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AN IMPROVEMENT OF THE NON-EXISTENCE REGION FOR LIMIT CYCLES OF THE BOGDANOV-TAKENS SYSTEM

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Dedicated to Professor Hiroshi Kokubu on the occasion of 60th birthday

Abstract. In this paper, an improvement of the global region for the non-existence of limit cycles of the Bogdanov-Takens system, which is well--known in the Bifurcation Theory, is given by two ideas. The first is to apply the existence of the algebraic invariant curve of the system to the Bendixson-Dulac criterion, and the second is to consider a necessary condition in order that a closed orbit of the system includes two equilibrium points. In virtue of these methods, it shall be shown that our previous result and the result of Gasull et al. are improved partially.

1. Introduction

We consider the following system called Bogdanov-Takens system (for instance see [1, 5, 8, 10] or [12]) having a cups of order 2:

$$\begin{cases} \dot{x} = y \\ \dot{y} = (x + \mu_2)y + x^2 + \mu_1 \,, \end{cases} \tag{1.1}$$

where the dot $(\dot{})$ denotes differentiation, μ_1 and μ_2 are real parameters.

The system is a classical Liénard system and has been well-known in the Bifurcation Theory. Several local results for the orbits of the system have been given in the bifurcation diagram (for instance see [7, 9, 11] or [12]) as follows.

Proposition 1. *Let $|\mu_1|$ and $|\mu_2|$ are sufficient small. If the parameter pair (μ_1, μ_2) belongs to the set*

$$D = \Big\{ (\mu_1, \mu_2) \mid \mu_1 < -\frac{49}{25}\mu_2^2 \ \text{or}\ \mu_1 > -\mu_2^2 \Big\} \,,$$

then system (1.1) has no limit cycles.

Remark that the above result has been given under the local condition for the parameters.

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Our aim is to give a global condition for the non-existence of limit cycles of system (1.1). Previously, we gave the following result in [5].

Proposition 2. *If the parameter pair* (μ_1, μ_2) *belongs to the region*

$$E = \begin{cases} \text{(i)} & \mu_1 \geq 0 \\ \text{(ii)} & \mu_1 < 0 \quad \textit{and} \quad \mu_2 \leq 0 \\ \text{(iii)} & \mu_1 \geq -\mu_2^2 \\ \text{(iv)} & \mu_1 \leq -(\mu_2+1)^2\,, \end{cases}$$

then system (1.1) *has no limit cycles.*

E(i) and E(ii) are trivial conditions given by the criterion for the classification of the equilibrium points (for instance see [4] or [13]).

E(iii) is a global condition given in [5]. It was proved by using the tool in [3] that if some plane curve defined in the Liénard system has no intersecting points itself, then the system has no limit cycles.

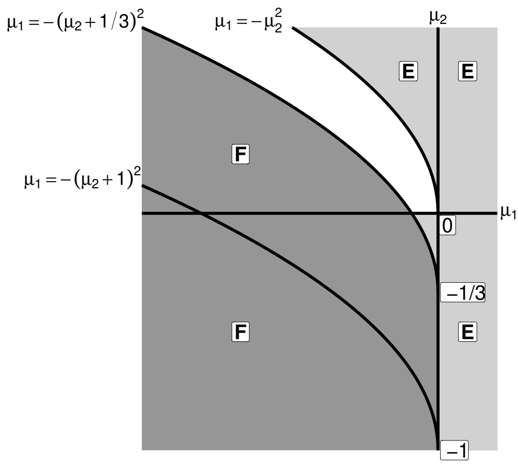
We in [4] proved that system (1.1) has the algebraic invariant curve of the form $y = -x + \sqrt{-\mu_1}$ if $\mu_1 = -(\mu_2+1)^2$. E(iv) is a significant condition given by combinating the existence of the algebraic invariant curve with the Bendixson-Dulac criterion (for the detail see [4]). This idea is also used in Lemma 5.

In this paper, we give the global result extended the region E. Our main result is stated as follows.

Theorem 3. *Let*

$$F = \left\{ (\mu_1, \mu_2) \mid \mu_1 \leq -(\mu_2+\epsilon)^2 \;\; \textit{and} \;\; \epsilon \geq \frac{1}{3} \right\}.$$

If the parameter pair (μ_1, μ_2) *belongs to the set* $E \cup F$, *then system* (1.1) *has no limit cycles.*



Non-existence region $E \cup F$

We note that the condition E(iv) is contained to F. From the above result, the shaped region in the previous page expresses an improved global region for the non-existence of limit cycles of system (1.1).

In the Section 2, we shall prepare several facts for the transformed system of system (1.1) and prove Theorem 3 in the next section. In the final section, we discuss on the relation of Theorem 3 and the result of Gasull et al. [1] improved the Perko's Conjectures [10]. It shall be shown that our result improves them partially in Theorem 10.

2. Preliminaries

From Proposition 2, we assume $\mu_1 < 0$ and $\mu_2 > 0$ for the parameter pair (μ_1, μ_2) to simplify the discussion.

By setting $x = z - \sqrt{-\mu_1}$, system (1.1) is transformed into the system

$$\begin{cases} \dot{z} = y \\ \dot{y} = -(-z + \sqrt{-\mu_1} - \mu_2)y - (-z^2 + 2\sqrt{-\mu_1}z)\,. \end{cases} \tag{2.1}$$

For system (2.1), several facts have been given in [5].

Lemma 4. *System* (2.1) *has exactly two equilibrium points* $\mathrm{O}(0,0)$ *and* $\mathrm{A}(2\sqrt{-\mu_1}, 0)$, *and these indices are* $+1$ *and* -1, *respectively.*

Since the equilibrium point A is saddle, we note from the Poincaré index theorem (for instance see [11], [13] or [14]) that if there exists a closed orbit C of system (2.1), then it must contain the only one equilibrium point O in the inside of C.

Lemma 5. *Let* $\epsilon = \sqrt{-\mu_1} - \mu_2$ *and the set*

$$D_1 = \left\{(z, y) \mid y < -\epsilon(z - 2\sqrt{-\mu_1})\right\}.$$

Then system (2.1) *has no limit cycles on* D_1.

Proof. We shall use the well-known Bendixson-Dulac theorem (for instance see [14]) for this purpose. We set $X(z,y) = y$ and $Y(z,y) = -(-z+\epsilon)y - (-z^2 + 2\sqrt{-\mu_1}z)$ and define the 'Bendixson-Dulac function $B(z,y)$' by $B(z,y) = -(y+z-2\sqrt{-\mu_1})^{-1}$, where the denominator is the algebraic invariant curve of system (2.1) for $\epsilon = 1$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial z}(BX) + \frac{\partial}{\partial y}(BY) \\ &\quad = \frac{1}{(y + z - 2\sqrt{-\mu_1})^2}\left\{y + \epsilon(z - 2\sqrt{-\mu_1})\right\} \\ &\quad < 0 \end{aligned}$$

on D_1. This completes the proof. $\square$

Let

$$l^+ = \{(z,y) \mid y + \epsilon(z - 2\sqrt{-\mu_1}) = 0,\ z > 0\},$$
$$l^- = \{(z,y) \mid y + \epsilon(z - 2\sqrt{-\mu_1}) = 0,\ z \leq 0\}$$

and $D_2 = \mathbb{R}^2 - (D_1 \cup l^+ \cup l^-)$.

To prove Theorem 3, we divide the proof in two cases (I) $\epsilon \geq 1$ and (II) $0 < \epsilon < 1$.

Case (I):

Lemma 6. *Let $\epsilon \geq 1$. Then an orbit of system* (2.1) *passing through l^+* (resp. l^-) *must cross l^+* (resp. l^-) *from D_1* (resp. D_2) *to D_2* (resp. D_1).

Proof. Let us express an orbit γ of system (2.1) as the pair $(\phi(t), \psi(t))$. Suppose that γ meets the line l^+. If γ meets l^+ at a point $(\phi(t_0), \psi(t_0))$ with $t_0 \geq 0$, then we have

$$\frac{\psi^{'}(t_0)}{\phi^{'}(t_0)} = -\epsilon + \Big(1 - \frac{1}{\epsilon}\Big)\phi(t_0) \geq -\epsilon$$

for $\epsilon \geq 1$. Thus, the point $(\phi(t), \psi(t))$ crosses the line l^+ from D_1 to D_2 at $t = t_0$. Similarly, we see that the point $(\phi(t), \psi(t))$ crosses the line l^- from D_2 to D_1 at $t = t_0$. □

Case (II): From the similar discussion to Lemma 6, we have the following

Lemma 7. *Let $0 < \epsilon < 1$. Then an orbit of system* (2.1) *passing through l^+* (resp. l^-) *must cross l^+* (resp. l^-) *from D_2* (resp. D_1) *to D_1* (resp. D_2).

Proof. We use the same signs as Lemma 6. If γ meets l^+ at a point $(\phi(t_0), \psi(t_0))$ with $t_0 \geq 0$, then we have $\psi^{'}(t_0)/\phi^{'}(t_0) < -\epsilon$ for $0 < \epsilon < 1$. Thus, the point $(\phi(t), \psi(t))$ crosses the line l^+ from D_2 to D_1 at $t = t_0$. Similarly, we see that the point $(\phi(t), \psi(t))$ crosses the line l^- from D_1 to D_2 at $t = t_0$. □

3. Proof of Theorem 3

First, we remark from Lemma 5 that a non-trivial closed orbit C of system (2.1) can not stay in D_1. We divide the proof in two cases.

Case (I): If $\epsilon \geq 1$, from Lemma 6, the orbit C of system (2.1) must pass through l^+ (resp. l^-) from D_1 (resp. D_2) to D_2 (resp. D_1). Thus, it must contain two equilibrium points O and A. This contradicts to the Poincaré index theorem.

Case (II): If $0 < \epsilon < 1$, from Lemma 7, the orbit C passes through l^+ (resp. l^-) from D_2 (resp. D_1) to D_1 (resp. D_2). On the other hand, we have the following fact by the similar calculation to [6].

Lemma 8. *Let the function $y = y(x)$ be the orbit of system* (1.1) *starting from the initial point $(0, 2\epsilon\sqrt{-\mu_1})$. Then the inequality*

$$y(2\sqrt{-\mu_1}) < 2\Big(1 - \frac{1}{3\epsilon}\Big)(-\mu_1)$$

holds for $\epsilon > 0$.

Proof. Since

$$y(2\sqrt{-\mu_1}) - y(0) = \int_0^{2\sqrt{-\mu_1}} \frac{dy}{dz}dz = \int_0^{2\sqrt{-\mu_1}} \Big\{z - \epsilon - \frac{-z(z-2\sqrt{-\mu_1})}{y}\Big\}dz\,,$$

we get

$$\begin{aligned} y(2\sqrt{-\mu_1}) &< 2\epsilon\sqrt{-\mu_1} + \int_0^{2\sqrt{-\mu_1}} \Big\{z - \epsilon - \frac{-z(z-2\sqrt{-\mu_1})}{2\epsilon\sqrt{-\mu_1}}\Big\}dz \\ &< 2\Big(1 - \frac{1}{3\epsilon}\Big)(-\mu_1) \end{aligned}$$

for $\epsilon > 0$. □

From the above lemma, we remark that the condition $\epsilon \geq 1/3$ is a necessary condition in order that the orbit $y = y(x)$ starting from the initial point $(0, 2\epsilon\sqrt{-\mu_1})$ intersects the half-line $m^+ = \{(z, y) \mid z = 2\sqrt{-\mu_1},\ y \geq 0\}$. Thus, if $1/3 \leq \epsilon < 1$, from Lemma 7, Lemma 8 and the uniqueness of the solution for the initial value problem, the closed orbit C must contain two equilibrium points O and A. This also contradict to the Poincaré index theorem.

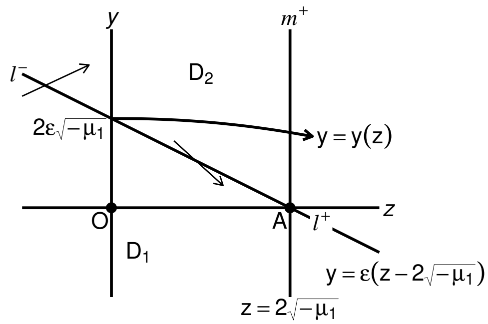


Figure for $1/3 \leq \epsilon < 1$

Therefore, we conclude from the Cases (I) and (II) that if $\epsilon = \sqrt{-\mu_1} - \mu_2 \geq 1/3$, then system (1.1) has no limit cycles.

The proof of the theorem is completed now. □

Example. Consider system (1.1) with $\mu_1 = -19$ and $\mu_2 = 4$. The pair (μ_1, μ_2) does not belong to the set E, but the set F. This is an example which can not apply Proposition 2. Using Theorem 3, it follows that this system has no limit cycles.

4. Relation with previous works

To discuss on the relation Theorem 3 and the result of Gasull et al. [1], we use the same signs as [1]. Let $\mu_1 = -m^2$ $(m > 0)$ and $\mu_2 = b > 0$ for system (1.1).

Using b and m, the set F in Theorem 3 is written as the following

$$F = \left\{ (m, b) \mid b \leq m - \frac{1}{3} \right\}.$$

Our purpose in this section is to improve the results (Theorems 1 and 2) in [1] partially. The following is known in [1, 2, 8] or [10].

Proposition 9. *No limit cycles of system* (1.1) *exist if and only if* $b \leq b^*(m)$ *or* $b \geq m$*, for an unknown function* $b^*(m)$*.*

We note that the condition $b \geq m$ coincides with E(iii) in Proposition 2. Perko in [10] gave the function $b^*(m) = 5m/7 + O(m^2)$. Gasull et al. [2] improved it and gave the global lower and upper bounds of $b^*(m)$ (see the Theorem 2 in [1]) as follows.

$$\max\left(\frac{5m}{7}, m - 1\right) < b^*(m) < \min\left(\frac{(5 + \frac{37}{12}m)m}{7 + \frac{37}{12}m}, m - 1 + \frac{25}{7m}\right).$$

We consider on the positions of two curves $b = m - 1/3$ and $b = b^*(m)$. If $m \geq 12/5$, then we have

$$\min\left(\frac{(5 + \frac{37}{12}m)m}{7 + \frac{37}{12}m}, m - 1 + \frac{25}{7m}\right) \leq m - \frac{1}{3}.$$

Thus, we get $b^*(m) < m - 1/3$. If $0 < m \leq 7/6$, then it holds that

$$\left(m - \frac{1}{3} \leq\right) \frac{5m}{7} < b^*(m) < \frac{(5 + \frac{37}{12}m)m}{7 + \frac{37}{12}m}. \tag{4.1}$$

Moreover, remark that there exists $m^* \in (7/6, 12/5)$ such that $b^*(m^*) = m^* - 1/3$. Thus, we have $b^*(m) > m - 1/3$ for $0 < m < m^*$ and $b^*(m) \leq m - 1/3$ for $m^* \leq m < 12/5$. Therefore, we conclude by combinating Theorem 3 with the Theorem 2 in [1] the following

Theorem 10. *Let*

$$G_1 = \left\{ (m, b) \mid b \leq m - \frac{1}{3} \;\; \textit{and} \;\; m \geq m^* \right\},$$

$$G_2 = \left\{ (m, b) \mid b \leq b^*(m) \;\; \textit{and} \;\; 0 < m < m^* \right\},$$

$$G_3 = \left\{ (m, b) \mid b \geq m \right\},$$

where $b^*(m)$ *is the function satisfying the inequality* (4.1) *and* $m^* \in (7/6, 12/5)$*.*

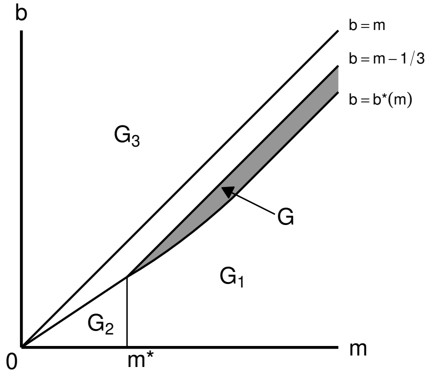
If the parameter pair (m, b) *belongs to the set* $G_1 \cup G_2 \cup G_3$*, then system* (1.1) *has no limit cycles.*

Remark that the set $G = \left\{ (m, b) \mid b^*(m) \leq b \leq m - \frac{1}{3} \right\}$ is an improved region for the non-existence of limit cycles.

From the above theorem, [5] and [8], we have the following

Corollary 11. *Let the parameter pair* (m, b) *belongs to the set* $\mathbb{R}^2 - (G_1 \cup G_2 \cup G_3)$. *If the limit cycle for system* (1.1) *exists, then it is at most one, hyperbolic and unstable.*

The shaped region G below expresses an improved non-existence region corresponds to the Figure 2 in [1].



An improved non-existence region G

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