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PERIODIC TRAVELING WAVES IN THE SYSTEM OF LINEARLY COUPLED NONLINEAR OSCILLATORS ON 2D-LATTICE

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Abstract. In this paper we obtain results on existence of non-constant periodic traveling waves with arbitrary speed $c > 0$ in infinite system of linearly coupled nonlinear oscillators on a two-dimensional lattice. Sufficient conditions for the existence of such solutions are obtained with the aid of critical point method and linking theorem.

1. Introduction

In the present paper we study equations that describe the dynamics of an infinite system of linearly coupled nonlinear oscillators on a two dimensional lattice. Let $q_{n,m}(t)$ be a generalized coordinate of the (n,m)-th oscillator at time t. It is assumed that each oscillator interacts linearly with its four nearest neighbors. The equations of motion of the system are of the form

$$\begin{aligned} \ddot{q}_{n,m}(t) &= c_1\big(q_{n+1,m}(t) + q_{n-1,m}(t) - 2q_{n,m}(t)\big) \\ &\quad + c_2\big(q_{n,m+1}(t) + q_{n,m-1}(t) - 2q_{n,m}(t)\big) - U'\big(q_{n,m}(t)\big)\,, \quad (n,m) \in \mathbb{Z}^2\,, \end{aligned} \tag{1}$$

where $q_{n,m}$ is a sequence of real functions, c_1, $c_2 \in \mathbb{R}$, $U \in C^1(\mathbb{R};\mathbb{R})$ is an external on-site potential. Equations (1) form an infinite system of ordinary differential equations. This system, which is discrete in the spatial variables, is the infinite-dimensional Hamiltonian system and belongs to the so-called lattice dynamical systems. Note that as $U(r) = K(1-\cos r)$ we have the discrete sine–Gordon type equations or a two-dimensional version of the Frenkel–Kontorova model [13]. Equations of such type are of interest in view of numerous applications in physics [1,12,13], material science [14], image processing [17], biology [11,24], chemical reaction theory [27], etc.

There are many papers devoted to numerical simulation of lattice systems (see [12,13,18,20,31,32,33]). But the question of the existence of solutions in such systems is extremely important.

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Among the solutions of such equations, traveling waves deserve special attention. In papers [5, 8, 10, 19, 21] traveling waves for infinite systems of linearly and nonlinearly coupled oscillators on 2D-lattice are studied, while [3,4,23,28] deal with traveling waves for such systems on 1D-lattice. In particular, in [23] certain results of such type are obtained by means of bifurcation theory, while in [3,4,5,8,10,28] the existence of periodic and solitary traveling waves is studied by means of critical point theory.

S.N. Chow, J. Mallet-Paret and W. Shen [16] studied the existence and stability of traveling waves in lattice dynamical systems, in particular, in lattice ordinary differential equations (lattice ODEs) and in coupled map lattices (CMLs). They proved the asymptotic stability with asymptotic phase shift of the traveling wave solution (under appropriate spectral conditions) and the existence of traveling waves in CML's which arise as time-discretizations of lattice ODEs. They also showed that these results apply to the discrete Nagumo equation

$$\dot{x}_n(t) = d\big(x_{n+1}(t) + x_{n-1}(t) - 2x_n(t)\big) - g\big(x_n(t)\big)\,, \quad n \in \mathbb{Z},\ d \in \mathbb{R}\,.$$

Note that J.W. Cahn, J. Mallet-Paret and E.S. van Vleck [15] studied such equations on a two-dimensional spatial lattice. They obtained traveling wave solutions in each direction $e^{i\theta}$, and they explored the relation between the wave speed c, the angle θ, and the detuning parameter of the nonlinearity.

H.J. Hupkes, L. Morelli, P. Stehlík and V. Švígler [22] discussed a new type of traveling waves which make no sense in PDEs. They studied multichromatic traveling waves (fronts) for lattice Nagumo equation. Their results show that these multichromatic fronts can disappear and reappear as the diffusion coefficient is increased. These multichromatic waves can travel in parameter regimes where the monochromatic fronts are also free to travel.

G. Iooss and K. Kirchgässner [23] established the existence of traveling waves for linearly coupled oscillators with weak coupling. The motion of the chain, which they considered, is described by the system

$$\ddot{q}_n(t) = \alpha\big(q_{n+1}(t) + q_{n-1}(t) - 2q_n(t)\big) - V'\big(q_n(t)\big)\,, \quad n \in \mathbb{Z}\,,$$

where α is a small parameter, $V'(0) = 0$, $V''(0) = 1$.

In paper [3] almost the same system is studied. But in this paper α is not a small parameter, and another type of potentials, which satisfy the Ambrosetti-Rabinowitz condition, is considered. In paper [5] the system of linearly coupled nonlinear oscillators on $2D$ lattice is studied. Note that in [3] and [5] the profile function is only non-zero, the period of wave $2k \geq 2$, and the speed $c > c_0$ with some c_0 (so-called "supersonic waves"). To establish the existence of such waves, the mountain pass theorem is used. In [4], by means of the linking theorem, It is obtained a result on the existence of non-trivial $2k$-periodic traveling waves with arbitrary speed $c > 0$ and any $k \geq 1$ for chain of linearly coupled nonlinear oscillators.

P. Makita [28], using a variational technique, established the existence of periodic and homoclinic (solitary) traveling waves for a system of the form

$$\ddot{q}_n(t) + f'\big(q_n(t)\big) = V'\big(q_{n+1}(t) - q_n(t)\big) - V'\big(q_n(t) - q_{n-1}(t)\big)\,, \quad n \in \mathbb{Z}\,,$$

that describes the dynamics of an infinite chain of nonlinearly coupled nonlinear oscillators. In particular, he established the existence of non-constant supersonic traveling waves.

In papers [3,8,10,28], the method of periodic approximations is used to establish the existence of solitary traveling waves. Each solitary wave is obtained as a limit of periodic wave when the period goes to infinity.

By means of topological and variational methods, M. Fečkan and V. Rothos [19] established the existence of periodic traveling waves for a system of the form

$$\ddot{q}_{n,m}(t) = \Delta q_{n,m}(t) - f\big(q_{n,m}(t)\big)\,, \quad (n,m) \in \mathbb{Z}^2,$$

where $(\Delta q)_{n,m} = q_{n+1,m} + q_{n-1,m} + q_{n,m+1} + q_{n,m-1} - 4q_{n,m}$ is a two-dimensional discrete Laplacian, the nonlinearity f is odd and 2π-periodic.

L. Zhang and S. Guo [35] studied the existence and branching patterns of wave trains in a two-dimensional lattice with linear and nonlinear coupling between nearest particles and a nonlinear substrate potential. By means of invariant theory and singularity theory, they obtained the small amplitude solutions. They showed the impact of the direction φ of propagation and obtained the existence and branching patterns of wave trainsin a one-dimensional lattice by investigating the existence of traveling waves of the original two-dimensional lattice in the direction φ of propagation satisfying $\tan\varphi$ is rational.

G. Friesecke and K. Matthies [21] showed the existence of solitary traveling waves for a two-dimensional elastic lattice of particles interacting via harmonic springs between nearest and diagonal neighbors without external potential.

Paper [25] contains a result on the existence of heteroclinic traveling waves for the discrete sine -Gordon equation with linear interaction, while in [26] periodic, homoclinic and heteroclinic traveling waves for such systems with nonlinear interaction are studied. Papers [2, 6, 7, 9] are devoted to the existence of periodic, homoclinic and heteroclinic traveling waves for the discrete sine-Gordon type equations on 2D-lattice. To obtain the main results, variational techniques are also used here.

In contrast to the previous ones (see [8, 10, 19, 21]), in present paper we obtain the sufficient existence conditions of non-constant $2k$-periodic traveling waves with arbitrary speed $c > 0$ for any $k > 0$. Thus, there are both supersonic and subsonic waves. To obtain the main results, we used the fact that the corresponding equation of the traveling wave profile has a variational structure, i.e., it is the Euler-Lagrange equation for some action functional J_k. Using the linking theorem, the existence of non-trivial critical points of J_k, which are periodic solutions, is established.

2. Statement of a problem

We consider a system of oscillators with on-site potential of the form

$$U(r) = -\frac{a}{2}r^2 + V(r)\,.$$

Then (1) takes the form

$$\ddot{q}_{n,m}(t) = c_1\Delta_{(1)}q_{n,m}(t) + c_2\Delta_{(2)}q_{n,m}(t) + aq_{n,m}(t) - V'\big(q_{n,m}(t)\big)\,, \tag{2}$$

for $(n,m)\in\mathbb{Z}^2$, where

$$\left(\Delta_{(1)}q\right)_{n,m}=q_{n+1,m}+q_{n-1,m}-2q_{n,m}\,,$$

$$\left(\Delta_{(2)}q\right)_{n,m}=q_{n,m+1}+q_{n,m-1}-2q_{n,m}$$

are the discrete Laplacians for n and m, respectively. For $c_1=c_2=1$, the sum of these operators is the two-dimensional discrete Laplacian.

A traveling wave solution of Eq. (1) is a function of the form

$$q_{n,m}(t)=u(n\cos\varphi+m\sin\varphi-ct)\,,$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$\begin{aligned} c^2u''(s)&=c_1\big(u(s+\cos\varphi)+u(s-\cos\varphi)-2u(s)\big)\\ &+c_2\big(u(s+\sin\varphi)+u(s-\sin\varphi)-2u(s)\big)+au(s)-V'\big(u(s)\big)\,, \end{aligned} \tag{3}$$

The constant $c\neq 0$ is called the speed of the wave. If $c>0$, then the wave moves to the right, otherwise to the left.

In what follows, a solution of Eq. (1) is understood as a function u from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying this equation.

We consider the case of periodic traveling waves. The profile function of such wave satisfies the following periodicity condition

$$u(s+2k)=u(s)\,,\quad s\in\mathbb{R}\,, \tag{4}$$

where $k>0$ is some real number.

Note that the vector $\overrightarrow{l}(\cos\varphi,\sin\varphi)$ defines the direction of wave propagation and for the angles $\varphi=\frac{\pi k}{4}$, $k\in\mathbb{Z}$, Eq. (3) reduces to an equation that corresponds to the one-dimensional case. Thus, the results of the present paper contain those in [4] as special case and also establish the existence of waves that propagate in other directions.

3. Preliminaries

To equation (3) we associate the functional

$$J_k(u)=\int\limits_{-k}^{k}\left[\frac{c^2}{2}\big(u'(s)\big)^2-\frac{c_1}{2}\big(Au(s)\big)^2-\frac{c_2}{2}\big(Bu(s)\big)^2+\frac{a}{2}u^2(s)-V\big(u(s)\big)\right]ds\,, \tag{5}$$

where

$$(Au)(s):=u(s+\cos\varphi)-u(s)=\int\limits_{s}^{s+\cos\varphi}u'(\tau)d\tau\,,$$

$$(Bu)(s):=u(s+\sin\varphi)-u(s)=\int\limits_{s}^{s+\sin\varphi}u'(\tau)d\tau\,.$$

The functional J_k defined on the Hilbert space

$$E_k=\left\{u\in H^1_{\mathrm{loc}}(\mathbb{R}):\ u(s+2k)=u(s)\right\}$$

with the scalar product

$$(u,v)_k = \int_{-k}^{k} \big[u(s)v(s) + u'(s)v'(s)\big]\,ds$$

and corresponding norm

$$\|u\|_k = \left(\|u\|^2_{L^2(-k,k)} + \|u'\|^2_{L^2(-k,k)}\right)^{\frac{1}{2}} = \Big(\int_{-k}^{k} \big[(u(s))^2 + (u'(s))^2\big]\,ds\Big)^{\frac{1}{2}},$$

i.e. the Sobolev space of $2k$-periodic functions.

We always assume that

(h) *$V \in C^1(\mathbb{R};\mathbb{R})$, $V(0) = V'(0) = 0$ and $V'(r) = o(r)$ as $r \to 0$, and there exists $\mu > 2$ such that*

$$0 < \mu V(r) \le V'(r)r\,, \quad r \neq 0\,.$$

It is easy to verify that under assumption (h) there exist $d > 0$ and $d_0 \ge 0$ such that

$$V(r) \ge d|r|^\mu - d_0\,, \quad \mu > 2\,. \tag{6}$$

By direct calculation we obtain the following two lemmas (see [10], Lemma 3.2 and Lemma 3.3).

Lemma 1. *Under assumption (h) the functional J_k is C^1-functional on E_k and*

$$\begin{aligned}\langle J_k'(u),h\rangle = \int_{-k}^{k} \big[&c^2u'(s)h'(s) + c_1\big(u(s+\cos\varphi) + u(s-\cos\varphi) - 2u(s)\big)h(s)\\ &+ c_2\big(u(s+\sin\varphi) + u(s-\sin\varphi) - 2u(s)\big)h(s)\\ &+ au(s)h(s) - V'\big(u(s)\big)h(s)\big]\,ds\end{aligned}$$

for any u, $h \in E_k$.

Lemma 2. *Under assumption (h) the critical points of J_k in the space E_k are solutions of Eq.* (3) *satisfying* (4).

To obtain the main result, we need the linking theorem (see [29, 30, 34]).

Let H be a Hilbert space, $H = Y \oplus Z$. Let $\rho > r > 0$ and $z \in Z$ be such that $\|z\| = r$. Define

$$M := \{u = y + \lambda z : y \in Y, \|u\| \le \rho, \lambda \ge 0\}$$

and

$$M_0 := \{u = y + \lambda z : y \in Y, \|u\| = \rho \ \text{ and } \lambda \ge 0, \ \text{ or } \|u\| \le \rho, \lambda = 0\}\,,$$

i.e. M_0 is the boundary of M. Let

$$N := \{u \in Z : \|u\| = r\}\,.$$

Consider C^1-functional I on H and suppose that

$$\beta := \inf_{u\in N} I(u) > \alpha := \sup_{u\in M_0} I(u)\,.$$

In this situation we say that the functional I possesses the *linking geometry.*

Theorem 1 (**Linking**). *Suppose that the functional $I\colon H \to \mathbb{R}$ of class C^1 possesses the linking geometry and satisfies the Palais-Smale condition*

(PS) *if any sequence $\{u_n\} \subset H$ such that $I'(u_n) \to 0$ and the sequence $\{I(u_n)\}$ is bounded then $\{u_n\}$ contains a convergent subsequence.*

Let

$$b := \inf_{\gamma\in\Gamma} \max_{u\in M} I(\gamma(u)) \geq \beta\,,$$

where

$$\Gamma := \{\gamma \in C(M,H) : \gamma|_{M_0} = \mathrm{id}\}\,.$$

Then b is a critical value of I and

$$\beta \leq b \leq \sup_{u\in M} I(u)\,.$$

4. Main results

The main results of this paper are the following.

Theorem 2. *Assume (h) and suppose that $a > 0$. Then for every $k > 0$ and $c > 0$ Eq. (3) has a non-trivial solution u that satisfies condition (4).*

Theorem 3. *Assume (h) and suppose that $a = 0$. Then for every $k > 0$ and $c > 0$ Eq. (3) has a non-trivial solution u that satisfies condition (4). Moreover, this solution is non-constant for k large enough.*

Making use of the linking theorem, we prove the existence of non-trivial traveling waves with periodic profile function. For this, due to Lemma 2, it is enough to prove the existence of a non-trivial critical points of J_k.

Now we verify the conditions of the linking theorem for the functional J_k.

Lemma 3. *Under the assumptions of Theorem 2 the functional J_k satisfies Palais-Smale condition.*

Proof. Let $\{u_n\} \subset E_k$ be a Palais-Smale sequence of J_k at the level b, i.e. $J_k'(u_n) \to 0$ and $J_k(u_n) \to b$. Choose $\beta \in (\mu^{-1}, 2^{-1})$. Then, for n large enough, $\|J_k'(u_n)\|_{k,*} \leq 1$ and $|J_k(u_n)| \leq b+1$. Thus, for n large enough, we have

$$\begin{aligned}
b+1+\beta\|u_n\|_k \geq{}& J_k(u_n)-\beta\langle J_k'(u_n),u_n\rangle \\
&= \left(\frac{1}{2}-\beta\right)\int_{-k}^{k}\left[c^2(u_n'(s))^2-c_1(Au_n(s))^2-c_2(Bu_n(s))^2\right. \\
&\quad \left.+a(u_n(s))^2\right]ds-\int_{-k}^{k}\left[V(u_n(s))-\beta V'(u_n(s))u_n(s)\right]ds\,.
\end{aligned}$$

If $c_1\leq 0$ and $c_2\leq 0$, then

$$\begin{aligned}
J_k(u_n)-\beta\langle J_k'(u_n),u_n\rangle &\geq \left(\frac{1}{2}-\beta\right)\int_{-k}^{k}\left[c^2(u_n'(s))^2+a(u_n(s))^2\right]ds \\
&\geq \left(\frac{1}{2}-\beta\right)\alpha_0\|u_n\|_k^2\,,
\end{aligned}$$

where $\alpha_0=\min\{c^2;a\}$. Hence,

$$b+1+\beta\|u_n\|_k\geq\left(\frac{1}{2}-\beta\right)\alpha_0\|u_n\|_k^2\,,$$

and this implies that $\{u_n\}$ is bounded in E_k.

If $c_1>0$ and $c_2\leq 0$, then

$$\begin{aligned}
&J_k(u_n)-\beta\langle J_k'(u_n),u_n\rangle \\
&\quad\geq \left(\frac{1}{2}-\beta\right)\left(c^2\|u_n'\|^2_{L^2(-k,k)}-c_1\|Au_n\|^2_{L^2(-k,k)}+a\|u_n\|^2_{L^2(-k,k)}\right) \\
&\quad\quad+C(\beta\mu-1)\|Au_n\|^\mu_{L^\mu(-k,k)}-C_0\,.
\end{aligned}$$

Since for $\mu>2$

$$\|Au_n\|^2_{L^2(-k,k)}\leq C\|Au_n\|^2_{L^\mu(-k,k)}\leq K(\varepsilon)+\varepsilon\|Au_n\|^\mu_{L^\mu(-k,k)}\,,$$

where $K(\varepsilon)\to\infty$ as $\varepsilon\to 0$, we have

$$\begin{aligned}
b+1+\beta\|u_n\|_k \geq{}& \left(\frac{1}{2}-\beta\right)c^2\|u_n'\|^2_{L^2(-k,k)}+\left(\frac{1}{2}-\beta\right)a\|u_n\|^2_{L^2(-k,k)} \\
&-\left(\frac{1}{2}-\beta\right)c_1\varepsilon\|Au_n\|^\mu_{L^\mu(-k,k)}-\left(\frac{1}{2}-\beta\right)c_1K(\varepsilon) \\
&+C(\beta\mu-1)\|Au_n\|^\mu_{L^\mu(-k,k)}-C_0\,.
\end{aligned}$$

Choosing ε small enough, we have that

$$\begin{aligned}
b+1+\beta\|u_n\|_k \geq{}& \left(\frac{1}{2}-\beta\right)\left(c^2\|u_n'\|^2_{L^2(-k,k)}+a\|u_n\|^2_{L^2(-k,k)}\right) \\
&+C_1\|Au_n\|^\mu_{L^\mu(-k,k)}-C_0\geq\left(\frac{1}{2}-\beta\right)\alpha_0\|u_n\|_k^2+C_1\|Au_n\|^\mu_{L^\mu(-k,k)}-C_0\,,
\end{aligned}$$

where $\alpha_0 = \min\{c^2; a\}$. Since $\beta\mu - 1 > 0$, then $C_1 = C(\beta\mu - 1) > 0$, and we have

$$b + 1 + \beta\|u_n\|_k \geq \left(\frac{1}{2} - \beta\right)\alpha_0\|u_n\|_k^2 - C_0\,.$$

The last inequality implies that $\{u_n\}$ is bounded.

Similar reflections in cases where $c_1 \leq 0$, $c_2 > 0$ and $c_1 > 0$, $c_2 > 0$.

The boundedness of $\{u_n\}$ implies that, up to a subsequence (with the same denotation), $u_n \to u$ weakly in E_k, and hence, $Au_n \to Au$ and $Bu_n \to Bu$ weakly in E_k, and strongly in $L^2(-k,k)$ and $C([-k,k])$ (by the compactness of Sobolev embedding).

A straightforward calculation shows that

$$\begin{aligned}
c^2\|u_n - u\|_k^2 &= \int_{-k}^{k} \left[c^2(u_n'(s) - u'(s))^2 + c^2(u_n(s) - u(s))^2\right]\, ds \\
&= \langle J_k'(u_n), u_n\rangle + c_1\|Au_n - Au\|^2_{L^2(-k,k)} + c_2\|Bu_n - Bu\|^2_{L^2(-k,k)} \\
&\quad - a\|u_n - u\|^2_{L^2(-k,k)} + \int_{-k}^{k} [V'(u_n(s)) - V'(u(s))]\big(u_n(s) - u(s)\big)\, ds\,.
\end{aligned}$$

Obviously that all the terms on the right part converge to 0 (first and last terms converge to 0 by weak convergence, and other terms converge to 0 by strong convergence). Thus, $\|u_n - u\|_k \to 0$ as $n \to \infty$, and proof is complete. □

Lemma 4. *Assume* (h) *and suppose that* $a \geq 0$*. Then the functional* J_k *possesses the linking geometry.*

Proof. Consider the operator L defined by

$$\begin{aligned}
(Lu)(s) :=& - c^2u''(s) + c_1\big(u(s + \cos\varphi) + u(s - \cos\varphi) - 2u(s)\big) \\
&+ c_2\big(u(s + \sin\varphi) + u(s - \sin\varphi) - 2u(s)\big) + au(s)
\end{aligned}$$

with $2k$-periodic conditions. Elementary Fourier analysis shows that L is a self-adjoint operator in $L^2(-k,k)$, bounded below and that L has discrete spectrum which accumulates at $+\infty$, i.e. below zero there is a finite number of eigenvalues.

Let Z be the subspace of E_k generated by the functions with positive eigenvalues and Y be the subspace of E_k generated by the functions with non-positive eigenvalues. It is readily verified that $Y \perp Z$ and $E_k = Y \oplus Z$.

Denote by Q_k the quadratic part of the functional J_k

$$Q_k(u) = \frac{1}{2}\int_{-k}^{k} \left[c^2(u'(s))^2 - c_1(Au(s))^2 - c_2(Bu(s))^2 + a(u(s))^2\right]\, ds\,.$$

Obviously,

$$Q_k(y + z) = Q_k(y) + Q_k(z)\,,$$

where $y \in Y$, $z \in Z$.

Note that the quadratic form Q_k is positive definite on Z, i.e.

$$Q_k(u) \geq \alpha \|u\|_k^2 ,$$

with $\alpha > 0$. Assumption (h) implies that, given $\varepsilon > 0$, there exists $r_0 > 0$ such that $|V(r)| \leq \varepsilon r^2$ as $|r| \leq r_0$. Then

$$J_k(u) \geq Q_k(u) - \varepsilon \int_{-k}^{k} (u(s))^2 ds \geq Q_k(u) - \varepsilon \|u\|_k^2 \geq \delta \|u\|_k^2 ,$$

where $\delta > 0$. Hence, $J_k(u) > 0$ on $N = \{u \in Z : \ \|u\|_k = r\}$ provided $r > 0$ is small enough.

Now we fix $z \in Z$, $\|z\|_k = 1$ and set

$$M = \{u = y + \lambda z : \ y \in Y, \ \|u\|_k \leq \rho, \ \lambda \geq 0\} .$$

We have

$$J_k(y + \lambda z) = Q_k(y) + \lambda^2 Q_k(z) - \int_{-k}^{k} V\big(y(s) + \lambda z(s)\big)\, ds .$$

Then, since $Q_k(y) \leq 0$, by (6), we have

$$J_k(y + \lambda z) \leq \lambda^2 \gamma_0 + 2kd_0 - d\|y + \lambda z\|_{L^\mu(-k,k)}^\mu ,$$

where $\gamma_0 = Q_k(z)$. Since

$$\rho^2 = \|y + \lambda z\|_k^2 = \|y\|_k^2 + \lambda^2 ,$$

then $\lambda^2 \leq \rho^2$. Furthermore, on finite dimensional spaces all norms are equivalent. Hence,

$$\|y + \lambda z\|_{L^\mu(-k,k)} \geq c\|y + \lambda\|_k = c\rho ,$$

$$J_k(y + \lambda z) \leq \gamma_0 \rho^2 + 2kd_0 - dc^\mu \rho^\mu .$$

Since $\mu > 2$, the right hand part here is negative if ρ is large enough. Hence, $J_k(y + \lambda z) \leq 0$. If $u \in M_0$, $\|u\|_k \leq \rho$ and $\lambda = 0$, then $u = y \in Y$ and, obviously, $J_k(u) \leq 0$. Thus, J_k possesses the linking geometry. □

Proof of Theorem 2. Lemmas 3 and 4 show that J_k satisfies all conditions of linking theorem. Hence, J_k has non-trivial critical point $u \in E_k$, which is a C^2-solution of Eq. (3) that satisfies (4). The proof is complete. □

Lemma 5. *Under the assumptions of Theorem 3 the functional J_k satisfies Palais-Smale condition.*

Proof. We represent the functional J_k in the form

$$J_k(u) = \frac{1}{2} \Psi_k(u) - \int_{-k}^{k} V\big(u(s)\big)\, ds ,$$

where

$$\Psi_k(u) = \int_{-k}^{k} \left[c^2(u'(s))^2 - c_1(Au(s))^2 - c_2(Bu(s))^2\right] ds\,.$$

Let $\{u_n\} \subset E_k$ be a Palais-Smale sequence of J_k at the level b, i.e. $J_k'(u_n) \to 0$ and $J_k(u_n) \to b$ as $n \to \infty$. We prove by contradiction that it is bounded. Suppose that $\{u_n\}$ is unbounded. Then (possibly up to a subsequence) $\|u_n\|_k \to \infty$ as $n \to \infty$.

Let E_k^+ be the subspace of E_k generated by the functions of the operator L with positive eigenvalues, E_k^- be the subspace of E_k generated by the functions with negative eigenvalues and E_k^0 generated by the functions with zero eigenvalue. It is readily verified that they are all pairwise orthogonal and $E_k = E_k^+ \oplus E_k^0 \oplus E_k^-$. Then any function $u \in E_k$ can be represented as $u = u^+ + u^0 + u^-$, where $u^+ \in E_k^+$, $u^0 \in E_k^0$, $u^- \in E_k^-$.

Let

$$v_n = \frac{u_n}{\|u_n\|_k}\,,$$

then $\|v_n\|_k = 1$ and, up to a subsequence, $v_n \to v \in E_k$ weakly in E_k. Furthermore, $v_n \to v$ strongly in $L^p(-k,k)$, $p > 2$, and almost everywhere. Suppose that $v \neq 0$. It is readily verified that

$$\Psi_k(u^+ + u^0 + u^-) = \Psi_k(u^+) + \Psi_k(u^-),$$
$$\|u\|_k^2 = \|u^+\|_k^2 + \|u^0\|_k^2 + \|u^-\|_k^2$$

for $u \in E_k$ and $\|u^+\|_k \le \|u\|_k$, $\|u^-\|_k \le \|u\|_k$.

Since $u_n = \|u_n\|_k v_n$, we have

$$\frac{J_k(u_n)}{\|u_n\|_k} = \frac{1}{2}\Psi_k(v_n^+) + \frac{1}{2}\Psi_k(v_n^-) - \frac{1}{\|u_n\|_k^2}\int_{-k}^{k} V\big(\|u_n\|_k v_n(s)\big)\, ds\,. \tag{7}$$

Since $J_k'(u_n) \to 0$, then the right hand part here converges to zero. We show that the right hand side tends to $-\infty$. Indeed, $\Psi_k(v_n^+)$ is bounded, $\Psi_k(v_n^-) \le 0$, furthemore, due to (6), there exists $d > 0$ and $d_0 \ge 0$ such that $V(r) \ge d|r|^\mu - d_0$, where $\mu > 2$. Then

$$\begin{aligned}\frac{1}{\|u_n\|_k^2}\int_{-k}^{k} V\big(\|u_n\|_k v_n(s)\big)\, ds &\ge d\|u_n\|_k^{\mu-2}\int_{-k}^{k} |v_n(s)|^\mu ds - d_0\|u_n\|_k^{-2} \\ &= d\|u_n\|_k^{\mu-2}\cdot\|u_n\|_{L^\mu(-k,k)}^\mu - d_0\|u_n\|_k^{-2} \to +\infty\,,\end{aligned}$$

because $\|v_n\|_{L^\mu(-k,k)}^\mu \to \|v\|_{L^\mu(-k,k)}^\mu \neq 0$. Thus, the right hand part of (7) tends to $-\infty$ as $n \to \infty$. We got a contradiction. Hence, $v = 0$.

Since the space E_k decomposes into an orthogonal direct sum $E_k = E_k^+ \oplus E_k^0 \oplus E_k^-$, then $v_n^+ \to 0$, $v_n^0 \to 0$ and $v_n^- \to 0$ weakly in E_k. But E_k^0 and E_k^- are finite-dimensional spaces, and therefore $v_n^0 \to 0$ and $v_n^- \to 0$ strongly in E_k.

Suppose now that $v_n^+ \to 0$ strongly in E_k. Then

$$1 = \|v_n\|_k^2 = \|v_n^+\|_k^2 + \|v_n^0\|_k^2 + \|v_n^-\|_k^2 \to 0\,,$$

a contradiction. Hence, $v_n^+ \not\to 0$ strongly in E_k and $\|v_n^+\|_k \geq \varepsilon > 0$ for some $\varepsilon > 0$ (up to a subsequence).

Thus, we have

$$\begin{aligned}
0 \leftarrow\; & \frac{J_k(u_n)}{\|u_n\|_k^2} - \frac{1}{\mu\|u_n\|_k^2}\langle J_k'(u_n), u_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{-k}^{k}\left[c^2(v_n'(s))^2 - c_1(Av_n(s))^2 - c_2(Bv_n(s))^2\right] ds \\
&\quad - \frac{1}{\|u_n\|_k^2}\int_{-k}^{k}\Big[V(u_n(s)) - \frac{1}{\mu}V'(u_n(s))u_n(s)\Big] ds \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Psi_k(v_n) + \frac{1}{\|u_n\|_k^2}\int_{-k}^{k}\Big[\frac{1}{\mu}V'(u_n(s))u_n(s) - V(u_n(s))\Big] ds \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\big(\Psi_k(v_n^+) + \Psi_k(v_n^-)\big) \\
&\quad + \frac{1}{\|u_n\|_k^2}\int_{-k}^{k}\Big[\frac{1}{\mu}V'(u_n(s))u_n(s) - V(u_n(s))\Big] ds\,.
\end{aligned}$$

Due to (h), the integral on the right-hand side of the last equality is nonnegative. Moreover, $\Psi_k(v_n^-) \to 0$ and on the space E_k^+ the quadratic part Ψ_k is positive definite, i.e. $\Psi_k(v_n^+) \geq \varepsilon_0 > 0$, because $\|v_n^+\|_k \geq \varepsilon$. And we have

$$\begin{aligned}
\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Psi_k(v_n^+) + \frac{1}{\|u_n\|_k^2}\int_{-k}^{k}\Big[\frac{1}{\mu}V'\big(u_n(s)\big)u_n(s) - V\big(u_n(s)\big)\Big] ds \\
\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\varepsilon_0 > 0\,.
\end{aligned}$$

But the left hand part converges to zero, and we have a contradiction. Hence, the sequence $\{u_n\}$ is bounded. And this implies (see the proof of Lemma 3), up to a subsequence, that $\|u_n - u\|_k \to 0$. The proof is complete. □

Proof of Theorem 3. Lemmas 4 and 5 show that J_k satisfies all conditions of linking theorem. Hence, J_k has non-trivial critical point $u \in E_k$, which is a C^2-solution of Eq. (3) that satisfies (4).

We show that $u \neq$ Const. Indeed, let $u(s) = \alpha \neq 0$ be a constant solution of Eq. (3). Then from Eq. (3) with $a = 0$ we have that $V'(\alpha) = 0$, but this contradicts the condition (h). The proof is complete. □

Note that in Theorem 2, in contrast to Theorem 3, the existence of only non-trivial solutions was established. And the existence of non-constant solutions in the case $a > 0$ remains open.

Example 1. Consider the potential

$$V(r) = \frac{d}{p}|r|^p,\ d > 0,\ p > 2\,. \tag{8}$$

Then (2) takes the form

$$\begin{aligned} c^2u''(s) &= c_1(u(s+\cos\varphi) + u(s-\cos\varphi) - 2u(s)) \\ &+c_2(u(s+\sin\varphi) + u(s-\sin\varphi) - 2u(s)) + au(s) - d|u(s)|^{p-2}u(s)\,. \end{aligned} \tag{9}$$

It is easy to see that the potential (8) satisfies the assumption (h). Hence, if $a \geq 0$, then for every $k > 0$ and $c > 0$ Eq. (9) has a non-trivial $2k$-periodic solution, i.e., there exists a non-trivial $2k$-periodic traveling wave. Moreover, if $a = 0$, then this solution is non-constant for k large enough.

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