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ON EULER METHODS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Numerical methods for fractional differential equations have specific properties with respect to the ones for ordinary differential equations. The paper discusses Euler methods for Caputo differential equation initial value problem. The common properties of the methods are stated and demonstrated by several numerical experiments. Python codes are available to researchers for numerical simulations.

1. Introduction

Fractional differential equations have become an area of mathematics being widely investigated in recent years. Several approaches to fractional derivative were formulated, e.g. Grünwald–Letnikov, Riemann–Liouville, Atangana–Baleanu, Riesz, Caputo and others. The Caputo approach has become popular in connection with formulation of fractional differential equations initial value problems. It was due to the form of initial conditions where integer order derivatives are employed. Thence physical meaning of such conditions is more clear and understandable. For basics of fractional calculus and its applications we refer to [1], [7]. The numerical analysis of these tasks has been developed following the fractional calculus advancement. A survey of some numerical approaches can be found in monograph [6] and the references therein. In the paper we introduce the basic Euler numerical schemes to the Caputo type differential equations. We restrict our discussions to a scalar problem with fractional order derivative $\alpha \in (0,1)$ but it is possible to generalize the following considerations to $\alpha > 1$ and also to a vector counterpart of the problem. The aim of the paper is to introduce the methods, to mention their properties and mainly demonstrate them by numerical experiments. Moreover, we present the Python codes of the methods to be utilized by researchers.

We consider the initial value problem

$$\begin{cases} D^\alpha y(t) = f\big(t, y(t)\big)\,, \\ y(0) = y_0\,, \end{cases} \tag{1.1}$$

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where $\alpha \in (0,1)$ and $t \in [0, t_f]$. Caputo differential operator is introduced as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s\,, \qquad \alpha \in (0,1)\,.$$

First of all we mention conditions, which locally ensure existence and uniqueness of a solution of initial value problem (1.1) (see [6, p. 97]).

Theorem 1.1. *Let $M := [0, \chi^\star] \times [y_0 - \delta, y_0 + \delta]$ with $\chi^\star > 0$ and $\delta > 0$. Let $f\colon M \to \mathbb{R}$ be a bounded function on M satisfying the Lipschitz condition with respect to the second variable $|f(t,u) - f(t,v)| \le L|u - v|$ with $L > 0$ independent of t, u, v. Let $\chi = \min\{\chi^\star, (\delta\Gamma(1+\alpha)/|f|)^{1/\alpha}\}$. Then there exists a unique solution $y : [0,\chi] \to \mathbb{R}$ of (1.1).*

We often need to employ numerical methods to obtain approximation of the solution since an analytical form of the solution can be found in a very narrow class of initial value problems.

2. Numerical methods

There are two common approaches to obtain a numerical formula for problem (1.1): direct discretization of the Caputo derivative and transformation of a fractional differential equation to a fractional integral equation with a subsequent discretization of the fractional integral. The transformation is based on application of α Caputo integral to (1.1), which gives

$$y(t) = y_0 + D^{-\alpha} f\big(t, y(t)\big) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, y(s)\big)\,\mathrm{d}s\,. \tag{2.1}$$

We use the second approach. In the following, we consider an equidistant mesh with stepsize $h > 0$ ($h = t_f/N$), i.e. $t_0 = 0$, $t_n = nh, n = 0,1,2,\dots,N$. Then y_n denotes a numerical approximation of the exact solution value $y(t_n)$.

2.1. **Fractional forward Euler method.** The fractional forward Euler formula (FFEM) solving (1.1) is introduced as follows:

$$y_{n+1} = y_0 + h^{\alpha} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j)\,, \quad n = 0,1,\dots,N-1\,,$$

where

$$b_{j,n+1} = \frac{(n-j+1)^{\alpha} - (n-j)^{\alpha}}{\Gamma(1+\alpha)}\,, \quad n = 0,1,\dots,N-1\,,\ \ j = 0,1,\dots,n\,. \tag{2.2}$$

2.2. **Fractional backward Euler method.** The fractional backward Euler formula (FBEM) solving (1.1) is introduced as follows:

$$y_{n+1} = y_0 + h^{\alpha} \sum_{j=0}^{n} b_{j,n+1} f(t_{j+1}, y_{j+1}), \quad n = 0,1,\dots,N-1,$$

where $b_{j,n+1}$ is given by (2.2).

As we can see, the integral (2.1) is approximated by the left-point and right-point rule in the case of FFEM and FBEM, respectively. In the following we mention some crucial properties of these methods.

2.3. **Properties of FFEM and FBEM.** First we introduce an assertion which classifies FFEM and FBEM as first order methods (see [6, p. 103]).

Theorem 2.1. *Let $y(t)$ be a solution of* (1.1)*, $f(t,y)$ satisfy the Lipschitz condition with respect to the second argument and $f(t,y(t))$, $y(t) \in C^1[0,t_f]$. Let $y_n, n = 0,1,2,\ldots,N$ be the approximations of $y(t_n)$ by FFEM (resp. FBEM). Then*

$$|y(t_n) - y_n)| \leq Kh, \qquad n = 0,1,2,\ldots,N,$$

where $K > 0$ is a constant independent of h and n.

Both the methods are convergent and stable considering the assumptions of Theorem 2.1 are fulfilled. The stability means that small perturbations in the initial conditions would not lead to large errors in the numerical solution (see [6, p. 102]).

3. Numerical experiments

Example 3.1. Consider the following initial value problem

$$\begin{cases} D^\alpha y(t) = -y(t)\,, & t \in [0,10] \\ y(0) = 1\,, \end{cases} \tag{3.1}$$

which may serve as a test initial value problem for A-stability. Its analytical solution is well known: $y(t) = E_\alpha(-t^\alpha)$, where $E_\alpha(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is the Mittag-Leffler function. The numerical solution obtained by both the Euler's schemes are shown in Figures 1-5 with fractional derivative order α equal to 0.05, 0.25, 0.5, 0.75, 0.95, in sequence. For these values of α there are shown solutions considered on meshes with various stepsizes h (0.01, 0.5, 1 and 2). We can observe that, in analogy with Euler's methods for ordinary differential equations, the backward fractional Euler method has much better asymptotic stability behavior then the forward one. All the presented parameter settings lead to a positive decreasing solution in the case of FBEM. On the contrary, in the case of FFEM, larger stepsizes cause a more prominent oscillation of the numerical solution.

Example 3.2. Consider the following initial value problem

$$\begin{cases} D^{1/3} y(t) = \frac{9t^{5/3}}{10\Gamma(2/3)}, & t \in [0,10]. \\ y(0) = 1. \end{cases} \tag{3.2}$$

The exact solution of this problem is $y(t) = t^2/2 + 1$. The numerical solution for various stepsizes h (0.01, 0.5, 1 and 2) is shown in Figure 6. Propagation of absolute error by FFEM and FBEM is shown in Table 1 and 2, respectively. The computed absolute errors coincide with the estimate introduced in Theorem 2.1.

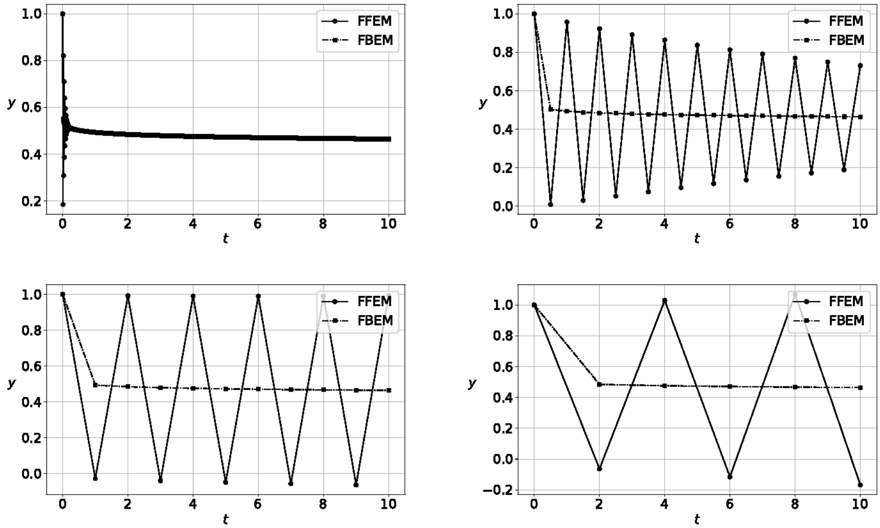

FIG. 1: Numerical solution of (3.1) with $\alpha = 0.05$ and $h = 0.01, 0.5, 1, 2$.

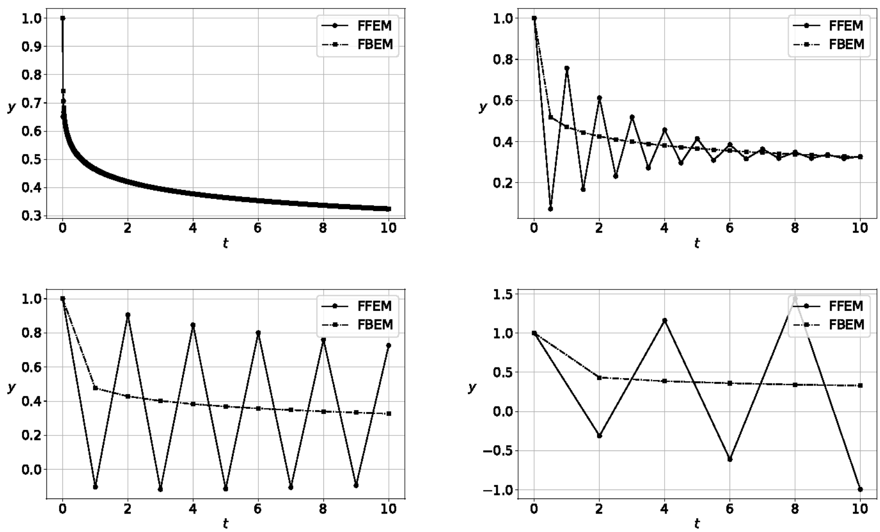

FIG. 2: Numerical solution of (3.1) with $\alpha = 0.25$ and $h = 0.01, 0.5, 1, 2$.

Remark 3.3. One can easily check, that $y(t) = t^2/2 + 1$ is also the solution of the problem

$$\begin{cases} D^{1/3}y(t) = \frac{9t(2y(t)-2)^{2/3}}{10\Gamma(2/3)}, & t \in [0, 10]. \\ y(0) = 1. \end{cases} \tag{3.3}$$

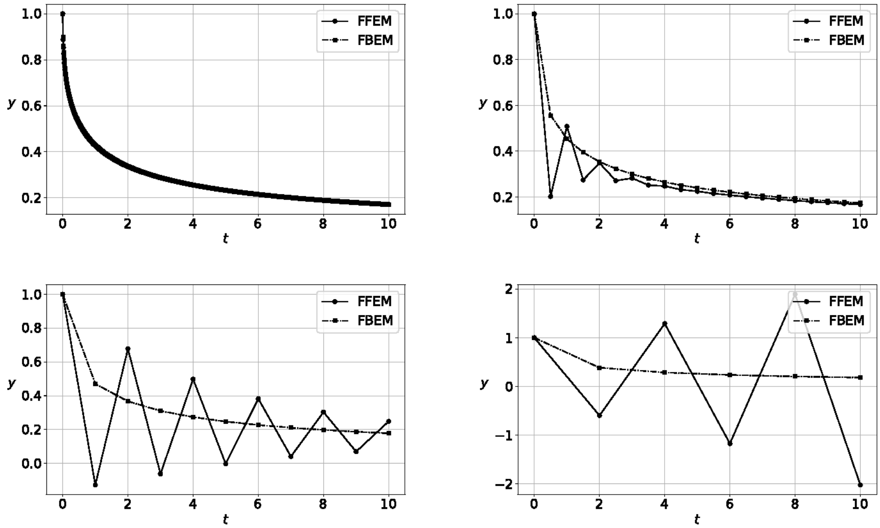

FIG. 3: Numerical solution of (3.1) with $\alpha = 0.5$ and $h = 0.01, 0.5, 1, 2$.

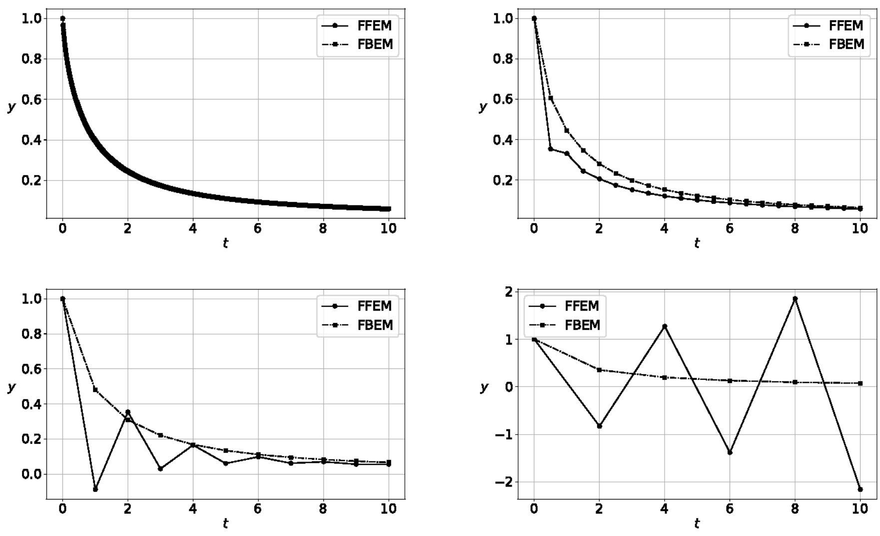

FIG. 4: Numerical solution of (3.1) with $\alpha = 0.75$ and $h = 0.01, 0.5, 1, 2$.

Nevertheless, both the fractional Euler's methods give the constant numerical solution $y_n = 1$, $n = 0, 1, 2, \ldots$, which corresponds to the trivial constant solution

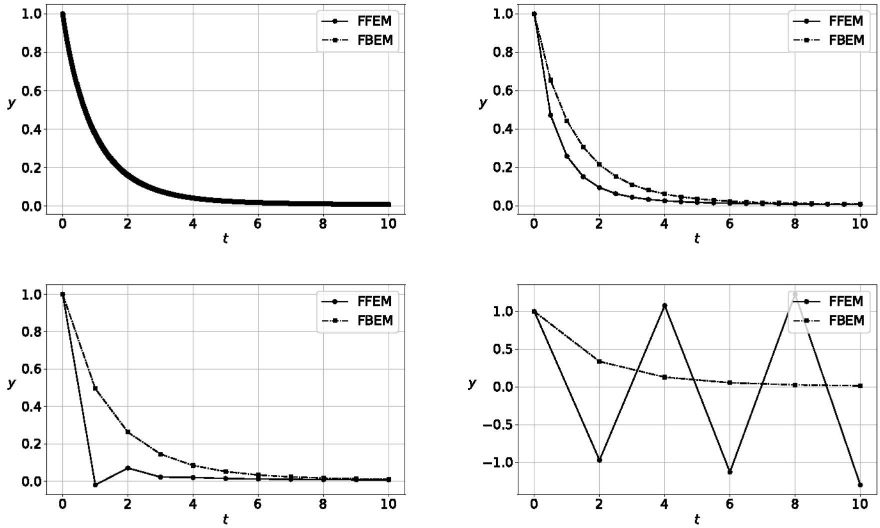

FIG. 5: Numerical solution of (3.1) with $\alpha = 0.95$ and $h = 0.01, 0.5, 1, 2$.

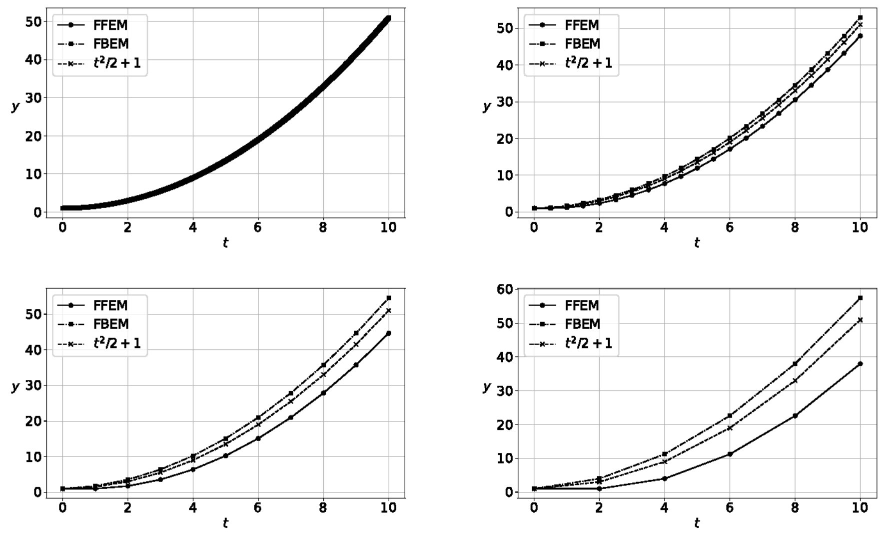

FIG. 6: Numerical solution of (3.2) for $h = 0.01, 0.5, 1, 2$.

$y(t) = 1$ of (3.3). The uniqueness of solution is not ensured for this initial value problem.

t	$h = 0.1$	$h = 0.01$	$h = 0.001$
0	0.0000e-00	0.0000e-00	0.0000e-00
1	6.3616e-02	5.7214e-03	5.3422e-04
2	1.2314e-01	1.1157e-02	1.0544e-03
3	1.8108e-01	1.6523e-02	1.5714e-03
4	2.3814e-01	2.1849e-02	2.0865e-03
5	2.9463e-01	2.7149e-02	2.6004e-03
6	3.5069e-01	3.2429e-02	3.1134e-03
7	4.0641e-01	3.7694e-02	3.6257e-03
8	4.6187e-01	4.2947e-02	4.1374e-03
9	5.1710e-01	4.8189e-02	4.6487e-03
10	5.7214e-01	5.3422e-02	5.1595e-03

TAB. 1: Absolute errors of FFEM for (3.2).

t	$h = 0.1$	$h = 0.01$	$h = 0.001$
0	0.0000e-00	0.0000e-00	0.0000e-00
1	3.5315e-02	4.2737e-03	4.6575e-04
2	7.6014e-02	8.8387e-03	9.4555e-04
3	1.1818e-01	1.3474e-02	1.4286e-03
4	1.6119e-01	1.8148e-02	1.9135e-03
5	2.0475e-01	2.2848e-02	2.3996e-03
6	2.4873e-01	2.7568e-02	2.8866e-03
7	2.9303e-01	3.2303e-02	3.3743e-03
8	3.3760e-01	3.7051e-02	3.8625e-03
9	3.8239e-01	4.1809e-02	4.3513e-03
10	4.2737e-01	4.6575e-02	4.8405e-03

TAB. 2: Absolute errors of FBEM for (3.2).

4. Concluding remarks

In the previous sections we have introduced forward and backward fractional Euler methods. We have compared numerical solutions obtained by these methods under various parameter settings. The computational costs significantly increase during the process since the computation of value y_{n+1} depends on approximation of the definite integral over $[0, t_n]$ in the case of FFEM and $[0, t_{n+1}]$ in the case of FBEM. The whole history of the solution must be taken into account. This phenomenon is common for all reasonable numerical schemes due to the Caputo fractional derivative nature. On this account there were considered parallel algorithms for efficient employment of available processors to deal with the vast number of computations to get the solution in reasonable time, see, e.g. [2], [8].

In the literature, we can also find some simplifying numerical approaches which neglect a part of the process' history. However, such techniques appeared to be unreliable in general, see [5] and references therein.

Python codes by the author related to the above discussed problems are available at: *https://math.fme.vutbr.cz/Home/kundrat/software*. With respect to the above formulated problems it is enough to modify the input part of the code to get numerical solution of own initial value problems.

As another valuable source of numerical programs, the paper [4] may serve. It introduces various numerical schemes and offers link to their Matlab codes. The paper also serves as a manual to these programs.

Finally, in connection with numerical methods for fractional differential equations, we mention recent paper [3] where open problems in fractional calculus are stated. Particularly error analysis of numerical methods for fractional differential equations is found unsatisfactory and it offers many research challenges.

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