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POROUS MEDIA EQUATION ON LOCALLY FINITE GRAPHS

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Abstract. In this paper, we consider two typical problems on a locally finite connected graph. The first one is to study the Bochner formula for the Laplacian operator on a locally finite connected graph. The other one is to obtain global nontrivial nonnegative solution to porous-media equation via the use of Aronson-Benilan argument. We use the curvature dimension condition to give a characterization two point graph. We also give a porous-media equation criterion about stochastic completeness of the graph. There is not much work in the direction of the study of nonlinear heat equations on locally finite connected graphs.

1. Introduction

In this paper, we study some typical problems related to heat equations and porous-media equation on a locally finite connected graph. We do believe that the study of nonlinear heat equations on locally finite connected graphs is an important subject as this happens in Riemannian geometry. After some thinking, we immediately realize that the Sobolev type inequality on graphs [3] plays a key role in such a research. However, Sobolev type inequality on graphs is not a topic of this paper. We first study the Bochner formula for the Laplacian operator on a locally finite connected graph. Our Bochner formula is new and should be useful in the study of eigenvalue estimates of the Laplacian operators on graphs. Similar but different form of Bochner formula has been formulated in [1] and [5]. Once we have the Bochner formula, we may use it to study the global behavior of the bounded solution to the heat equation on the graph. The Bochner formula is of independent interest. The last question under our consideration of this paper is to obtain global positive solution to porous-media equation via the use of Aronson-Benilan argument [6]. This is a hard question since we may not have the Sobolev compactness imbedding theorem and it is not easy to obtain

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the global solution from the exhaustion domain method. We can overcome this difficulty by using the Aronson-Benilan type estimate of the bounded solutions to the Porous-media equation. We also give a porous-media equation criterion about stochastic completeness of the graph. Our main results are the Bochner formula (7), Theorem 3 and Theorem 4 below. Theorem 3 may not be new and we include proof here for completeness. The curvature dimension condition is important in Theorem 3 in Section 3.

Here is the plan of the paper. In Section 2, we introduce the working spaces and the Laplace operator on them. Since we shall use the distance function in the curvature dimension condition, we give some useful comments about the the relation between stochastically complete and mean curvature. We study the maximum principle for both heat equation and the porous media equation. In Section 3, we obtain the Bochner formula for heat equation and we also give a characterization of the two point graph. In Section 4, we consider the locally bounded global solution to the porous-media equation. Some comments about McKean type results are in Section 5.

2. The Maximum principles

We start by recalling some definitions and the maximum principle for the bounded solution to heat equation. Let $G = (V(G), E(G))$ be an infinite, locally finite, connected graph without loops or multiple edges where $V = V(G)$ is the set of vertices of G and $E = E(G)$ is the set of edges. We still write $x \in G$ when x is a vertex of G. We use the notation $x \sim y$ to indicate the edge connects the vertex x to its neighbor vertex y. We equip V with the symmetric weight $\mu_{xy} \geq 0$ associated to the edge $x \sim y$ such that $\mu_{xy} > 0$ for each edge $x \sim y$ and $\sum_{x\sim y} \mu_{xy} > 0$ is finite for each $x \in V$. Let $d_x = \sum_{x\sim y} \mu_{xy} > 0$. We always assume that $\mu_{xy} = \mu_{yx}$ for each edge $x \sim y$ which make edges be unoriented. We call such a graph the short name the weighted graph.

As in [7], we let m be a measure on V with full support (i.e., m is a map $m\colon V \to (0, \infty)$). Then, (V, m) is a measure space. We define the space of all square summable functions on G,

$$l^2(V, m) = \{f\colon V \to R; \sum_{x\in V} m(x) f(x)^2 < \infty\}$$

with the inner product

$$(f, g) = \sum_{x\in V} m(x) f(x) g(x)\,.$$

Define on $l^2(V, m)$ the Laplacian operator for the function f,

$$\Delta f(x) = \frac{1}{m(x)} \sum_{x\sim y} \mu_{xy}\big(f(y) - f(x)\big)\,. \tag{1}$$

and the norm of the gradient of the function f by

$$|\nabla f|^2(x) = \frac{1}{m(x)} \sum_{x\sim y} \mu_{xy}(f(y) - f(x))^2\,.$$

Note that

$$\big(\Delta f(x)\big)^2 \le \frac{d_x}{m(x)}\,|\nabla f|^2(x)\,.$$

We now consider the case when $m(x) = d_x$ for all $x \in G$. Recall that it is showed in Theorem 2.5 in [14] that on such G the heat kernel $p_t(x,y)$ always exists and when $\sum_{y\in V} p_t(x,y) = 1$ for any $x \in V$, $G = (V,E)$ is called stochastically complete. We now recall the definition of mean curvature of a metric sphere. Fix $x_0 \in G$ and let $r(x) = d(x,x_0)$. Let

$$d_+(x) = \sum_{\{y\sim x;r(y)=r(x)+1\}} \mu_{xy}$$

and

$$d_-(x) = \sum_{\{y\sim x;r(y)=r(x)-1\}} \mu_{xy}$$

be the sums of weights of vertices which are in the distances $r(x)\pm 1$. Define the *mean curvature* $H(x)$ of the sphere of radius $r(x)$ to x_0 by

$$H(x) = \Delta r(x) = \frac{1}{d_x}\sum_{x\sim y} \mu_{xy}\big(r(y)-r(x)\big)\,.$$

It can be verified that

$$H(x) = \frac{d_+(x)-d_-(x)}{d_x}\,.$$

We remark that under the assumption that there exists some $x_0 \in G$ and a constant $C \ge 0$ such that $H(x) \le C$ on V, G is stochastically complete (see Theorem 25 in [7] and see also [14]). As noticed by the the unknown referee that this result also works well when we consider the more general definition of the Laplacian defined in (1) on V,

$$\Delta f(x) = \frac{1}{m(x)}\sum_{x\sim y} \mu_{xy}\big(f(y)-f(x)\big)$$

for arbitrary $m\colon V \to (0,\infty)$ and $\sum_{x\sim y}\mu_{xy} = m(x)$. Then we have the following maximum principle ([7], [11], [13]).

Theorem 1. *Assume that G is stochastically complete. Let $u_0(x)$ be any bounded function on G. Then any bounded solution $u(t,x)$ to the heat equation*

$$u_t = \Delta u$$

with initial data $u(0) = u_0$ satisfies

$$\sup_G |u(t,x)| \le \sup_G |u_0(x)|$$

for every $t \ge 0$.

The proof of the result above is standard. By Theorem 1 we can derive the uniqueness of bounded solution to the heat equation on V. In fact the claim follows by considering differences of bounded solutions with same initial condition. Actually we can extend the maximum principle to positive solution to the porous media type equation

$$u_t = \Delta \log u\,, \quad \text{in} \quad (0,\infty)\times V \tag{2}$$

with the bounded non-negative initial data u_0. In fact we have the following result, which improve our previous version as pointed out by the unknown referee.

Theorem 2. *On the weighted graph $G=(V,E)$, the following two conditions are equivalent.*

(1) *G is stochastically complete.*

(2) *Any positive bounded solution $v\colon V\times[0,\infty)\to(0,\infty)$ to $v_t\le\Delta\log v$ satisfies*

$$\sup_{G\times[0,\infty)} v(t,x) = \sup_G v_0(x) \tag{3}$$

for every $t\ge 0$.

Proof. We first consider (1) $\Rightarrow$ (2). By considering $\frac{v(t,x)}{\sup_G v_0(x)}$ and rescaling the time variable, we may assume $\sup_G v_0(x)=1$. Set $u(t,x)=\log v(t,x)$ and $u_0(x)=\log v_0(x)\le 0$. Then u satisfies that

$$e^u u_t \le \Delta u. \tag{4}$$

Let $w=u_+$. Applying the Kato inequality [11] (and the proof given there for $m(x)=d_x$ works well for the Laplacian (1)) we have

$$e^w w_t \le \Delta w\,.$$

Let $f=\int_0^\infty e^{-t}wdt$. Then we have

$$\Delta f = \int_0^\infty e^{-t}\Delta w dt \ge \int_0^\infty e^{-t}(e^w)_t\,dt = -1+\int_0^\infty e^{-t}e^w\,dt$$

and the right term is

$$-1+\int_0^\infty e^{-t}e^w\,dt = \int_0^\infty e^{-t}(e^w-1)\,dt \ge f\,.$$

Stochastic completeness of G and boundedness and non-negativity of f imply that $f=0$ on V. Then we have $w=0$ and $u\le 0$, which implies (3).

We then consider (2) $\Rightarrow$ (1). We argue by contradiction. The stochastic incompleteness of G (see Theorem 3.1 in [14] or Theorem 25 in [7]) implies that for some $\lambda<0$ and some nontrivial positive bounded w such that

$$-e^{-1}\Delta w = \lambda w\,.$$

Let $u=e^{-\lambda t}w-p_t w$. Then u is a nontrivial non-negative bounded solution to $u_t=e^{-1}\Delta u\ge 0$ with the initial data $u_0(x)=0$. We may assume that $u\le 1$, which implies that $e^u\le e$. Then we have

$$e^u u_t \le \Delta u\,.$$

Let $v = e^u$. We have $v_t \leq \Delta \log v$ and $v(x,0) = e^0 = 1 < \sup_{G\times[0,\infty)} v(t,x)$, which gives a contradiction. Thus we have the desired result. $\square$

The maximum principle above gives us a comparison lemma for the porous media equation (2). We shall use this fact in Section 5.

3. Bochner formula for heat equation and a characterization of two point graph

In this section we let $m(x) = d_x$ for the locally finite graph. The main results in this section are the Bochner type formula and a characterization of two point graph.

We introduce the Bochner formula and the curvature dimension condition following the method of Bakry-Emery (see [9]). We define

$$\Gamma(f,g) = \frac{1}{2}\{\Delta(fg)(x) - f(x)\Delta g(x) - g(x)\Delta f(x)\}$$

and

$$\Gamma_2(f,g) = \frac{1}{2}\{\Delta\Gamma(fg)(x) - \Gamma(f,\Delta g)(x) - \Gamma(g,\Delta f)(x)\}\,.$$

Then, by direct computation,

$$\Delta f^2(x) = 2f(x)\Delta f(x) + |\nabla f|^2(x)\,,$$

$$\Gamma(f,g)(x) = \frac{1}{2d_x}\sum_{y\sim x}\mu_{xy}\big(f(y)-f(x)\big)\big(g(y)-g(x)\big)\,,$$

$$\Gamma(f,f)(x) = \frac{1}{2}|\nabla f|^2(x)\,,$$

$$\Gamma_2(f,f)(x) = \frac{1}{4}|D^2 f|^2(x) - \frac{1}{2}|\nabla f|^2(x) + \frac{1}{2}(\Delta f)^2(x)\,, \tag{5}$$

where we have defined

$$|D^2 f|^2(x) := \frac{1}{d_x}\sum_{y\sim x}\frac{\mu_{xy}}{d_y}\sum_{z\sim y}\mu_{yz}|f(x)-2f(y)+f(z)|^2\,. \tag{6}$$

We define

$$(\nabla f,\nabla\Delta f)(x) = \frac{1}{d_x}\sum_{y\sim x}\mu_{xy}\big(f(y)-f(x)\big)\big(\Delta f(y)-\Delta f(x)\big)\,.$$

We now compute the Bochner formula for the function f.

$$-\Delta|\nabla f|^2(x) = -|D^2 f|^2(x) + \frac{2}{d_x}\sum_{y\sim x}\frac{\mu_{xy}}{d_y}\sum_{z\sim y}\mu_{yz}\big(f(x)-2f(y)+f(z)\big)(f(x)-f(y).$$

Set

$$I = \frac{2}{d_x}\sum_{y\sim x}\frac{\mu_{xy}}{d_y}\sum_{z\sim y}\mu_{yz}\big(f(x)-2f(y)+f(z)\big)\big(f(x)-f(y)\big)\,.$$

Note that

$$
\begin{aligned}
I &= 2|\nabla f|^2(x) + \frac{2}{d_x}\sum_{y\sim x}\mu_{xy}\big(f(x) - f(y)\Delta f(y)\big) \\
&= 2|\nabla f|^2(x) + \frac{2}{d_x}\sum_{y\sim x}\mu_{xy}\big(f(x) - f(y)\big(\Delta f(y) \\
&\quad - \Delta f(x)\big) + \Delta f(x)\frac{2}{d_x}\sum_{y\sim x}\mu_{xy}\big(f(x) - f(y)\big) \\
&= 2|\nabla f|^2(x) + 2|\Delta f|^2(x) - 2(\nabla f, \nabla\Delta f)(x)\,.
\end{aligned}
$$

Then we have the following Bochner formula.

Assertion: *On the locally finite, connected graph $G = (V(G), E(G))$, which may be an infinite set without loops or multiple edges, for any function f on G, we have the Bochner formula*

$$
-\Delta|\nabla f|^2(x) = -|D^2 f|^2(x) + 2|\nabla f|^2(x) + 2|\Delta f|^2(x) - 2(\nabla f, \nabla\Delta f)(x)\,. \tag{7}
$$

We now consider the bounded solution $f(t,x) = P_t(f_0)$ to the heat equation

$$
f_t = \Delta f
$$

with initial data f_0 on the locally finite connected graph $G = (V, E)$. Recall that

$$
(\partial_t - \Delta)f^2(t,x) = -|\nabla f|^2(t,x) \le 0\,. \tag{8}
$$

Using the maximum principle for the stochastic complete, locally finite, connected graph, we have

$$
f^2(t,x) \le \sup_V f^2(0,x)\,,
$$

which may be useful for other purpose. Since $f(t,x)$ is uniformly bounded in t, we know that there exists $t_j \to \infty$ such that

$$
f(t_j, x) \to f_\infty(x)
$$

for each $x \in G$, and $\lim_{t_j\to\infty} |\nabla f(t_j,x)|^2 \to 0$, which implies that $f_\infty(x) = \text{const}$.

Using (7) we get that

$$
(\partial_t - \Delta)\Big(\frac{1}{2}|\nabla f|^2(t,x)\Big) = -\frac{1}{2}|D^2 f|^2(t,x) + |\nabla f|^2(x) + |\Delta f|^2(t,x)\,. \tag{9}
$$

We believe that this formula is useful in the study of Schrodinger type equations on graphs.

In the rest of this section, we give a characterization of two point graph as below. Assume now on G the curvature dimension condition that for any $f\colon V \to \mathbb{R}$,

$$
\Gamma_2(f,f)(x) \ge \frac{1}{m}(\Delta f)^2(x) + \frac{k}{2}|\nabla f|^2(x)\,, \tag{10}
$$

for some uniform constants $m > 0$ and $k \in \mathbb{R}$. One example of such graph is the two point graph $G = (V(G), E(G))$ with $V = \{x, y\}$. Note that

$$
\Delta f(x) = f(y) - f(x)\,, \quad |\nabla f|^2(x) = (f(y) - f(x))^2\,,
$$

and $|D^2 f|^2(x) = 4|f(y) - f(x)|^2$. Then

$$\Gamma_2(f,f) = (f(y) - f(x))^2$$

and then the condition (10) above is

$$(f(y) - f(x)^2 \geq \frac{1}{m}(f(y) - f(x))^2 + \frac{k}{2}(f(y) - f(x))^2\,,$$

which is in turn equivalent to $1 \geq \frac{1}{m} + \frac{k}{2}$ for $m > 0$, i.e., $k \leq 2(1 - \frac{1}{m})$.

We have the following result.

Theorem 3. *Let G be a locally finite connected graph. Assume* (10) *is true for $k \geq 2 - 2/m$. Moreover, we assume that the vertex measure equals the degree d_x. Then the graph has at most two vertices.*

Proof. To prove this conclusion, we fix $p \in G$ and let $f(x) = d(x,p)$. Note that

$$\Delta f(p) = 1 = |\nabla f|^2(p)\,,$$

and then

$$\Gamma_2(f,f)(p) = \frac{1}{4}|D^2 f(p)|^2\,.$$

Applying the curvature dimension condition to the function f we have

$$\frac{1}{4}|D^2 f(p)|^2 \geq \frac{1}{m} + \frac{k}{2}\,.$$

From the definition relation (6) we also have the estimate that

$$\frac{1}{4}|D^2 f(p)|^2 \leq 1\,.$$

Then we have

$$\frac{1}{m} + \frac{k}{2} \leq 1.$$

By our assumption we know that

$$\frac{1}{m} + \frac{k}{2} = 1$$

which in turn gives us that

$$\frac{1}{4}|D^2 f(p)|^2 = 1\,.$$

This then implies the point p has at most one neighbor point. This then completes the proof of Theorem 3. $\square$

4. Global solution to the porous-media equation

Given any positive bounded function $u_0 : V \to \mathbb{R}_+$ bounded below by a positive constant. We consider the global existence of the positive solution $u(t,x)$ to the porous-media equation

$$u_t = \Delta \log u\,, \quad \text{in} \quad (0,\infty) \times V \tag{11}$$

with the initial data $u(0) = u_0$.

Take any connected finite subgraph $\Omega \subset V$. We may first consider (11) in $(0,\infty)\times\Omega$ with initial data and boundary condition u_0. Let $f=\frac{1}{2}\log u$. Then $u=e^{2f}$ and it satisfies the equivalent problem

$$e^f(e^f)_t = \Delta f\,, \quad \text{in} \quad (0,\infty)\times V\,. \tag{12}$$

Actually we can get the local in time solution u_Ω to (11) (respectively $f_\Omega = \frac{1}{2}\log u_\Omega$ to (12)) in $(0,T)\times\Omega$ (for some $T>0$) by using the discrete Morse flow method [12].

For $N>1$ an integer and any $T>0$, let

$$h=T/N\,, \quad t_n = nh\,, \quad n=0,1,2,\dots,N\,.$$

Assume that we have constructed $f_j\in L^2(\Omega)$, $0\le j\le n-1$, and f_{n-1} is a minimizer of the functional

$$I_{n-1}(f)=\frac{1}{2h}\int_\Omega |e^f - e^{f_{n-2}}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla f|^2\,dx$$

on the space $H=\{f\in L^2(\Omega)\mid f-f_0=0 \text{ on } \partial\Omega\}$. Note that H is a closed convex subset of $L^2(\Omega)$. Define

$$I_n(f)=\frac{1}{2h}\int_\Omega |e^f - e^{f_{n-1}}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla f|^2\,dx$$

on H. It is clear that the infimum is finite and by applying the Poincaré inequality to $f-f_0$ (see [3]), any minimizing sequence is bounded in H. Then, since Ω is finite, we know that the minimizing sequence is uniformly bounded on Ω and by extracting a subsequence, we may assume that the minimizing sequence converges on Ω to a limit and we conclude that I_n has a minimizer f_n in H which satisfies

$$\frac{1}{h}(e^f - e^{f_{n-1}})e^f = \Delta f$$

along with the uniform energy bound

$$\frac{1}{2h}\int_\Omega |e^{f_n} - e^{f_{n-1}}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla f_n|^2\,dx \le \frac{1}{2}\int_\Omega |\nabla f_{n-1}|^2\,dx \le C\,. \tag{13}$$

We remark that f can not be a constant provided f_{n-1} is nontrivial.

We define $f_N(t)\in L^2$ for $t\in[0,T]$ such that, for $n=1,\dots,N$,

$$f_N(t)=f_n, \quad t\in[t_{n-1},t_n]\,.$$

We further define, for $n=1,\dots,N$,

$$\partial_t e^{f_N}(t)=\frac{1}{h}(e^{f_n}-e^{f_{n-1}})\,, \quad t\in[t_{n-1},t_n]\,.$$

Then f_N satisfies

$$e^{f_N}\partial_t e^{f_N}(t)=\Delta f_N$$

in $\Omega\times(0,T)$. Note that the energy bound (13) implies that

$$\int_0^T\int_\Omega |\partial_t(e^{f_N})|^2 + \sup_t \int_\Omega |\nabla f_N|^2\,dx \le 5C.$$

We may use the Poincare inequality [3] that

$$\lambda_1(\Omega)\int_\Omega u^2 \le \int\int_\Omega |\nabla u|^2$$

for $u = f_N - f_0$ to get the uniform $L^2(\Omega)$ bound of $\{f_N\}$. Taking a subsequence of $\{f_N\}$ that converges in $L_t^\infty H$, one obtains a limit $f \in L_t^\infty H$ that satisfies

$$e^f \partial_t e^f = \Delta f$$

in distribution sense in the domain $\Omega \times (0,T)$. Since Δf is bounded, we know $\partial_t e^f$ is well-defined and the equation holds point-wise. Applying the maximum principle we know that e^{2f} is uniformly bounded by both the initial data and the boundary data.

To get the globally defined solution, we need the linear upper bound for $u_\Omega = e^{2f}$ and we follow a well-known argument due to Aronson and Benilan [6].

Let $\lambda > 1$. Define

$$w_\lambda(t,x) = \lambda u_\Omega(\lambda^{-1}t, x)\,.$$

Then $w_\lambda(t,x)$ satisfies (11) in $(0,T)\times\Omega$ with the initial data and boundary condition $\lambda u_0(x)$, which is bigger than $u_0(x)$. By using the comparison principle we know that

$$w_\lambda(t,x) > u_\Omega(t,x) \quad \text{in} \quad (0,T)\times\Omega\,.$$

Set

$$v_\lambda(t,x) = w_\lambda(t,x) - u_\Omega(t,x)\,.$$

Then

$$\frac{\partial}{\partial\lambda} v_\lambda(t,x) \ge 0\,, \quad \text{in} \quad (0,T)\times\Omega\,,$$

or equivalently $u_t \le t^{-1}u$ for $u = u_\Omega$, which by integration, implies that $u_\Omega(t,x) \le C(1+t)$ where $C > 0$ is a constant depending only on u_0. Hence we can extend the global solution $u_\Omega(t,x)$. Take $\Omega = \Omega_j$ where $V = \bigcup \Omega_j$, $\Omega_j \subset \Omega_{j+1}$ are exhaustion finite subgraphs of V. Then we get a sequence of solutions $\{u_j\}$ defined on $\Omega_j \times (0,\infty)$. By taking diagonal subsequence we can get a sub-convergence sequence on any finite subset of V, still denoted by $\{u_j\}$ and a global (locally bounded) positive solution $u(t,x)$ of (11) with the initial data u_0 such that

$$u(t,x) = \lim_{j\to\infty} u_j(t,x)\,,$$

locally in $(0,\infty)\times V$.

In summary, we then have proven the result below.

Theorem 4. *For any bounded positive function $u_0\colon V \to \mathbb{R}_+$ bounded below by a positive constant, there exists a global nontrivial positive solution to* (11) *with initial data u_0.*

The uniqueness question to (11) is an interesting (may be very difficult) problem and it may be considered by using the maximum principle. We leave it open to interested readers.

5. McKean type eigenvalue estimate

We now study the McKean type eigenvalue estimate for the principal eigenvalue λ_1 of the Lapalacian operator defined in [10] on the locally finite graph. We recall the definition below. Let $\lambda_1(R)$ be the first eigenvalue of the Laplacian operator defined on the ball $B_R(x_0)$ with Dirichlet boundary condition on the boundary $\partial B_R(x_0)$. Then the sequence $(\lambda_1(R))$ is a decreasing sequence in R and the principal eigenvalue $\lambda_1 = \lim_{R\to\infty}\lambda_1(R)$ is well-defined. It is not hard to see that λ_1 does not depend on the based point x_0. Recall that for any finite subset $\Omega\subset G$, we define Ω^c the complement set of Ω in G. We let $\mu(\Omega)=\int_\Omega 1=\sum_{x\in\Omega} d_x$ and define the measure of the edge boundary $\partial\Omega$ by

$$\mu(\partial\Omega)=|\partial\Omega|=\sum_{\xi=xy\in E(G),x\in\Omega,y\in\Omega^c}\mu_{xy}\,.$$

It can be directly verify [3] that for any f in the finite set Ω,

$$\int_\Omega \Delta f=\sum_{x\in\Omega}\sum_{y\in\Omega^c}(f(y)-f(x))\mu_{xy}\,. \tag{14}$$

We also recall the general integration by part formula. Let $\Omega\subset V$ be a finite subset of V. Define

$$U_1(\Omega)=\{y\in V;\mathrm{dist}(y,\Omega)\le 1\}.$$

Theorem 5. *For any function f,h, we have*

$$\int_\Omega(\Delta f,h)=-\frac{1}{2}\sum_{x,y\in U_1(\Omega)}\mu_{xy}\nabla_{xy}f\nabla_{xy}h \tag{15}$$

where $\nabla_{xy}f=f(y)-f(x)$.

Note that the right hand side of (15) can be written as

$$=-\frac{1}{2}\sum_{x,y\in\Omega}\mu_{xy}\nabla_{xy}f\nabla_{xy}h+\sum_{x\in\Omega,y\in\Omega^c}\mu_{xy}h(x)\nabla_{xy}f\,.$$

We recall the definition of the Cheeger constant $h(\mu)$ [3] based at x_0 defined by

$$h(\mu)=\inf_{\Omega\subset V;x_0\in\Omega,\mu(\Omega)\le\frac{1}{2}\mu(V)}\frac{|\partial\Omega|}{\mu(\Omega)}\,.$$

Then as in the Riemannian geometry case we have

$$\lambda_1\ge\frac{h(\mu)^2}{2}\,.$$

The following result is about McKean type result such that the constant a in Theorem 6 below may depend on the base point x_0.

Theorem 6. *Assume that $H(x)\ge a>0$ for some x_0 and some positive constant a. Then $\lambda_1\ge a^2/2$.*

Proof. Take $f(x) = r(x) = d(x, x_0)$ in (14) on any finite set Ω containing x_0. Then using $H(x) \geq a$ we know that $\int_\Omega H(x) \geq a\mu(\Omega)$ and right side is equals to $\mu(\partial\Omega)$, which implies that

$$\mu(\partial\Omega) = \int_\Omega H(x) \geq a\mu(\Omega)\,.$$

Hence $h(\mu) \geq a$, which implies our conclusion.

□

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