iv) without substantial improvements, containing numerous open questions, not discussed here.

The principal idea for ii) is the implementation of certain damage factor $\mathfrak{D}$, following [13], relying on the nonlocal approach suggested by [7].The ill-possedness of such approach for most engineering formulations, if applied to the evaluation of strains or stresses in general, criticized by [8], unlike the preliminary existence analysis of [1] (valid for a pure Dirichlet boundary value problem), can be fortunately avoided here thanks to a careful (rather complicated) choice of $\mathfrak{D}$. The long tradition of an intuitive need of such regularization, motivated by micromechanical considerations, can be documented on [2]. For iii) the model of cohesive interfaces by [23] for activation and development of cracks for pre-defined potential crack positions can be used, applying the classical (extrinsic) formulation of the extended finite element method (XFEM) by [18], working with additional degrees of freedom (i. e. with new parameters for the evaluation of unknown function(s)) for adaptive local enrichment of standard FEM bases on all tips of macroscopic cracks and along them; alternatively the intrinsic version of XFEM by [10], modifying such bases in a more complicated way without increase of degrees of freedom, can be utilized. The method of discretization of time, based on the properties of Rothe seqences, will be considered as the first choice for any time t from a time interval $\mathcal{I} = [0, T]$ with a prescribed finite positive time T; for the discretization of a deformable body Ω, including all boundary and interface conditions, in the 3-dimensional Euclidean space $\mathcal{R}^3$, supplied by some fixed Cartesian coordinate system $x = (x_1, x_2, x_3)$, XFEM is then available.

2. A model problem

For simplicity (to avoid technical difficulties in proofs), let a deformable body Ω occupy a unit of a finite number of domains with Lipschitz continuous boundaries, whose unit consists of disjoint parts Θ (for Dirichlet boundary conditions), Γ (for Neumann boundary conditions) and Λ (for interface boundary conditions, required by iii) only); the pair of Cauchy initial conditions will be prescribed for $t = 0$. The standard notation of Lebesgue, Sobolev, Bochner–Sobolev, etc. (abstract) function spaces by [24], *Parts* 1 and 7, will be utilized. For the brevity we shall also introduce $H = L^2(\Omega)^3$, $M = L^\infty(\Omega)$, $X = L^2(\Gamma)^3$, $Z = L^2(\Lambda)$ and $V = \{v \in W^{1,2}(\Omega)^3 \colon\ v = o \text{ on } \Theta\}$ where o means the zero vector in $\mathcal{R}^3$, together with the notation of scalar products $(.\,,.)$ in H, $\langle .\,,.\rangle$ in X, $\langle .\,,.\rangle_*$ in Z and $(\!(.\,,.)\!)$ in $L^2(\Omega)^{3\times3}$; the symbol $[\,.\,]$ will be reserved for the jumps in normal components of values from V on Λ in the sense of traces, using some predefined orientation of unit normals $n = (n_1, n_2, n_3)$ to Λ, i.e. $[v] = v^+ - v^- \in Z$, v^+ and v^- understood as $v_1n_1 + v_2n_2 + v_3n_3$ from both sides of Λ for any $v \in V$. We shall also use the notation $\varepsilon(v) \in L^2(\Omega)^{3\times3}_{\mathrm{sym}}$ for the tensor of small strains, well-known in the linear theory of elasticity, assuming $\varepsilon_{ij}(v) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ for all $i, j \in \{1, 2, 3\}$; all upper dots will be applied as abbreviations of $\partial/\partial t$.

The weak formulation of the linear momentum conservation reads

$$(v, \rho(\ddot{u} + \alpha\dot{u})) + (\!(\varepsilon(v), \sigma)\!) = (v, f) + \langle v, g\rangle - \langle [v], \tau\rangle_* \tag{2.1}$$

for any virtual displacement $v \in V$, related to the reference configuration of Ω, Θ, Γ and Λ; here $\rho \in M$ denotes the material density, not lesser, everywhere on Ω, than some prescribed positive constant, and $\alpha \in M$ the always non-negative mass damping factor (forcing certain energy dissipation even for i), to respect the reality of a physically open system), $f \in L^2(\mathcal{I}, H)$ refers to the applied volume forces and $g \in C_L(\mathcal{I}, X)$ to the applied surface forces (where C_L denotes the Lipschitz continuity). Moreover (2.1) contains an unknown time-dependent actual displacement $u \in W^{2,2,2,2}(\mathcal{I}, V, V, V^*)$ (using V^* dual to V: for more details see [24], *Part* 7.1, namely $u, \dot{u} \in L^2(\mathcal{I}, V)$, $\ddot{u} \in L^2(\mathcal{I}, V^*)$, cf. the Gelfand triple $V \subset H \subset V^*$) and some still undefined stresses $\sigma \in L^2(\mathcal{I}, L^2(\Omega)^{3\times3}_{\mathrm{sym}})$ and interface tractions $\tau \in L^2(\mathcal{I}, Z)$, always normal to Λ, which must be evaluated from appropriate constitutive relations.

The linearized theory of elasticity, applicable to i), works with the empiric Hooke law $\sigma = C\varepsilon(u)$ with $C \in L^\infty(\Omega)^{(3\times3)\times(3\times3)}_{\mathrm{sym}}$ where C contains 21 different material characteristics in general (for any Boltzmann continuum the symmetry can be derived from the conservation of angular momentum), reducible up to the pair of the well-known Lamé factors (or to the Young modulus and the Poisson coefficient in most engineering applications) in the isotropic case. In general, it is natural to assume $a\cdot Ca \geq ca\cdot a$ everywhere on Ω for any $a \in \mathcal{R}^{3\times 3}_{\rm sym}$ and a positive constant c independent of $x \in \Omega$ and a; the central dot denotes the standard scalar product in $\mathcal{R}^{3\times 3}$ here. In addition to α, the structural damping factor $\beta \in M$, not lesser, everywhere on Ω, than some prescribed positive constant, is required in the Kelvin parallel viscoelastic model, assuming $\sigma = C\varepsilon(u+\beta\dot{u})$. For ii) we shall use the seemingly slight modification of this relation in the form $\sigma = (1-\mathfrak{D})C\varepsilon(u+\beta\dot{u})$ where the most delicate step is the calculation of the nonlocal factor of irreversible damage $\mathfrak{D}$ with values in M, always between 0 (for no damage) and some positive constant ς lesser than 1 (to avoid the total damage iv)), and depending on u in a rather complicated way, which will be sketched later. In particular, we are allowed to set $\mathfrak{D} = 0$ for $t = 0$, as required by (2.2) below, as well as for $t < 0$ formally in the difference schemes (3.1) and (3.2). For iii) the assessment of the traction - separation law, activated by reaching sufficient level of deformation energy on Λ locally, is needed; for all details see [16]. Here, for simplicity, we introduce only some continuous real function $\mathfrak{T}$ and insert $\tau = \mathfrak{T}([u])$ into (2.1). Thus we obtain

$$(2.2)\quad (v,\rho(\ddot{u}+\alpha\dot{u})) + (\!(\varepsilon(v),(1-\mathfrak{D})C\varepsilon(u+\beta\dot{u}))\!) = (v,f) + \langle v,g\rangle - \langle [v],\mathfrak{T}([u])\rangle_*$$

for any $v \in V$ again.

Such approach has been applied in [32] to the quasi-static version of (2.2) and in [33] to certain dynamical simplification of (2.2), relying on [13] in both cases; its limitation, excluding any realistic description of iv), has been discussed by [31] in details. However, this limitation might be not crucial for engineering applications because most practical numerical simulations are expected to detect the risk of material destruction in advance, whereas the detailed quantitative description of its subsequent disintegration is less interesting. Thus the more significant step in the upgrade of these formulations is some reasonable incorporation of different material behaviour under tension and compression, typical just for quasi-brittle cracking in cement-based composites. We shall adopt the access, coming from [22], modified by [11], referred as the "Mazars' model" in software packages, for simplicity; some significant later improvements will be mentioned in *Section* **??**. The final aim will be the evaluation of $\mathfrak{D}$ for (2.2).

As the 1st step, to preserve the objectivity of our analysis, let us evaluate (at certain fixed time step $t \in \mathcal{I}$ in practice, as will be evident from Section 3)the scalar principal values ε_i with $i \in \{1,2,3\}$ of $\varepsilon(u)$, i.e. the triple of eigenvalues from the condition $\det(\varepsilon(u) - \varepsilon_i I) = 0$, I being the unit matrix in $\mathcal{R}^3$. As the 2nd step, we can evaluate an equivalent strain $\bar{\varepsilon}$, using some bounded continuous functions ω of 6 real non-negative (at most 3 non-zero) arguments, as $\tilde{\varepsilon} = \omega(-\varepsilon_{1-},\varepsilon_{1+},-\varepsilon_{2-},\varepsilon_{2+},-\varepsilon_{3-},\varepsilon_{3+})$ where ε_{i+} and ε_{i-} refer to the positive and negative parts of ε_i (for each $x \in \Omega$ locally); for an example of such admissible function ω see [5]. As the 3rd step, the nonlocal form of $\tilde{\varepsilon}$ reads

$$(2.3)\qquad \bar{\varepsilon}(x,t) = \int_\Omega \mathcal{K}(x,\tilde{x})\,\tilde{\varepsilon}(\tilde{x},t)\,\mathrm{d}\tilde{x}\,, \qquad \int_\Omega \mathcal{K}(x,\tilde{x})\,\mathrm{d}\tilde{x} = 1$$

for any $x \in \Omega$, still at a fixed $t \in \mathcal{I}$, using some regularizing kernel $\mathcal{K} \in L^2(\Omega \times \Omega)$ in the sense of [9] (the choice $\mathcal{K}(x,\tilde{x}) = \delta(x-\tilde{x})$, δ being the Dirac measure, forcing $\bar{\varepsilon} = \tilde{\varepsilon}$, is not allowed intentionally), e. g. the Gaussian one, recommended by [11]; for its numerical approximation using various radial basis functions cf. [25]. As the 4th step, we have to evaluate the trial value $\mathfrak{D}_\star$, using some non-decreasing continuous functions ϖ of 1 real argument, returning values between 0 and ς, as $\mathfrak{D}_\star = \varpi(\bar{\varepsilon})$; for an example of such admissible function ϖ see [21]. The final 5th step, forcing the irreversibility of damage, can be then written in the form $\mathfrak{D}(.,t) = \max_{0 \leq \xi \leq t} \mathfrak{D}_\star(.,\xi)$.

The practical design of ω, $\mathcal{K}$ and ϖ is typically not easy, being conditioned, beyond the scope of this paper, by the careful design of material parameters, by the analysis of related sensitivity and inverse problems and by the extensive experimental research. Here we remind only the following result, derived (in 2 different ways) by [6], *Part* 2.2: an arbitrary operator introduced by (2.3) is compact as an operator from $L^2(\Omega)$ to itself. In particular, for each $t \in \mathcal{I}$ a sequence $\{\tilde{\varepsilon}^m(.,t)\}_{m=1}^\infty$ with a weak limit $\tilde{\varepsilon}(.,t)$ is converted to a sequence $\{\bar{\varepsilon}^m(.,t)\}_{m=1}^\infty$ with a strong limit $\bar{\varepsilon}(.,t)$. We shall see in Section 3 that such property for the approximation of $\mathfrak{D}$ by a sequence of simple functions on $\mathcal{I}$ will be required.

3. Existence and convergence analysis

To be able to evaluate the time development of u, we need to set the Cauchy initial conditions $u(.,0) = o$ (no displacements occur in the reference configuration) and $\dot{u}(.,0) = w$ where the initial displacement rates $w \in V$ must be prescribed. Then the existence of solution of (2.2) can be verified in the constructive way, using the limit passage for a positive integer $m \to \infty$ from

$$\begin{aligned} &(v,\rho\tilde{u}^m) + (v,\alpha\rho\dot{u}^m) + (\!(\varepsilon(v),(1-\bar{\mathfrak{D}}^m_\times)C\varepsilon(\bar{u}^m))\!) \\ &+ (\!(\varepsilon(v),(1-\bar{\mathfrak{D}}^m_\times)\beta C\varepsilon(\dot{u}^m))\!) = (v,f^m) + \langle v,g^m\rangle - \langle [v],\mathfrak{T}([u^m_\times])\rangle_* \end{aligned} \tag{3.1}$$

on $\mathcal{I}$ where $h = T/m$ for brevity. The following approximations are used in (3.1): $u^m(t) = u^m_{s-1} + (u^m_s - u^m_{s-1})(t-(s-1)h)/h$ (linear Lagrange splines) and $\bar{u}^m(t) = u^m_s$, assuming $(s-1)h < t \leq sh$ (simple functions) with $s \in \{1,\dots,m\}$, whereas $u^m_\times(t) = u^m(t-h)$ and $\bar{u}^m_\times(t) = \bar{u}^m(t-h)$ (retarded versions of preceding types of functions), using $u^m_0 = o$ and $u^m_{-1} = -hw$; $\bar{\mathfrak{D}}^m_\times$ must be considered as $\mathfrak{D}$ evaluated for $\bar{u}^m_\times$. Such functions generate 4 different types of Rothe sequences $\{u^m\}_{m=1}^\infty$, $\{\bar{u}^m\}_{m=1}^\infty$, $\{u^m_\times\}_{m=1}^\infty$ and $\{\bar{u}^m_\times\}_{m=1}^\infty$; we shall need $\{\tilde{u}^m\}_{m=1}^\infty$ defined by $\tilde{u}^m = (\dot{u}^m - \dot{u}^m_\times)/h$, too. Using the values u^m_{-1}, u^m_0 and u^m_s with $s \in \{1,\dots,m\}$ for a fixed m, we can omit all upper indices m for brevity and rewrite (3.1) to its more transparent form for step-by-step evaluations of u_s from formally linear elliptic equations (whose solvability can be verified applying the Lax-Milgram theorem) at all times $t = sh$

$$\begin{aligned} &(v,\rho\mathcal{D}^2u_s) + (v,\alpha\rho\mathcal{D}u_s) + (\!(\varepsilon(v),(1-\mathfrak{D}_{s-1})C\varepsilon(u_s))\!) \\ &+ (\!(\varepsilon(v),(1-\mathfrak{D}_{s-1})C\varepsilon(\mathcal{D}u_s))\!) = (v,f_s) + \langle v,g_s\rangle - \langle [v],\mathfrak{T}([u_{s-1}])\rangle_*\,, \end{aligned} \tag{3.2}$$

taking f_s and g_s as the mean values of $f(.,t)$, $g(.,t)$ over t between $(s-1)h$ and sh (which is the Clément quasi-interpolation by [24], *Part* 8.2) and using the obvious notation of the 1st and 2nd relative differences $\mathcal{D}u_s = (u_s - u_{s-1})/h$ and $\mathcal{D}^2 u_s = (\mathcal{D}u_s - \mathcal{D}u_{s-1})/h$.

In particular, for $v = 2h\mathcal{D}u_s$ in (3.2) we can derive some useful a priori estimates. Namely, using $s \in \{1,\dots,r\}$ as the Einstein summation index for an arbitrary fixed $r \in \{1,\dots,m\}$, for the left-hand side of (3.2) we have

$$\begin{aligned}
(3.3)\quad 2h(\mathcal{D}u_s, \rho\mathcal{D}^2 u_s) &= (u_r - u_{r-1}, \rho(u_r - u_{r-1}))/h^2 - (w, \rho w) \\
&\quad + (u_s - 2u_{s-1} + u_{s-2}, \rho(u_s - 2u_{s-1} + u_{s-2}))/h^2\,, \\
2h(\mathcal{D}u_s, \alpha\rho\mathcal{D}u_s) &= 2(u_s - u_{s-1}, \alpha\rho(u_s - u_{s-1}))/h\,, \\
2h(\!(\varepsilon(\mathcal{D}u_s), (1-\mathfrak{D}_{s-1})\beta C\varepsilon(\mathcal{D}u_s))\!) &= (\!(\varepsilon(u_s - u_{s-1}), (1-\mathfrak{D}_{s-1})\beta C\varepsilon(u_s - u_{s-1}))\!)/h\,, \\
2h(\!(\varepsilon(\mathcal{D}u_s), (1-\mathfrak{D}_{s-1})C\varepsilon(u_s))\!) &= (\!(\varepsilon(u_r), (1-\mathfrak{D}_{r-1})C\varepsilon(u_r))\!) \\
+ (\!(\varepsilon(u_s - u_{s-1})(1-\mathfrak{D}_{s-1})C\varepsilon(u_s - u_{s-1}))\!) &- (\!(\varepsilon(u_{s-1})(\mathfrak{D}_{s-1}-\mathfrak{D}_{s-2})C\varepsilon(u_{s-1}))\!)\,.
\end{aligned}$$

To avoid very long formulae, for the right-hand side of (3.2) it is sufficient, applying the Cauchy–Schwarz and Young inequalities with any positive ϵ, using the standard norms in H and V, to present the estimates

$$\begin{aligned}
(3.4)\ 2h(\mathcal{D}u_s, f_s) &\le \frac{\epsilon}{h}\sum_{s=1}^{r}\|u_s - u_{s-1}\|_H^2 + \frac{h}{\epsilon}\sum_{s=1}^{r}\|f_s\|_H^2\,, \\
2h\langle\mathcal{D}u_s, g_s\rangle &= 2\langle u_r, g_r\rangle - 2\langle u_{s-1}, g_s - g_{s-1}\rangle \\
&\le \gamma\epsilon\|u_r\|_V^2 + \frac{1}{\epsilon}\|g_r\|_X^2 + \gamma\epsilon h\sum_{s=1}^{r}\|u_{s-1}\|_V^2 + \frac{1}{\epsilon h}\sum_{s=1}^{r}\|g_s - g_{s-1}\|_X^2\,, \\
-2h\langle[\mathcal{D}u_s], \mathfrak{T}([u_{s-1}])\rangle_* &\le 4\gamma\epsilon h\sum_{s=1}^{r}\|u_s\|_V^2 + \frac{4\gamma}{\epsilon h}\|u_s - u_{s-1}\|_V^2
\end{aligned}$$

where the positive constant γ stems from the trace theorem (duplicated on Λ, unlike Γ). Using the upper estimates (3.4) and the similar lower ones generated by (3.3) (all details must be left to the curious reader), the discrete Gronwall lemma, applied to (3.2), gives

$$(3.5)\qquad \|\mathcal{D}u_r\|_H^2 + h^2\sum_{s=1}^{r}\|\mathcal{D}^2 u_s\|_H^2 + h\sum_{s=1}^{r}\|\mathcal{D}u_s\|_V^2 + \|u_r\|_V^2 \le \mathcal{C}$$

where $\mathcal{C}$ is some fixed positive constant (sufficiently large, depending on T).

In terms of (3.1), the a priori estimate (3.5) can be interpreted as follows: $h\{\tilde{u}^m\}_{m=1}^\infty$ is bounded in $L^2(I,H)$ and $\{\dot{u}^m\}_{m=1}^\infty$ is bounded in $L^2(I,V)$, whereas $\{\bar{u}^m\}_{m=1}^\infty$ and $\{\bar{u}_\times^m\}_{m=1}^\infty$ are bounded in $L^\infty(I,V)$. The Eberlein - Shmul'yan theorem then guarantees that, up to subsequences, $\{\dot{u}^m\}_{m=1}^\infty$ has a weak limit $u' \in L^2(I,V)$, whereas $\{\bar{u}^m\}_{m=1}^\infty$ and $\{\bar{u}_\times^m\}_{m=1}^\infty$ have their weak limits $\bar{u}, \bar{u}_\times \in L^\infty(I,V)$. Finally, using these results together with (3.1), we can see that $\{\tilde{u}^m\}_{m=1}^\infty$ has a weak limit $u'' \in L^2(I,V^*)$. For any $t \in \mathcal{I}$ let us now define

$$(3.6)\, u(.,t) = \int_0^t u'(.,\xi)\,\mathrm{d}\xi\,,\quad \hat{u}(.,t) = w + \int_0^t u''(.,\xi)\,\mathrm{d}\xi\,,\quad \hat{u}^m(.,t) = w + \int_0^t \tilde{u}^m(.,\xi)\,\mathrm{d}\xi\,,$$

valid for any positive integer m in the last case. Since the Aubin - Lions lemma yields also the strong convergence of $\{\bar{u}^m\}_{m=1}^\infty$ to $\bar{u}$ and of $\{\bar{u}_\times^m\}_{m=1}^\infty$ to $\bar{u}_\times$ in $L^2(I,H)$ (in both cases), it is not difficult to identify u with both $\bar{u}$ and $\bar{u}_\times$ and $\dot{u}$ with both u' and $\hat{u}$: namely the square of the norm of $u^m - \bar{u}^m$ in $L^2(I,H)$ can be estimated, using (3.5), from above as the sum of all $\|u_s - u_{s-1}\|_H^2 = h^2\|\mathcal{D}u_s\|_H^2 \le \mathcal{C}h^2$ over $s \in \{1,\dots,m\}$, lesser or equal to $\mathcal{C}Th$, vanishing with $h \to 0$, thus $u = \bar{u}$, etc. Even the (seemingly strange) 2nd left-hand-side additive term of (3.5) is required here for the identification of $\hat{u}$ with $\dot{u}$: considering $\{\hat{u}^m\}_{m=1}^{\infty}$ by the last relation of (3.6), we obtain $\hat{u}^m(t) - \dot{u}^m(t) = (t - sh)\mathcal{D}^2 u_s$, thus the square of the norm of $\hat{u}^m - \dot{u}^m$ in $L^2(I,H)$ is just the sum of all $\|(t-sh)\mathcal{D}^2 u_s\|_H^2$, whose upper bound is $\mathcal{C}h/3$, tending to zero with $h \to 0$. This enables us to perform the limit passage from (3.1) to (2.2) where $u, \dot{u} \in L^2(\mathcal{I}, V)$ and $\ddot{u} \in L^2(\mathcal{I}, V^*)$, which can be expressed as $u \in W^{2,2,2,2}(\mathcal{I}, V, V, V^*)$.

4. Conclusions and generalizations

The formulation and analysis of a model problem, supplied by the main ideas of proofs, in this paper was intended as the demonstration of numerical considerations for a class of initial and boundary value problems for partial differential equations of evolution, connected with extraordinarily significant tasks of modelling and simulation of behaviour of advanced materials, building end structures. Some preliminary computational results, referring namely to [32] and [33], for i), ii) and iii) have been presented (with numerous illustrative figures) and discussed in [34]. The much more detailed analysis is under development, including e. g. such processes as dynamics of multiple contacts / impacts of deformable bodies with potential micro- and macro-cracking by [26], requiring an explicit time integration scheme like [4], as well as an appropriate parallel / distributed computing platform.

The approach of [26] demonstrates also the need of incorporation of a complete set of conservation laws by the 1st principle of classical thermodynamics in formulations covering several physical processes (known as “multiphysics” in the unofficial language of scientific computing), together with the compatibility of constitutive relations with the 2nd and 3rd ones, respecting the finite (not only linearized small) strains and related stresses. Numerous inspirations can be found in the reviews of crack branching [28] and of XFEM-based simulations [17], opening the possibility of computational modelling of physically realistic development of all active interfaces Λ. Since only the linear elastic and viscous components have been combined in our model problem, one natural generalization could be the proper analysis of plastic zones, observed namely on crack tips in practice, combining the intuitive engineering approach of [12] with the deep mathematical analysis by [19]. Also the thermodynamical study of initiation and propagation of anisotropic damage, introduced by [14], revised by [35] and [15], taking $\mathfrak{D}$ as matrix characteristics, should belong to the research priorities for the next years.

References

[1] Altan, S., *Existence in nonlocal elasticity*, Arch. Mech. **47** (1989), 25–36.

[2] Bažant, Z.P., *Why continuum damage is nonlocal: micromechanics arguments*, J. Eng. Mech. **117** (1991), 1070–1089.

[3] Bermúdez de Castro, A., *Continuum Thermomechanics*, Birkhäuser, Basel, 2005.

[4] Bybordiani, M., Dias da Costa, D., *A consistent finite element approach for dynamic crack propagation with explicit time integration*, Comput. Methods Appl. Mech. Eng. **376** (2021), 1–32, 113652.

[5] de Vree, J.H.P., Brekelmans, W.A.M., van Gils, M.A.J., *Comparison of nonlocal approaches in continuum damage mechanics*, Comput. Struct. **55** (1995), 581–588.

[6] Drábek, P., Milota, I., *Methods of Nonlinear Analysis*, Birkhäuser, Basel, 2013.

[7] Eringen, A.C., *Theory of Nonlocal Elasticity and Some Applications*, Tech. report, Princeton University, Princeton, 1984.

[8] Evgrafov, A., Bellido, J.-C., *From nonlocal Eringen's model to fractional elasticity*, Math. Mech. Solids **24** (2019), 1935–1953.

[9] Fasshauer, G.E., Ye, Q., *Reproducing kernels of generalized Sobolev spaces via a Green function approach with distributional operators*, Numer. Math. **119** (2011), 585–611.

[10] Fries, T.P., Belytschko, T., *The intrinsic XFEM: a method for arbitrary discontinuities without additional unknowns*, Int. J. Numer. Methods Eng. **68** (2006), 1358–1385.

[11] Giry, C., Dufour, F., Mazars, J., *Stress-based nonlocal damage model*, Int. J. Solids Struct. **48** (2011), 3431–3443.

[12] Hashiguchi, K., *Elastoplasticity Theory*, Springer Berlin, 2014.

[13] Havlásek, P., Grassl, P., Jirásek, M., *Analysis of size effect on strength of quasi-brittle materials using integral-type nonlocal models*, Eng. Fract. Mech. **157** (2016), 72–85.

[14] Ju, J.W., *Isotropic and anisotropic damage variables in continuum damage*, J. Eng. Mech. **116** (1990), 2764–2770.

[15] Kamińska, I., Szwed, A., *A thermodynamically consistent model of quasibrittle elastic damaged materials based on a novel Helmholtz potential and dissipation function*, MDPI Materials **14** (2021), 1–30, 6323.

[16] Kozák, V., Chlup, Z., Padělek, P., Dlouhá, I., *Prediction of the traction separation law of ceramics using iterative finite element modelling*, Solid State Phenomena **258** (2017), 186–189.

[17] Li, H., Li, J., Yuan, H., *A review of the extended finite element method on macrocrack and microcrack growth simulations*, Theor. Appl. Fract. Mech. **97** (2018), 236–249.

[18] Mariani, S., Perego, U., *Extended finite element method for quasi-brittle fracture*, Int. J. Numer. Meth. Engn. **58** (2003), 103–126.

[19] Mielke, A., Roubíček, T., *Rate-Independent Systems*, Springer, New York, 2015.

[20] Mousavi, S.M., *Dislocation-based fracture mechanics within nonlocal and gradient elasticity of bi-Helmholtz type*, Int. J. Solids Struct. **87** (2016), 92–93, 105–120.

[21] Peerlings, R.H.J., R.de Borst, Brekelmans, W.A.M., Geers, M., *Gradient enhanced damage modelling of concrete fracture*, Int. J. Numer. Anal. Methods Geomech. **3** (1998), 323–342.

[22] Pijaudier-Cabot, G., Mazars, J., *Damage models for concrete*, Handbook of Materials Behavior Models (Lemaitre, J., ed.), Academic Press, Cambridge (Massachusetts, USA), 2001, pp. 500–512.

[23] Pike, M.G., Oskay, C., *XFEM modeling of short microfiber reinforced composites with cohesive interfaces*, Finite Elem. Anal. Des. **106** (2015), 16–31.

[24] Roubíček, T., *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Basel, 2005.

[25] Skala, V., *A practical use of radial basis functions interpolation and approximation*, Investigación Operacional **37** (2016), 137–144.

[26] Štekbauer, H., Němec, I., Lang, R., Burkart, D., ValaSte22, J., *On a new computational algorithm for impacts of elastic bodies*, Appl. Math. **67** (2022), 28 pp., in print.

[27] Sumi, Y., *Mathematical and Computational Analyses of Cracking Formation*, Springer, Tokyo, 2014.

[28] Sun, Y., Edwards, M.G., Chen, B., Li, C., *A state-of-the-art review of crack branching*, Eng. Fract. Mech. **257** (2021), 1–33, 108036.

[29] Szabó, B., Babuška, I., *Finite Element Analysis: Method, Verification and Validation*, J. Wiley & Sons, Hoboken, 2021.

[30] Turner, M.J., Clough, R.W., Martin, H.C., Top, L.J., *Stiffness and deflection analysis of complex structures*, Journal of the Aeronautical Sciences **23** (1956), 805–823.

[31] Vala, J., *On a computational smeared damage approach to the analysis of strength of quasi-brittle materials*, WSEAS Trans. Appl. Theor. Mech. **16** (2021), 283–292.

[32] Vala, J., Kozák, V., *Computational analysis of quasi-brittle fracture in fibre reinforced cementitious composites*, Theor. Appl. Fract. Mech.. **107** (2020), 1–8, 102486.

[33] Vala, J., Kozák, V., *Nonlocal damage modelling of quasi-brittle composites*, Appl. Math. **66** (2021), 701–721.

[34] Vala, J., Kozák, V., Jedlička, M., *Scale bridging in computational modelling of quasi-brittle fracture of cementitious composites*, Solid State Phenomena **325** (2021), 56–64.

[35] Vilppo, J., Kouhia, R., Hartikainen, J., Kolari, K., Fedoroff, A., Calonius, K., *Anisotropic damage model for concrete and other quasi-brittle materials*, Int. J. Solids Struct. **225** (2021), 1–13, 111048.

[36] Zlámal, M., *On the finite element method*, Numer. Math. **12** (1968), 394–409.

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