ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 189–197

BARTZ-MARLEWSKI EQUATION WITH GENERALIZED LUCAS COMPONENTS

Hayder R. Hashim

Abstract. Let $\{U_n\} = \{U_n(P,Q)\}$ and $\{V_n\} = \{V_n(P,Q)\}$ be the Lucas sequences of the first and second kind respectively at the parameters $P \geq 1$ and $Q \in \{-1, 1\}$. In this paper, we provide a technique for characterizing the solutions of the so-called Bartz-Marlewski equation

$$x^2 - 3xy + y^2 + x = 0\,,$$

where $(x, y) = (U_i, U_j)$ or (V_i, V_j) with $i,\ j \geq 1$. Then, the procedure of this technique is applied to completely resolve this equation with certain values of such parameters.

1. Introduction

The Lucas sequences $\{U_n\}$ and $\{V_n\}$ are defined by the following recurrences:

$$U_0 = 0,\ U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2}\,, \tag{1}$$

$$V_0 = 2,\ V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2}\,, \tag{2}$$

where $n \geq 2$ and $\gcd(P, Q) = 1$. In fact, these sequences are simply called the Lucas sequences, and their numbers are called the generalized Lucas numbers. With certain values of P and Q, these sequences describe well known binary recurrence sequences, e.g. $V_n(1,-1) = L_n$, $U_n(1,-1) = F_n$, $U_n(2,-1) = P_n$, $V_n(2,-1) = Q_n$, $U_n(1,-2) = J_n$, $V_n(1,-2) = \bar{J}_n$, $U_n(6,1) = B_n$ and $V_n(6,1) = \bar{B}_n$ that respectively represent the nth terms of the sequences of Lucas, Fibonacci, Pell, Pell-Lucas, Jacobsthal, Jacobthal-Lucas, Balacing and Balacing-Lucas numbers. One of the interesting results related to these sequences is presented by the identity

$$V_n^2 = DU_n^2 + 4Q^n\,,$$

where $D = P^2 - 4Q$ denotes the discriminant of the sequences. For more details related to these sequences and their properties, one can see e.g. [5] and [8].

In fact, nowadays many authors have investigated the solutions of some Diophantine equations (that have infinity many solutions over rational integers) in the

2020 *Mathematics Subject Classification*: primary 11D45; secondary 11B39.
Key words and phrases: Lucas sequences, Diophantine equation.
Received May 26, 2022, revised August 2022. Editor C. Greither.
DOI: 10.5817/AM2022-3-189

numbers of some types of Lucas sequences. For instance, Luca and Srinivasan [6] completely solved the so-called Markoff equation

$$x^2 + y^2 + z^2 = 3xyz\,, \tag{3}$$

where $(x, y, z) = (F_i, F_j, F_k)$ under the condition that $i, j, k \geq 2$ and $i \leq j \leq k$. Furthermore, Togbé, Kafle and Srinivasan [4] obtained the triples (P_i, P_j, P_k) that satisfy equation (3) under the same conditions. Recently with Tengely [2], we investigated the positive solutions with Fibonacci terms for the Jin-Schmidt equation, which is a generalization of equation (3). Moreover, with Szalay, Tengely [3] we studied the solutions $(x, y, z) = (U_i, U_j, U_k)$ or (V_i, V_j, V_k) with $i, j, k \geq 1$ of another generalization of Markoff equation that is called the Markoff-Rosenberger equation. One can find such results in relation to the Markoff equation and its generalizations in [9]. In this paper, we deal with an other Diophantine equation that has the form

$$x^2 - kxy + y^2 + x = 0 \quad \text{for some} \quad k \in \mathbb{N}\,,$$

which was originally noticed in 1999 by Marlewski and his student Bartz [1] while working on solutions of some problems during the LI Mathematical Olympiad. Indeed, with some computer experiments, they observed that the latter equation has infinitely many solutions in the positive integers x and y with $k = 3$. This observation remained as a conjecture until 2004 in which Marlewski and Zarzycki [7] used the Pell equation theory with some Computer Algebra System called (DERIVE 5) to prove the validity of this observation and conjecture. Respecting to the original founders of the observation, here we call the following equation by the Bartz-Marlewski equation:

$$x^2 - 3xy + y^2 + x = 0\,. \tag{4}$$

Therefore, in this paper we mainly give a technique for investigating the solutions $(x, y) = (U_i, U_j)$ or (V_i, V_j) with $i, j \geq 1$ of the Bartz-Marlewski equation (4) where $P \geq 1$ and $Q = \pm 1$. For the simplicity of presenting our results, we assume that $D > 1$. Our argument is mainly based on bounding the values of the indices i and j. Indeed, as applications we show that if $(x, y) = (V_i, V_j)$ is a solution of equation (4) then a nonzero upper bounds for i and j can be effectively determined only with $1 \leq P \leq 20; Q = -1$ and $1 \leq P \leq 21; Q = 1$. Furthermore, with these ranges of parameters we investigated the set of solutions of equation (4). On the other hand, if the pair $(x, y) = (U_i, U_j)$ satisfies equation (4) then we show that $i \leq 2$ with $P \geq 22; Q = -1$ and $P \geq 21; Q = 1$. Here, we also determine the complete set of such solutions.

2. Auxiliary results

Here, we present some results concerning the Lucas sequences that we use later in the proofs of our main results. In fact, the characteristics polynomial of any of these sequences is defined by

$$X^2 - PX + Q = 0\,,$$

whose roots are

$$\alpha = \frac{P+\sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P-\sqrt{D}}{2},$$

which clearly lead to the following facts: $\alpha = \sqrt{D} + \beta$ and $\alpha = P - \beta$. Hence, the general terms of the Lucas sequences can be written in the following forms which are called by Binets formulas:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for} \quad n \geq 0, \tag{5}$$

and

$$V_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0. \tag{6}$$

Lemma 1. *If $P \geq 1$ and $Q = \pm 1$ such that $D > 1$, then $\alpha > 1$ and $\beta = \frac{Q}{\alpha}$.*

Proof. The first statement can be easily justified since $P \geq 1$ and $D > 1$. Hence,

$$\alpha = \frac{P+\sqrt{D}}{2} > 1.$$

However, in order to prove the second statement, we start from the right hand side, i.e.

$$\begin{aligned} \frac{Q}{\alpha} &= \frac{Q}{\frac{P+\sqrt{D}}{2}} = \frac{2Q}{P+\sqrt{D}} = \frac{2Q}{P+\sqrt{D}} \cdot \frac{P-\sqrt{D}}{P-\sqrt{D}} \\ &= \frac{2Q(P-\sqrt{D})}{P^2 - D} = \frac{2Q(P-\sqrt{D})}{4Q} = \frac{P-\sqrt{D}}{2} \\ &= \beta, \end{aligned}$$

and this completes the proof of Lemma 1. □

Note that for later use and the simplicity of presenting our results, we also assume that $\alpha^I > 1$ for any $I \in \mathbb{Z}$.

Remark 1. Suppose that T_n represents the general term of the Lucas sequences U_n or V_n. To determine all the pairs $(x, y) = (T_i, T_j)$ with $i, j \geq 1$ satisfying the Bartz-Marlewski equation (4), the first step in our argument is mainly based on computing upper bounds for i and j (such that $1 \leq i \leq j$). Therefore, to resolve the equation completely dropping out the condition that $i \leq j$, we have to apply this argument on the following equations separately:

$$X^2 - 3XY + Y^2 + X = 0, \tag{7}$$

$$X^2 - 3XY + Y^2 + Y = 0, \tag{8}$$

where $(X, Y) = (T_i, T_j)$ with $1 \leq i \leq j$. In fact, one can easily observe that the complete set of solutions of equation (8) can be defined by the set $\{(b, a) : (a, b) \text{ is a solution of equation (7)}\}$. Therefore, later in the proofs of our results we mainly apply our argument on equation (7) and obtain the complete set of its solutions, and then the solutions of equation (8) can be determined easily. Finally, the complete set of solutions of the Bartz-Marlewski equation is derived from all the obtained solutions of the latter equations for which they satisfy equation (4).

3. Main results

Theorem 1. *Let $P \geq 1$ and $Q \in \{-1, 1\}$ such that $D > 1$. If $(x, y) = (U_i, U_j)$ with $j \geq i \geq 1$ is a solution of equation (4), then*

$$i < \frac{\ln|21D|}{\ln|\alpha|} = \ell_1 \quad \text{and} \quad j < \mu_1 \,, \tag{9}$$

where $\mu_1 = 3\ell_1$. Furthermore, if $P \geq 22; Q = -1$ and $P \geq 21; Q = 1$, then $i \leq 2$.

Proof. Following the conclusion of Remark 1, we mainly deal with the equation

$$U_i^2 - 3U_iU_j + U_j^2 + U_i = 0 \,. \tag{10}$$

We firstly insert the corresponding Binet's formulas defined in (5) in the latter equation to obtain

$$\left(\frac{\alpha^i - \beta^i}{\alpha - \beta}\right)^2 - 3\left(\frac{\alpha^i - \beta^i}{\alpha - \beta}\right)\left(\frac{\alpha^j - \beta^j}{\alpha - \beta}\right) + \left(\frac{\alpha^j - \beta^j}{\alpha - \beta}\right)^2 + \left(\frac{\alpha^i - \beta^i}{\alpha - \beta}\right) = 0 \,,$$

which can be further written as

$$\frac{1}{D}(\alpha^i - \beta^i)^2 - \frac{3}{D}(\alpha^i - \beta^i)(\alpha^j - \beta^j) + \frac{1}{D}(\alpha^j - \beta^j)^2 + \frac{1}{\sqrt{D}}(\alpha^i - \beta^i) = 0 \tag{11}$$

since $\sqrt{D} = \alpha - \beta$. Simplifying equation (11) gives that

$$\begin{aligned} \frac{1}{D}\alpha^{2j} =& \frac{1}{D}\big(- \alpha^{2i} + 2\alpha^i\beta^i - \beta^{2i} + 3\alpha^{i+j} - 3\alpha^i\beta^j - 3\alpha^j\beta^i + 3\beta^{i+j} + 2\alpha^j\beta^j \\ & - \beta^{2j}\big) + \frac{1}{\sqrt{D}}\big(- \alpha^i + \beta^i\big). \end{aligned}$$

From Lemma 1, we have that $\beta = \frac{Q}{\alpha}$. Hence, we obtain that

$$\begin{aligned} \frac{1}{D}\alpha^{2j} =& \frac{1}{D}\big(- \alpha^{2i} + 2\alpha^i\left(\frac{Q}{\alpha}\right)^i - \left(\frac{Q}{\alpha}\right)^{2i} + 3\alpha^{i+j} - 3\alpha^i\left(\frac{Q}{\alpha}\right)^j - 3\alpha^j\left(\frac{Q}{\alpha}\right)^i \\ & + 3\left(\frac{Q}{\alpha}\right)^{i+j} + 2\alpha^j\left(\frac{Q}{\alpha}\right)^j - \left(\frac{Q}{\alpha}\right)^{2j}\big) + \frac{1}{\sqrt{D}}\big(- \alpha^i + \left(\frac{Q}{\alpha}\right)^i\big) \,. \end{aligned}$$

Taking the absolute values for both sides of the latter equation with using the facts that $Q = \pm 1$ (i.e. $|Q| = 1$) and $D > 1$ (i.e $\sqrt{D} > 1$) leads to

$$\begin{aligned} \left|\frac{1}{D}\alpha^{2j}\right| <& \left|-\alpha^{2i}\right| + 2 + \left|-\alpha^{-2i}\right| + 3\left|\alpha^{i+j}\right| + 3\left|-\alpha^{i-j}\right| + 3\left|-\alpha^{j-i}\right| + 3\left|\alpha^{-i-j}\right| \\ & + 2 + \left|-\alpha^{-2j}\right| + \left|-\alpha^i\right| + \left|\alpha^{-i}\right| \,. \end{aligned}$$

Based on the facts that $1 \leq i \leq j$ (i.e. $-i, -j < j$ and $-j \leq -i < i$) and $\alpha > 1$, we get that

$$\begin{aligned} \left|\frac{1}{D}\alpha^{2j}\right| <& \left|\alpha^{i+j}\right| + 2\left|\alpha^{i+j}\right| + \left|\alpha^{i+j}\right| + 3\left|\alpha^{i+j}\right| + 3\left|\alpha^{i+j}\right| + 3\left|\alpha^{i+j}\right| + 3\left|\alpha^{i+j}\right| \\ & + 2\left|\alpha^{i+j}\right| + \left|\alpha^{i+j}\right| + \left|\alpha^{i+j}\right| + \left|\alpha^{i+j}\right| = 21\left|\alpha^{i+j}\right| \,. \end{aligned}$$

Multiplying the above inequality by $D\alpha^{-j-2i}$, we obtain that

$$\left|\alpha^{j-2i}\right| < 21D\left|\alpha^{-i}\right| , \tag{12}$$

which can written as

$$\left|\alpha^{i}\right| < \frac{21D}{\left|\alpha^{j-2i}\right|} < 21D \tag{13}$$

since $\left|\alpha^{j-2i}\right| > 1$ that is followed from the assumption that $\alpha^{I} > 1$ for all $I \in \mathbb{Z}$. Here, we conclude that $j - 2i > 0$ which means that as applications to our result presented in this theorem we have to determine the indices $i, j \geq 1$ satisfying equation (10) such that $j > 2i$. Furthermore, from inequalities (13) and (12) we respectively obtain that

$$i < \frac{\ln(21D)}{\ln(\alpha)} = \ell_1 \quad \text{and} \quad j < \frac{\ln(21D)}{\ln(\alpha)} + 2\ell_1 < 3\ell_1 = \mu_1 ,$$

and these prove the first statement of the theorem. Now, we deal with the second statement of the theorem starting with the case where $P \geq 22$ and $Q = -1$. Therefore,

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} > \frac{22 + \sqrt{22^2 + 4}}{2} > 22.045 .$$

Hence, the upper bound of i represented by ℓ_1 can be determined as follows:

$$\begin{aligned}\ell_1 = \frac{\ln(21D)}{\ln(\alpha)} &= \frac{\ln(21)}{\ln(\alpha)} + \frac{\ln(D)}{\ln(\alpha)} = \frac{\ln(21)}{\ln(\alpha)} + \frac{\ln((\alpha-\beta)^2)}{\ln(\alpha)} \\ &< \frac{\ln(21)}{\ln(22.045)} + 2\frac{\ln(\alpha-\beta)}{\ln(\alpha)} < 0.9993 + 2(1) = 2.9993\end{aligned}$$

as $\alpha > \beta$ and $\frac{\ln(\alpha-\beta)}{\ln(\alpha)} < 1$. Hence, $i \leq 2$. Similarly, we obtain $i \leq 2$ for $P \geq 21$ and $Q = 1$. This completes the proof of the second statement of the theorem. Thus, Theorem 1 is proved. □

Theorem 2. *Let $P \geq 1$ and $Q \in \{-1, 1\}$ such that $D > 1$. If $(x, y) = (V_i, V_j)$ with $j \geq i \geq 1$ is a solution of equation (4), then*

$$i < \frac{\ln|21|}{\ln|\alpha|} = \ell_2 \quad \textit{and} \quad j < \mu_2 , \tag{14}$$

where $\mu_2 = 3\ell_2$. Furthermore, equation (4) has no such solutions if $P \geq 21; Q = -1$ and $P \geq 22; Q = 1$.

Proof. We indeed prove this theorem following our approach used in the proof of Theorem 1. In other words, we mainly insert the corresponding Binet's formulas presented in (6) in the equation

$$V_i^2 - 3V_iV_j + V_j^2 + V_i = 0 . \tag{15}$$

After some simplifications with using the facts that $\beta = \frac{Q}{\alpha}$ with $|Q| = 1$, we get that

$$\left|\alpha^{j-2i}\right| < 21\left|\alpha^{-i}\right| . \tag{16}$$

Again, since we assumed that $\alpha^I > 1$ for all $I \in \mathbb{Z}$ that implies that $\left|\alpha^{j-2i}\right| > 1$ we obtain that

$$\left|\alpha^i\right| < \frac{21}{\left|\alpha^{j-2i}\right|} < 21 \tag{17}$$

Again, as applications to our result presented in this theorem we have to determine the indices $i, j \geq 1$ satisfying equation (10) such that $j > 2i$. Thus, from inequality (17) we have that

$$i < \frac{\ln(21)}{\ln(\alpha)} = \ell_2\,.$$

Moreover, combining the latter inequality with inequality (16) implies that

$$j < \frac{\ln(21)}{\ln(\alpha)} + 2\ell_2 < 3\ell_2 = \mu_2\,,$$

and the first statement of the theorem is proved. Now, we deal with the second statement in which we show that equation (4) has no such solutions if $P \geq 21; Q = -1$ and $P \geq 22; Q = 1$. In other words, we show that $\ell_2 \leq 1$ (i.e. $i < 1$) where $P \geq 21; Q = -1$ and $P \geq 22; Q = 1$. If $P \geq 21$ and $Q = -1$, we get that

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} > \frac{21 + \sqrt{21^2 + 4}}{2} > 21.0475\,.$$

Therefore,

$$\ell_2 = \frac{\ln(21)}{\ln(\alpha)} < \frac{\ln(21)}{\ln(21.0475)} < 0.9993\,,$$

which leads to $i \leq 0$, and this contradicts that i is a nonzero positive integer. Similarly, if $P \geq 22$ and $Q = 1$ we have that $\alpha \geq 21.9544$ which gives that $\ell_2 < 0.9856$, i.e. $i \leq 0$. Therefore, the second statement is proved. Hence, Theorem 2 is completely proved. □

4. Applications

In this section, we present a theorem that respectively contains two applications to our main results presented by Theorem 1 and Theorem 2. The first part of the theorem is an application to Theorem 1 in which we obtain the positive solutions $(x, y) = (U_i, U_j)$ with $i, j \geq 1$ of equation (4) with the parameters $1 \leq P \leq 21; Q = -1$ and $1 \leq P \leq 20; Q = 1$. In the second part of the theorem which is an application to Theorem 2, we investigate the solutions $(x, y) = (V_i, V_j)$ with $i, j \geq 1$ of equation (4) for the values of P and Q with which the equation is solvable, i.e. $1 \leq P \leq 20; Q = -1$ and $1 \leq P \leq 21; Q = 1$. Furthermore, as we remarked in the proofs of our main results that we have to determine the solutions corresponding to the indices i and j such that $j \geq i \geq 1$ and $j > 2i$. Then, as we mentioned in Remark 1 we permute the components of the obtained solutions in order to get the complete form for the set of solutions to the Bartz-Marlewski equation (4).

Theorem 3. *Let $1 \le P \le 21; Q = -1$ and $1 \le P \le 20; Q = 1$ such that $D > 1$. If $(x, y) = (U_i, U_j)$, then the Bartz-Marlewski equation (4) has no more solutions other than $(P, Q, x, y) = (1, -1, 1, 2)$. Furthermore, the Bartz-Marlewski equation contains no solutions $(x, y) = (V_i, V_j)$ if $1 \le P \le 20; Q = -1$ and $1 \le P \le 21; Q = 1$ with $D > 1$.*

Proof. For the first part of the theorem, we firstly determine the upper bounds for the indices i,j in the equation

$$U_i^2(P,Q) - 3U_i(P,Q)U_j(P,Q) + U_j^2(P,Q) + U_i(P,Q) = 0 \tag{18}$$

using the result of Theorem 1 presented by (9). Below, we give the computations of these bounds for P and Q with which $D, \alpha > 1$ in the ranges $1 \le P \le 20; Q = 1$ and $1 \le P \le 21; Q = -1$.

(P,Q)	$\lfloor \ell_1 \rfloor$	$\lfloor \mu_1 \rfloor$	(P,Q)	$\lfloor \ell_1 \rfloor$	$\lfloor \mu_1 \rfloor$
$(1,-1)$	9	29	$(21,-1)$	3	9
$(2,-1)$	5	17	$(3,1)$	4	14
$(3,-1)$	4	14	$(4,1)$	4	12
$(4,-1)$	4	12	$(5,1)$	3	11
$(5,-1)$	3	11	$(6,1)$	3	11
$(6,-1)$	3	11	$(7,1)$	3	10
$(7,-1)$	3	10	$(8,1)$	3	10
$(8,-1)$	3	10	$(9,1)$	3	10
$(9,-1)$	3	10	$(10,1)$	3	9
$(10,-1)$	3	9	$(11,1)$	3	9
$(11,-1)$	3	9	$(12,1)$	3	9
$(12,-1)$	3	9	$(13,1)$	3	9
$(13,-1)$	3	9	$(14,1)$	3	9
$(14,-1)$	3	9	$(15,1)$	3	9
$(15,-1)$	3	9	$(16,1)$	3	9
$(16,-1)$	3	9	$(17,1)$	3	9
$(17,-1)$	3	9	$(18,1)$	3	9
$(18,-1)$	3	9	$(19,1)$	3	9
$(19,-1)$	3	9	$(20,1)$	3	9
$(20,-1)$	3	9	-	-	-

Now, we consider the case where we have $(P, Q) = (1, -1)$, and the other cases are handled similarly. Here, we have $i \le 9$ and $j \le 29$. With the help of the SageMath software [10], one can easily find the values of i and j such that $1 \le i \le 9$ and $1 \le j \le 29$ with $j > 2i$ satisfying equation (18). Hence, we obtain that $i = 1$ and $j = 3$ which imply that $(x, y) = (1, 2)$. Indeed, if we permute the component of this pair of solution we do get another solution. Therefore, in this case we get $(1,2)$ is the only pair of solution to equation (4). Following the same approach with the other values of P and Q in the above table, we obtain no more solutions. Thus, the first part of the theorem is proved.

Next, we deal with the other part of the theorem starting with determining the minimum upper bounds of the index i and the index j in the equation

$$V_i^2(P,Q) - 3V_i(P,Q)V_j(P,Q) + V_j^2(P,Q) + V_i(P,Q) = 0 \tag{19}$$

where $1 \le P \le 21; Q = 1$ and $1 \le P \le 20; Q = -1$ such that $D, \alpha > 1$. This can be done using the result of Theorem 2 presented by (14), and we provide a summary for the computations in the following table:

(P,Q)	$\lfloor \ell_2 \rfloor$	$\lfloor \mu_2 \rfloor$	(P,Q)	$\lfloor \ell_2 \rfloor$	$\lfloor \mu_2 \rfloor$
$(1,-1)$	6	18	$(3,1)$	3	9
$(2,-1)$	3	10	$(4,1)$	2	6
$(3,-1)$	2	7	$(5,1)$	1	5
$(4,-1)$	2	6	$(6,1)$	1	5
$(5,-1)$	1	5	$(7,1)$	1	4
$(6,-1)$	1	5	$(8,1)$	1	4
$(7,-1)$	1	4	$(9,1)$	1	4
$(8,-1)$	1	4	$(10,1)$	1	3
$(9,-1)$	1	4	$(11,1)$	1	3
$(10,-1)$	1	3	$(12,1)$	1	3
$(11,-1)$	1	3	$(13,1)$	1	3
$(12,-1)$	1	3	$(14,1)$	1	3
$(13,-1)$	1	3	$(15,1)$	1	3
$(14,-1)$	1	3	$(16,1)$	1	3
$(15,-1)$	1	3	$(17,1)$	1	3
$(16,-1)$	1	3	$(18,1)$	1	3
$(17,-1)$	1	3	$(19,1)$	1	3
$(18,-1)$	1	3	$(20,1)$	1	3
$(19,-1)$	1	3	$(21,1)$	1	3
$(20,-1)$	1	3	-	-	-

Again, with the help of SageMath, we see that for any parameters P and Q in the above table there are no values for i and j satisfying equation (19) such that $1 \le i \le \lfloor \ell_2 \rfloor$ and $1 \le j \le \lfloor \mu_2 \rfloor$ with $j \ge 2i$. Hence, the Bartz-Marlewski equation (4) contains no solutions $(x,y) = (V_i, V_j)$. Thus, Theorem 3 is completely proved. $\square$

Acknowledgement. The author is very grateful to the referee for the valuable suggestions.

References

[1] Bartz, E., Marlewski, A., *A computer search of solutions of a certain Diophantine equation*, Pro Dialog (in Polish, English summary) **10** (2000), 47–57.

[2] Hashim, H.R., Tengely, Sz., *Solutions of a generalized Markoff equation in Fibonacci numbers*, Math. Slovaca **70** (2020), no. 5, 1069–1078. DOI: https://doi.org/10.1515/ms-2017-0414 10.1515/ms-2017-0414 10.1515/ms-2017-0414 10.1515/ms-2017-0414

[3] Hashim, H.R., Tengely, Sz., Szalay, L., *Markoff-Rosenberger triples and generalized Lucas sequences*, Period. Math. Hung. (2021).

[4] Kafle, B., Srinivasan, A., Togbé, A., *Markoff equation with Pell component*, Fibonacci Quart. **58** (2020), no. 3, 226–230.

[5] Koshy, T., *Polynomial extensions of two Gibonacci delights and their graph-theoretic confirmations*, Math. Sci. **43** (2018), no. 2, 96–108 (English).

[6] Luca, F., Srinivasan, A., *Markov equation with Fibonacci components.*, Fibonacci Q. **56** (2018), no. 2, 126–129 (English).

[7] Marlewski, A., Zarzycki, P., *Infinitely many positive solutions of the Diophantine equation* $x^2 - kxy + y^2 + x = 0$, Comput. Math. Appl. **47** (2004), no. 1, 115–121 (English). DOI: 10.1016/S0898-1221(04)90010-7

[8] Ribenboim, P., *My numbers, my friends. Popular lectures on number theory*, New York, NY: Springer, 2000 (English).

[9] Srinivasan, A., *The Markoff-Fibonacci numbers*, Fibonacci Q. **58** (2020), no. 5, 222–228 (English).

[10] Stein, W. A. et al., *Sage Mathematics Software (Version 9.0)*, The Sage Development Team, 2020, `http://www.sagemath.org`.

Faculty of Computer Science and Mathematics,
University of Kufa,
P.O.Box 21, 54001 Al Najaf, Iraq
E-mail: `hayderr.almuswi@uokufa.edu.iq`