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LOCALLY FUNCTIONALLY COUNTABLE SUBALGEBRA OF $\mathcal{R}(L)$

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Abstract. Let $L_c(X) = \{f \in C(X) \colon \overline{C_f} = X\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leqslant \aleph_0$. In this paper, we present a pointfree topology version of $L_c(X)$, named $\mathcal{R}_{\ell c}(L)$. We observe that $\mathcal{R}_{\ell c}(L)$ enjoys most of the important properties shared by $\mathcal{R}(L)$ and $\mathcal{R}_c(L)$, where $\mathcal{R}_c(L)$ is the pointfree version of all continuous functions of $C(X)$ with countable image. The interrelation between $\mathcal{R}(L)$, $\mathcal{R}_{\ell c}(L)$, and $\mathcal{R}_c(L)$ is examined. We show that $L_c(X) \cong \mathcal{R}_{\ell c}\big(\mathfrak{O}(X)\big)$ for any space X. Frames L for which $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$ are characterized.

1. Introduction

In this paper, all spaces are assumed to be Tychonoff, all frames are completely regular, and all rings are commutative with an identity element.

The notation $C(X)$ denotes the ring of all real-valued continuous functions on a topological space X (see [12]). Let $C_c(X)$ (resp. $C^F(X)$) denote the ring of all continuous functions of $C(X)$ with the countable (resp. finite) image. The ring $C_c(X)$ was introduced and studied in [10]. This subalgebra has more attendance recently; see, for example, [1, 4, 11, 14, 17, 18]. In [16], the authors introduced and studied the ring $\mathcal{R}_c(L)$ as the pointfree topology version of $C_c(X)$ (see also [6, 8, 9]). By $L_c(X)$, we mean the ring of all continuous functions that C_f is dense in X for $f \in C(X)$, where $C_f = \bigcup\{U \colon U \in \mathfrak{O}(X) \text{ and } |f(U)| \leqslant \aleph_0\}$; see [15]. Note that $C_c(X)$ is the largest subring of $C(X)$ whose elements have the countable image and that the subring $L_c(X)$ of $C(X)$ lies between $C_c(X)$ and $C(X)$. This motivates us to introduce this subring in a pointfree topology, named, $\mathcal{R}_{\ell c}(L)$.

A brief outline of this paper is as follows. In Section 2, we review, some definitions and results of frames and continuous functions.

In Section 3, we present a new subring of $\mathcal{R}(L)$ that contains $\mathcal{R}_c(L)$. We define $\mathcal{R}_{\ell c}(L)$ the set of all $\alpha \in \mathcal{R}(L)$ such that $(C_\alpha)^* = \bot$, where C_α is the join of all elements $a \in L$ with $\alpha|_a \in \mathcal{R}_c({\downarrow} a)$ (see Definition 3.1). We show that $\mathcal{R}_{\ell c}(L)$ is a subring of $\mathcal{R}(L)$. We observe that $\mathcal{R}_{\ell c}(L)$ enjoys most of the important properties

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that are shared by $\mathcal{R}(L)$ and $\mathcal{R}_c(L)$. Next, we introduce other subrings of $\mathcal{R}(L)$ (see Definition 3.18) and study their relations with $\mathcal{R}(L)$, $\mathcal{R}_c(L)$, and $\mathcal{R}_{\ell c}(L)$ (see Proposition 3.21).

In Section 4, we prove the equality of $\mathcal{R}_{\ell c}(L)$ and $\mathcal{R}(L)$ under certain conditions (see Propositions 4.5 and 4.7). Analogous to the main objective of research in the context $\mathcal{R}(L)$, we will try to study some useful facts about $\mathcal{R}_{lc}(L)$ and algebraic properties of $\mathcal{R}_{\ell c}(L)$ (see Proposition 4.13).

In the final section, we study the constant functions that are obtained from the restriction of a frame map $\alpha \in \mathcal{R}(L)$ to the codomain M for every sublocale M of L, and we denote $\mathcal{R}_{(M,\text{constant})}(L)$ to be the set of all $\alpha \in \mathcal{R}(L)$ such that $\alpha|^M \in \mathcal{R}^1(M)$. A relation between $\mathcal{R}_{(M,\text{constant})}(L)$ and $\mathcal{R}_c(L)$ is investigated.

2. Preliminaries

2.1. Functionally and locally functionally countable subalgebra of $C(X)$. We know $L_c(X) = \{f \in C(X)\colon \overline{C_f} = X\}$, where $C_f = \bigcup\{U\colon U \in \mathfrak{O}(X)$ and $|f(U)| \leqslant \aleph_0\}$. In [15], it was proved that $L_c(X)$ is a subalgebra as well as a sublattice of $C(X)$ containing $C_c(X)$, and this subring is called *the locally functionally countable subalgebra* of $C(X)$. The properties of the subalgebra $L_c(X)$ were mentioned in [15]. Similar to the above definition, $L_F(X)$ and $L_1(X)$ are the locally functionally finite and constant, respectively.

2.2. Frames and their homomorphism. Our notation and terminology for frames and locales will be that of [13] and [19]. We shall not discourse at length upon the rudiments of pointfree topology here, however, we recall some basic notion.

A *frame* (or *locale*) is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s\colon s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote by $\perp$ and $\top$, respectively, the bottom and the top elements of L. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}(X)$. An element $p \neq \top$ is a *prime* in a frame L if $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. The set of all prime elements of L is denoted by ΣL.

Every frame is a complete Heyting algebra with the Heyting implication given by

$$a \to b = \bigvee\{x \in L\colon a \wedge x \leq b\}.$$

The *pseudocomplement* of $a \in L$ is the element $a^* = a \to \perp = \bigvee\{x \in L\colon x \wedge a = \perp\}$. If $a \vee a^* = \top$, then a is said to be *complemented.*

Recall from [3] (see also [2]) that the frame of reals $\mathcal{L}(\mathbb{R})$ is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s)\colon p < r < s < q\}$.

(R4) $\top = \bigvee\{(p, q)\colon p, q \in \mathbb{Q}\}$.

For every $p,\, q \in \mathbb{Q}$, put

$$\langle p, q\rangle := \{x \in \mathbb{Q} \colon p < x < q\} \quad \text{and} \quad \rrbracket p, q\llbracket := \{x \in \mathbb{R} \colon p < x < q\}\,.$$

Corresponding to every operation $\diamond\colon \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $\diamond \in \{+, \cdot, \wedge, \vee\}$) we define an operation on $\mathcal{R}(L)$, denoted by the same symbol $\diamond$, by

$$\alpha \diamond \beta\,(p, q) = \bigvee \big\{\alpha(r, s) \wedge \beta(u, w) \colon \langle r, s\rangle \diamond \langle u, w\rangle \subseteq \langle p, q\rangle\big\},$$

where $\langle r, s\rangle \diamond \langle u, w\rangle \subseteq \langle p, q\rangle$ means that for each $r < x < s$ and $u < y < w$, we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}(L)$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$. An element α of $\mathcal{R}(L)$ is said to be bounded if there exist $p, q \in \mathbb{Q}$ such that $\alpha(p, q) = \top$. The set of all bounded elements of $\mathcal{R}(L)$ is denoted by $\mathcal{R}^*(L)$, which is a sub-f-ring of $\mathcal{R}(L)$. The *cozero map* is the map $\operatorname{coz}\colon \mathcal{R}(L) \to L$, defined by

$$\operatorname{coz}(\alpha) = \bigvee \big\{\alpha(p, 0) \vee \alpha(0, q) \colon p, q \in \mathbb{Q}\big\}\,.$$

A *cozero element* of L is an element of the form $\operatorname{coz}(\alpha)$ for some $\alpha \in \mathcal{R}(L)$ (see [3]). The cozero part of L, denoted by $\operatorname{Coz}(L)$, is the set of all cozero elements. It is well known that L is completely regular if and only if $\operatorname{Coz}(L)$ generates L. The homomorphism $\tau\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{O}(\mathbb{R})$ given by $(p, q) \mapsto \rrbracket p, q\llbracket$ is an isomorphism (see [3, Proposition 2]).

For a topology space X and every $A \subseteq X$ and $f \in C(X)$, we have $(f|_A)^{-1}\colon \mathfrak{O}(\mathbb{R}) \longrightarrow \mathfrak{O}(A)$ with $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$ for every $U \in \mathfrak{O}(\mathbb{R})$. Also, for every $\alpha \in \mathcal{R}(L)$ and every $a \in L$, we have $\alpha|_a\colon \mathfrak{L}(\mathbb{R}) \to {\downarrow} a$ with $\alpha|_a(p, q) = \alpha(p, q) \wedge a$.

An element $\alpha \in \mathcal{R}(L)$ is said to have the *pointfree countable image* if there is a countable subset S of $\mathbb{R}$ with $\alpha \blacktriangleleft S$ (we say α *overlap* of S), where $\alpha \blacktriangleleft S$ means that $\tau(u) \cap S = \tau(v) \cap S$ implies $\alpha(u) = \alpha(v)$ for any $u, v \in \mathfrak{L}(\mathbb{R})$. In [16], it is shown that for any $\alpha \in \mathcal{R}(L)$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent:

(1) $\alpha \blacktriangleleft S$,

(2) $\tau(p, q) \cap S = \tau(v) \cap S$ implies $\alpha(p, q) = \alpha(v)$, for any $v \in \mathfrak{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$, and

(3) $\tau(p, q) \cap S \subseteq \tau(v) \cap S$ implies $\alpha(p, q) \leq \alpha(v)$, for any $v \in \mathfrak{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$.

For any frame L, we put

$$\mathcal{R}_c(L) := \{\alpha \in \mathcal{R}(L) \colon \alpha \text{ has the pointfree countable image }\}.$$

For any completely regular frame L, the set $\mathcal{R}_c(L)$ is a sub-f-ring of $\mathcal{R}(L)$. The ring $\mathcal{R}_c(L)$ is introduced as the pointfree version of $C_c(X)$ (see [16]). Also, $\mathcal{R}^F(L)$ is the pointfree version of $C^F(X)$. We denote the set of all constant functions of $\mathcal{R}(L)$ by $\mathcal{R}^1(L)$.

2.3. **Sublocales.** A *sublocale* of a locale L is a subset $S \subseteq L$ such that

(i) for every $A \subseteq S$, $\bigwedge A \in S$, and

(ii) for every $a \in L$ and $s \in S$, $a \to s \in S$.

The lattice of all sublocales of L is denoted by $\mathcal{S}(L)$. The meet in this lattice is intersection. The join of any collection $\{S_i \colon i \in I\} \subseteq \mathcal{S}(L)$ is given by

$$\bigvee_i S_i = \Big\{ \bigwedge M \colon M \subseteq \bigcup_i S_i \Big\} .$$

The lattice $\mathcal{S}(L)$, partially ordered by inclusion, is a *coframe*. The smallest sublocale of L is $\mathsf{O} = \{\top\}$, which is called the *void* sublocale. Indeed the largest is L.

3. The subalgebra $\mathcal{R}_{lc}(L)$ of $\mathcal{R}(L)$

In this section, we introduce the pointfree topology version of the ring $L_c(X)$. We begin with the following definition.

Definition 3.1. For every $\alpha \in \mathcal{R}(L)$, we put

$$\mathscr{C}_\alpha = \big\{ a \in L \colon \alpha|_a \in \mathcal{R}_c(\downarrow a) \big\} \quad \text{and} \quad C_\alpha = \bigvee \mathscr{C}_\alpha .$$

We say that an element α of $\mathcal{R}(L)$ has *the pointfree locally countable image* if $(C_\alpha)^* = \bot$. We put

$$\mathcal{R}_{\ell c}(L) := \big\{ \alpha \in \mathcal{R}(L) \colon \alpha \text{ has the pointfree locally countable image} \big\} .$$

Also, $\mathcal{R}_{\ell c}(L)$ is called the pointfree locally functionally countable image subring of $\mathcal{R}(L)$.

We show that this definition is a conservative extension for continuous functions on topological spaces. Throughout this article, for $f \in C(X)$ and the isomorphism $\tau \colon \mathcal{L}(\mathbb{R}) \longrightarrow \mathfrak{O}(\mathbb{R})$, the frame map $f^{-1} \circ \tau \colon \mathcal{L}(\mathbb{R}) \longrightarrow \mathfrak{O}(X)$ is denoted by f_τ. Note that for any $p < q$ in $\mathbb{Q}$, $f_\tau(p,q) = f^{-1}(\rrbracket p, q \llbracket)$ and $f_\tau|_U = (f|_U)_\tau$ for every $U \in \mathfrak{O}(X)$. Therefore, $C_f = C_{f_\tau}$ for every $f \in C(X)$. By this fact, the following proposition holds.

Proposition 3.2. *If $f \in C(X)$, then $f \in L_c(X)$ if and only if $f_\tau \in \mathcal{R}_{lc}\big(\mathfrak{O}(X)\big)$.*

Recall from [3] that for any space X, there is a one-one onto map **Frm** $(\mathcal{L}(\mathbb{R}), \mathfrak{O}(X)) \to \mathbf{Top}(X, \mathbb{R})$ given by the correspondence $\varphi \longmapsto \widetilde{\varphi}$ such that

$$p < \widetilde{\varphi}(x) < q \quad \text{if and only if} \quad x \in \varphi(p,q)$$

whenever $p < q$ in $\mathbb{Q}$ (also, see [5]). This means that $\mathcal{R}(\mathfrak{O}X) \cong C(X)$ for any topological space X. Here, we give a counterpart of this result.

Proposition 3.3. *For any space X, $\mathcal{R}_{\ell c}\big(\mathfrak{O}(X)\big) \cong L_c(X)$.*

Proof. We define $\theta \colon L_c(X) \to \mathcal{R}_{\ell c}\big(\mathfrak{O}(X)\big)$ by $\theta(g) = g_\tau$. By Proposition 3.2, θ is well defined and injective. Let $\alpha \in \mathcal{R}_{lc}\big(\mathfrak{O}(X)\big)$. Then $\alpha \circ \tau^{-1} \colon \mathfrak{O}(\mathbb{R}) \longrightarrow \mathfrak{O}(X)$ is a frame map, and hence by [5, Theorem 1], there exists a unique continuous function $f \colon X \longrightarrow \mathbb{R}$, such that $f^{-1} = \alpha \circ \tau^{-1}$. Therefore, $\theta(f) = f^{-1} \circ \tau = \alpha$. Now let $p, q \in \mathbb{Q}$ and let $U \in \mathfrak{O}(X)$. Then

$$(f^{-1}|_U)\big(\tau(p,q)\big) = f^{-1}(\rrbracket p, q \llbracket) \wedge U = \alpha\big(\tau^{-1}(\rrbracket p, q \llbracket)\big) \wedge U = \alpha(p,q) \wedge U = \alpha|_U(p,q) .$$

Therefore, $(f^{-1}|_U) \circ \tau = \alpha|_U$. Now, since $f|_U$ has the countable image, then $(f^{-1}|_U)^{-1} \circ \tau = \alpha|_U$ has a pointfree countable image ([16, Proposition 3.11]). Therefore, $C_\alpha = C_f$ and hence $f \in L_c(X)$. □

Lemma 3.4. *For every $\alpha \in \mathcal{R}(L)$ and every $a \in L$, if $\alpha \in \mathcal{R}_c(L)$, then $\alpha|_a \in \mathcal{R}_c(\downarrow a)$.*

Proof. It is evident. □

By this lemma, it manifests that $\mathcal{R}^F(L) \subseteq \mathcal{R}_c(L) \subseteq \mathcal{R}_{\ell c}(L) \subseteq \mathcal{R}(L)$.

Remark 3.5. Note that the equality between these objects may not necessarily hold. For example, let the basic neighborhood of x be the set $\{x\}$, for each point $x \geq \sqrt{2}$ and for the rest of the real numbers (i.e., $x < \sqrt{2}$), let the basic neighborhoods be the usual open intervals containing x. This is a topology τ on $\mathbb{R}$ and in this case, we put $X = \mathbb{R}$. Clearly, X is a completely regular Hausdorff space, which is finer than the usual topology of $\mathbb{R}$. Consider the function $f\colon X \to \mathbb{R}$ defined by $f(x) = x$ for $x \geq \sqrt{2}$ and $f(x) = \sqrt{2}$, otherwise, so we have $f \in L_c(X) \setminus C_c(X)$ (for more details, see [15]). Proposition 3.3 implies $f_\tau \in \mathcal{R}_{\ell c}\big(\mathfrak{O}(X)\big) \setminus \mathcal{R}_c\big(\mathfrak{O}(X)\big)$ (because $C_c(X) \cong \mathcal{R}_c\big(\mathfrak{O}(X)\big)$, by [6, Lemma 3.16]). Now, consider the identity function $\mathrm{id}\colon X \to \mathbb{R}$, which is continuous. Then $\mathrm{id} \in C(X) \setminus L_c(X)$. It follows from Proposition 3.3 that $\mathrm{id}_\tau \in \mathcal{R}\big(\mathfrak{O}(X)\big) \setminus \mathcal{R}_{\ell c}\big(\mathfrak{O}(X)\big)$.

We need the following lemmas to show that $\mathcal{R}_{\ell c}(L)$ is a sub-f-ring and $\mathbb{R}$-subalgebra of $\mathcal{R}(L)$. The proof is routine, so we omit it.

Lemma 3.6. *Let $\alpha, \beta \in \mathcal{R}(L)$ and $a, b \in L$ be given. Then the following statements hold:*

(1) *If $\diamondsuit \in \{+, \cdot, \wedge, \vee\}$, then $\alpha\diamondsuit\beta|a = \alpha|_a\diamondsuit\beta|_a$.*

(2) *If $a \leq b$ and $\alpha|_b \in \mathcal{R}_c(\downarrow b)$, then $\alpha|_a \in \mathcal{R}_c(\downarrow a)$.*

Lemma 3.7. *If α, $\beta \in \mathcal{R}(L)$, then $C_{\alpha\diamondsuit\beta} \geq C_\alpha \wedge C_\beta$ for every $\diamondsuit \in \{+, \cdot, \wedge, \vee\}$.*

Proof. Let a, $b \in L$ such that $\alpha|_a \in \mathcal{R}_c(\downarrow a)$ and $\beta|_b \in \mathcal{R}_c(\downarrow b)$. By part (2) of Lemma 3.6, we have $\alpha|_{a\wedge b}, \beta|_{a\wedge b} \in \mathcal{R}_c(\downarrow a \wedge b)$. Now, by part (1) of Lemma 3.6, we have

$$\alpha\diamondsuit\beta|_{a\wedge b} = \alpha|_{a\wedge b}\diamondsuit\beta|_{a\wedge b} \in \mathcal{R}_c\big(\downarrow\big(a \wedge b\big)\big)$$

for every $\diamondsuit \in \{+, \cdot, \wedge, \vee\}$. Therefore,

$$\begin{aligned} C_\alpha \wedge C_\beta &= \bigvee \big\{a \wedge b\colon a, b \in L, \alpha|_a \in \mathcal{R}_c(\downarrow a), \beta|_b \in \mathcal{R}_c(\downarrow b)\big\} \\ &\leq \bigvee \{c \in L\colon \alpha\diamondsuit\beta|_c \in \mathcal{R}_c(\downarrow c)\} \\ &= C_{\alpha\diamondsuit\beta}, \end{aligned}$$

for every $\diamondsuit \in \{+, \cdot, \wedge, \vee\}$. □

It is evident that for every $0 \neq r \in \mathbb{R}$, $\alpha|_a \blacktriangleleft S$ if and only if $r\alpha|_a \blacktriangleleft \{rx\colon x \in S\}$ for every $\alpha \in \mathcal{R}(L)$ and every $a \in L$. By this fact and Lemma 3.7, the following proposition holds.

Proposition 3.8. *It follows that $\mathcal{R}_{\ell c}(L)$ is a sub-f-ring and an $\mathbb{R}$-subalgebra of $\mathcal{R}(L)$.*

Remark 3.9. Recall that $|\alpha| = \alpha \vee (-\alpha)$ for every $\alpha \in \mathcal{R}(L)$. For every $p, q \in \mathbb{Q}$, we have

$$|\alpha|(p,q) = |\alpha|(p,-) \wedge |\alpha|(-,q) = \begin{cases} \perp & \text{if} \quad q \leq 0, \\ -\alpha(p,q) \vee \alpha(p,q) & \text{if} \quad p \geq 0, \\ \alpha(-q,q) & \text{if} \quad p < 0 < q\,. \end{cases}$$

Now, let us state the results in relation to the absolute value function and the rings $\mathcal{R}_c(L)$ and $\mathcal{R}_{lc}(L)$.

Proposition 3.10. *If S is a subset of $\mathbb{R}$ and $|\alpha| \blacktriangleleft S$, then $|\alpha| \blacktriangleleft S \cap [0,\infty)$ for every $\alpha \in \mathcal{R}(L)$.*

Proof. Put $S_1 := S \cap [0,\infty)$. Let (p,q), $v \in \mathcal{L}(\mathbb{R})$ with $\tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$ be given. We show that $|\alpha|(p,q) \leq |\alpha|(v)$ by considering several cases. Therefore, $|\alpha| \blacktriangleleft S_1$.

First case. If $p \geq 0$, then $\tau(p,q) \cap S = \tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1 = \tau(v) \cap S$, which follows that $|\alpha|(p,q) \leq |\alpha|(v)$.

Second case. If $q \leq 0$, then $\perp = |\alpha|(p,q) \leq |\alpha|(v)$.

Third case. If $0 \in \tau(p,q) \cap S_1$, then $0 \in \tau(v) \cap S_1$. Therefore, there exists an element $n \in \mathbb{N}$ such that $\tau(\frac{-1}{n}, \frac{1}{n}) \subseteq \tau(v) \cap \tau(p,q)$. On the other hand, we have

$$\tau(p,q) = \tau\Big(p, \frac{-1}{n+1}\Big) \cup \tau\Big(\frac{-1}{n}, \frac{1}{n}\Big) \cup \tau\Big(\frac{1}{n+1}, q\Big),$$

and so $|\alpha|(p,q) = |\alpha|(p, \frac{-1}{n+1}) \vee |\alpha|(\frac{-1}{n}, \frac{1}{n}) \vee |\alpha|(\frac{1}{n+1}, q)$.

- Since $\tau(\frac{-1}{n}, \frac{1}{n}) \subseteq \tau(v)$, then $|\alpha|(\frac{-1}{n}, \frac{1}{n}) \leq |\alpha|(v)$.
- Since $\tau(p, \frac{-1}{n+1}) \cap S_1 \subseteq \tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$, case (2) implies $|\alpha|(p, \frac{-1}{n+1}) \leq |\alpha|(v)$.
- Since $\tau(\frac{1}{n+1}, q) \cap S_1 \subseteq \tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$, , case (1) implies $|\alpha|(\frac{1}{n+1}, q) \leq \alpha|(v)$.

Therefore, $|\alpha|(p,q) \leq |\alpha|(v)$.

Fourth case. If $0 \in \tau(p,q)$ and $0 \notin S_1$, then $0 \notin S$. Since $\big((-\infty,0) \cup (0,\infty)\big) \cap S = \mathbb{R} \cap S$, then $|\alpha|\big((-,0) \vee (0,-)\big) = |\alpha|(\top) = \top$. Therefore

$$\begin{aligned} |\alpha|(p,q) &= |\alpha|(p,q) \wedge \top \\ &= |\alpha|(p,q) \wedge \big(|\alpha|(-,0) \vee |\alpha|(0,-)\big) \\ &= |\alpha|\big((p,q) \wedge (-,0)\big) \vee |\alpha|\big((p,q) \wedge (0,-)\big) \\ &= |\alpha|(p,0) \vee |\alpha|(0,q)\,. \end{aligned}$$

- Since $\tau(p,0) \cap S_1 \subseteq \tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$, case (2) implies $|\alpha|(p,0) \leq |\alpha|(v)$.
- Since $\tau(0,q) \cap S_1 \subseteq \tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$, case (1) implies $|\alpha|(0,q) \leq |\alpha|(v)$.

So, given the above relations, it follows that $|\alpha|(p,q) \leq |\alpha|(v)$. $\square$

Proposition 3.11. *If $S \subseteq [0,\infty)$ and $|\alpha| \blacktriangleleft S$, then $\alpha \blacktriangleleft S \cup \{-x : x \in S\}$ for every $\alpha \in \mathcal{R}(L)$.*

Proof. Put $S_1 := S \cup \{-x : x \in S\}$, and let $(p,q), v \in \mathcal{L}(\mathbb{R})$ with $\tau(p,q) \cap S_1 \subseteq \tau(v) \cap S_1$ be given. For every $v \in \mathcal{L}(\mathbb{R})$, set $v^+ = \tau^{-1}\big(\tau(v) \cap (0,\infty)\big)$ and $v^- = \tau^{-1}\big(\tau(v) \cap (-\infty,0)\big)$. We show that $|\alpha|(p,q) \leq |\alpha|(v)$ by considering several cases. Therefore, $|\alpha| \blacktriangleleft S_1$.

First case. If $p \geq 0$, then

$$\begin{aligned}\tau(p,q) \cap S &= \tau(p,q) \cap S_1 \\ &= \tau(p,q) \cap S_1 \cap (0,\infty) \\ &\subseteq \tau(v) \cap S_1 \cap (0,\infty) \\ &= \tau(v) \cap S \cap (0,\infty) \\ &= \tau(v^+) \cap S\,.\end{aligned}$$

Hence $|\alpha|(p,q) \leq |\alpha|(v^+)$. Therefore, Remark 3.9 implies

$$\begin{aligned}\alpha(p,q) &= \alpha(p,q) \wedge \big(-\alpha(p,q) \vee \alpha(p,q)\big) \\ &= \alpha(p,q) \wedge |\alpha|(p,q) \\ &\leq \alpha(p,q) \wedge |\alpha|(v^+) \\ &= \alpha(p,q) \wedge \Big(\alpha\big(\tau^{-1}(\{-x : x \in v^+\})\big) \vee \alpha(v^+)\Big) \\ &= \alpha\big((p,q) \wedge \tau^{-1}(\{-x : x \in v^+\})\big) \vee \big(\alpha(p,q) \wedge \alpha(v^+)\big) \\ &= \alpha(p,q) \wedge \alpha(v^+) \\ &\leq \alpha(v^+) \\ &\leq \alpha(v)\,.\end{aligned}$$

Second case. If $q \leq 0$, then by Remark 3.9,

$$\alpha(p,q) = \alpha(p,q) \wedge \big(-\alpha(p,q) \vee \alpha(p,q)\big) = \alpha(p,q) \wedge |\alpha|(p,q) = \alpha(p,q) \wedge \perp \leq \alpha(v)\,.$$

The proofs of parts (3) and (4) are similar to those in parts (3) and (4) of the previous proposition. □

Proposition 3.12. *For every $\alpha \in \mathcal{R}(L)$, $\alpha \in \mathcal{R}_c(L)$ if and only if $|\alpha| \in \mathcal{R}_c(L)$.*

Proof. *Necessary.* If $\alpha \in \mathcal{R}_c(L)$, then $(-\alpha) \in \mathcal{R}_c(L)$ and hence $|\alpha| = \alpha \vee (-\alpha) \in \mathcal{R}_c(L)$, because $\mathcal{R}_c(L)$ is an f-ring.

Sufficiency. By Propositions 3.10 and 3.11, it is clear. □

For the proof of the next lemma, see [6, Lemma 3.7].

Lemma 3.13. *Let α be a unit element of $\mathcal{R}(L)$. Then $\alpha \in \mathcal{R}_c(L)$ if and only if $\alpha^{-1} \in \mathcal{R}_c(L)$.*

The previous propositions lead to the next result.

Proposition 3.14. *For every $\alpha, \beta \in \mathcal{R}(L)$, the following statements hold:*

(1) $C_{|\alpha|} = C_\alpha$.

(2) *If α is a unit element in $\mathcal{R}(L)$, then $C_{\alpha^{-1}} = C_\alpha$.*

(3) $C_{-\alpha} = C_\alpha$.

(4) *If $\alpha, \beta \in \mathcal{R}_{lc}(L)$, then $(C_\alpha \wedge C_\beta)^* = \bot$.*

Proof. (1) It is evident.

(2) Let α be a unit element $\mathcal{R}(L)$. It is enough to show that $\mathscr{C}_\alpha = \mathscr{C}_{\alpha^{-1}}$.

First, we show that for every $\bot \neq a \in L$ and the unit element $\alpha \in \mathcal{R}(L)$, we have $\alpha|_a$ is unit and $(\alpha|_a)^{-1} = \alpha^{-1}|_a$. Clearly, $\alpha|_a$ is unit, because

$$\operatorname{coz}(\alpha|_a) = \alpha|_a(-,0) \vee \alpha|_a(0,-) = a \wedge \operatorname{coz}(\alpha) = a \wedge \top = a = \top_{\downarrow a}.$$

For every $p, q \in \mathbb{Q}$,

$$\begin{aligned}
\alpha^{-1}|_a(p,q) &= \alpha^{-1}(p,q) \wedge a \\
&= \alpha\Big(\tau^{-1}\Big(\Big\{\frac{1}{x} : x \in \tau(p,q), x \neq 0\Big\}\Big)\Big) \wedge a \\
&= \alpha|_a\Big(\tau^{-1}\Big(\Big\{\frac{1}{x} : x \in \tau(p,q), x \neq 0\Big\}\Big)\Big) \\
&= (\alpha|_a)^{-1}(p,q)\,.
\end{aligned}$$

Therefore, $(\alpha|_a)^{-1} = \alpha^{-1}|_a$. Now, by this relation and Lemma 3.13, we have

$$a \in \mathscr{C}_\alpha \Leftrightarrow \alpha|_a \in \mathcal{R}_c(\downarrow a) \Leftrightarrow (\alpha|_a)^{-1} \in \mathcal{R}_c(\downarrow a) \Leftrightarrow \alpha^{-1}|_a \in \mathcal{R}_c(\downarrow a) \Leftrightarrow a \in \mathscr{C}_{\alpha^{-1}}\,.$$

(3) It is clear, because $(-\alpha)|_a = -(\alpha|_a)$ for every $\alpha \in \mathcal{R}(L)$ and every $a \in L$.

(4) The following relation completes the proof:

$$\begin{aligned}
(C_\alpha \wedge C_\beta)^* &= (C_\alpha \wedge C_\beta)^{***} = \big((C_\alpha \wedge C_\beta)^{**}\big)^* \\
&= \big((C_\alpha)^{**} \wedge (C_\beta)^{**}\big)^* = (\top \wedge \top)^* = \top^* = \bot\,.
\end{aligned}$$

□

We need the following lemma to show that $\mathcal{R}_{lc}(L)$ is a sublattice of $\mathcal{R}(L)$.

Proposition 3.15. *For every $\alpha \in \mathcal{R}(L)$, the following statements hold:*

(1) *$\alpha \in \mathcal{R}_{\ell c}(L)$ if and only if $|\alpha| \in \mathcal{R}_{\ell c}(L)$.*

(2) *Let α be a unit element in $\mathcal{R}(L)$. Then $\alpha \in \mathcal{R}_{\ell c}(L)$ if and only if $\alpha^{-1} \in \mathcal{R}_{\ell c}(L)$.*

(3) *If $\alpha \in \mathcal{R}_{\ell c}(L)$, then $-\alpha \in \mathcal{R}_{\ell c}(L)$.*

Proof. It is clear by Proposition 3.14. □

Corollary 3.16. *It follows that $\mathcal{R}_{\ell c}(L)$ is a sublattice of $\mathcal{R}(L)$.*

Corollary 3.17. *For every $\alpha \in \mathcal{R}_{\ell c}(L)$, $\operatorname{coz}(\alpha) = \top$ if and only if α is a unit element in $\mathcal{R}_{\ell c}(L)$.*

Here, we introduce another subring of $\mathcal{R}(L)$.

Definition 3.18. For every $\alpha \in \mathcal{R}(L)$, we put

$$\mathscr{F}_\alpha = \{a \in L\colon \alpha|_a \in \mathcal{R}^F(\downarrow a)\} \quad \text{and} \quad F_\alpha = \bigvee \mathscr{F}_\alpha\,.$$

An element α of $\mathcal{R}(L)$ has *the pointfree locally finite image* if $(F_\alpha)^* = \bot$. We define

$$\mathcal{R}^F_\ell(L) := \{\alpha \in \mathcal{R}(L)\colon \alpha \text{ has the pointfree locally finite image}\}\,.$$

Also, for every $\alpha \in \mathcal{R}(L)$, we put

$$\mathfrak{l}_\alpha = \{a \in L\colon \alpha|_a \in \mathcal{R}^1(\downarrow a)\} \quad \text{and} \quad 1_\alpha = \bigvee \mathfrak{l}_\alpha\,.$$

An element α of $\mathcal{R}(L)$ has *the pointfree locally constant image* if $(1_\alpha)^* = \bot$. We define

$$\mathcal{R}^1_\ell(L) := \{\alpha \in \mathcal{R}(L)\colon \alpha \text{ has the pointfree locally constant image}\}\,.$$

One can easily see that $\mathcal{R}^1(L) \cong \mathbb{R}$.

Remark 3.19. Similar to Proposition 3.3, we can see that $C^F_\ell(X) \cong \mathcal{R}^F_\ell\big(\mathfrak{O}(X)\big)$ and $C^1_\ell(X) \cong \mathcal{R}^1_\ell\big(\mathfrak{O}(X)\big)$ for any space X. We note that Proposition 3.8 and Corollary 3.16 are also valid for $\mathcal{R}^F_\ell(L)$ and $\mathcal{R}^1_\ell(L)$.

Proposition 3.20. *For any frame L, $\mathcal{R}^F(L) \subseteq \mathcal{R}^F_\ell(L)$ and $\mathcal{R}^1(L) \subseteq \mathcal{R}^1_\ell(L)$.*

Proof. Let $\alpha \in \mathcal{R}^F(L)$ be given. Then there exists a finite subset S of $\mathbb{R}$ such that $\alpha \blacktriangleleft S$. Suppose that $a \in L$ and that $u, v \in \mathcal{L}(\mathbb{R})$ such that $\tau(u) \cap S = \tau(v) \cap S$. Since $\alpha \blacktriangleleft S$, we have $\alpha(u) = \alpha(v)$, and so $\alpha(u) \wedge a = \alpha(v) \wedge a$, which implies that $\alpha|_a(u) = \alpha|_a(v)$. Thus, $\alpha|_a \in \mathcal{R}^F(\downarrow a)$, and so $(F_\alpha)^* = (\bigvee L)^* = (\top)^* = \bot$. Hence, $\alpha|_a \in \mathcal{R}^F_\ell(\downarrow a)$. $\square$

For every $r \in \mathbb{R}$, in [16], it is shown that $\alpha = \mathbf{r}$ if and only if $\alpha \blacktriangleleft \{r\}$. By this fact, we end this section with the next result.

Proposition 3.21. *For any frame L, we have $\mathcal{R}^1_\ell(L) \subseteq \mathcal{R}^F_\ell(L) \subseteq \mathcal{R}_{\ell c}(L) \subseteq \mathcal{R}(L)$.*

Proof. Let $\alpha \in \mathcal{R}^1_\ell(L)$ and let $a \in \mathfrak{l}_\alpha$. Then $\alpha|_a \in \mathcal{R}^1(\downarrow a)$. Hence, $\alpha|_a = \mathbf{r}$ for some $r \in \mathbb{R}$, which implies that $\alpha|_a \blacktriangleleft \{r\}$. This shows that $\alpha|_a \in \mathcal{R}^F(\downarrow a)$ and so $a \in \mathscr{F}_\alpha$. Therefore, $\mathfrak{l}_\alpha \subseteq \mathscr{F}_\alpha$. By the assumptions, we have $(F_\alpha)^* \leq (1_\alpha)^* = \bot$, which shows that $\alpha \in \mathcal{R}^F_\ell(L)$. The inclusion $\mathcal{R}^F_\ell(L) \subseteq \mathcal{R}_{\ell c}(L)$ is clear, because every finite set is countable. $\square$

4. $\mathcal{R}_{\ell c}(L)$ versus $\mathcal{R}(L)$ and $\mathcal{R}_c(L)$

We are interested in characterization frames L for which $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$. First, we give some definitions and notations.

Definition 4.1. [7] Let L be a lattice. Then the element $\bot < p \in L$ is called a *particle* if $p \leq \bigvee_i a_i$, whenever $\bigvee_i a_i$ exists, implies $p \leq a_i$ for some i.

For any frame L, we put $P(L) := \{p \in L\colon p \text{ is a particle of } L\}$.

Lemma 4.2 ([20]). *We have $\mathcal{R}(\mathbf{2}) \cong \mathbb{R}$, where $\mathbf{2} = \{\bot, \top\}$.*

Remark 4.3. Recall from [15, Proposition 2.11] that if $\big(X, \mathfrak{O}(X)\big)$ is a completely regular and Hausdorff topology space such that $I(X)$, the set of isolated points of X, is dense in X, then $L_1(X) = L_F(X) = L_c(X) = C(X)$. Now, we study this result in frames. Also, note that $U \in \mathfrak{O}(X)$ is a particle if and only if $|U| = 1$; therefore $\overline{I(X)} = X$ if and only if $\Big(\bigvee P\big(\mathfrak{O}(X)\big)\Big)^* = \bot$.

Proposition 4.4. *Let L be a Boolean algebra and let $\big(\bigvee P(L)\big)^* = \bot$. Then*

$$\mathcal{R}^1_\ell(L) = \mathcal{R}^F_\ell(L) = \mathcal{R}_{\ell c}(L) = \mathcal{R}(L)\,.$$

Proof. By Proposition 3.21, we have $\mathcal{R}^1_\ell(L) \subseteq \mathcal{R}^F_\ell(L) \subseteq \mathcal{R}_{\ell c}(L) \subseteq \mathcal{R}(L)$.

Conversely, it is enough to show that $\mathcal{R}(L) \subseteq \mathcal{R}^1_\ell(L)$. Let $\alpha \in \mathcal{R}(L)$ be given. If p is a particle element, then p is an atom and by Lemma 4.2, we have $\mathcal{R}(\downarrow p) = \mathcal{R}(\mathbf{2}) \cong \mathbb{R}$. Therefore, $\alpha|_p \in \mathcal{R}(\downarrow p) \cong \mathbb{R}$ implies that there is an element $r \in \mathbb{R}$ such that $\alpha|_p = \mathbf{r}$, which more implies $p \in \mathfrak{l}_\alpha$. This shows that $P(L) \subseteq \mathfrak{l}_\alpha$ and so $(\bigvee \mathfrak{l}_\alpha)^* \leq \big(\bigvee P(L)\big)^* = \bot$. Therefore, $\alpha \in \mathcal{R}^1_\ell(L)$. □

Here, we give a condition that is $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$.

Proposition 4.5. *For every frame L, $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$ if and only if for every $\alpha \in \mathcal{R}(L)$ and every $\bot \neq a \in L$, there exists an element $b \neq \bot$ such that $b \leq a$ and $\alpha|_b \in \mathcal{R}_c(\downarrow b)$.*

Proof. *Necessity.* Assume that $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$, that $\alpha \in \mathcal{R}(L)$, and that $\bot \neq a \in L$. Then $(C_\alpha)^* = \bot$ and we conclude $a \wedge c_\alpha \neq \bot$. Therefore, there exists an element $x \in \mathscr{C}_\alpha$ such that $x \wedge a \neq \bot$. Now, Lemma 3.6 implies $\alpha|_{(x\wedge a)} \in \mathcal{R}_c\big(\downarrow(x \wedge a)\big)$.

Sufficiency. Assume that $\alpha \in \mathcal{R}(L)$ and that $\bot \neq a \in L$. Then there exists an element $\bot \neq x_a \in L$ such that $x_a \leq a$ and $\alpha|_{x_a} \in \mathcal{R}_c(\downarrow x_a)$. Hence, for any $a \in L$, we have $(C_\alpha)^* \leq \bigwedge_{a\in L} x_a^*$. If $t = \bigwedge_{a\in L} x_a^* \neq \bot$, then, there exists an element $\bot \neq x_t \in L$ such that $x_t \leq t = \bigwedge_{a\in L} x_a^*$ and $\alpha|_{x_t} \in \mathcal{R}_c(\downarrow x_t)$. Therefore $x_t \leq x_t^* \wedge x_t = \bot$, which is a contradiction. Hence $\bigwedge_{a\in L} x_a^* = \bot$, which implies that $(C_\alpha)^* = \bot$ and we conclude $\alpha \in \mathcal{R}_{\ell c}(L)$. □

Proposition 4.6. *Consider the following conditions:*

(1) $\mathcal{R}_c(L) = \mathcal{R}(L)$.

(2) *For every $a \in \Sigma L$, there exists an element $b \in L$ such that $a \leq b$ and $\mathcal{R}_c(\downarrow b) = \mathcal{R}(\downarrow b)$.*

Then (1) *implies* (2)*, and if L is Lindelöf and $\bigvee \Sigma L = \top$, then* (2) *implies* (1)*.*

Proof. (1) ⇒ (2) It is sufficient to take $b = \top$ for every $a \in \Sigma L$.

(2) ⇒ (1) Let L be Lindelöf and let $\bigvee \Sigma L = \top$. For every $a \in \Sigma L$, there exists an element $x_a \in L$ such that $a \leq x_a$ and $\mathcal{R}(\downarrow x_a) = \mathcal{R}_c(\downarrow x_a)$. The assumptions imply that $\bigvee_{a\in\Sigma L} x_a = \top$ and so there is a family $\{a_n\}_{n\in\mathbb{N}} \subseteq \Sigma L$ such that $\bigvee_{n\in\mathbb{N}} x_{a_n} = \top$, because L is Lindelöf. Now, let $\alpha \in \mathcal{R}(L)$ be given. For every $n \in \mathbb{N}$, since $\alpha|_{x_{a_n}} \in \mathcal{R}(\downarrow x_{a_n}) = \mathcal{R}_c(\downarrow x_{a_n})$, we infer that there exists a countable subset $S_n \subseteq \mathbb{R}$ such that $\alpha|_{x_{a_n}} \blacktriangleleft S_n$. Put $S := \bigcup_{n\in\mathbb{N}} S_n$. Suppose that (p, q), $v \in \mathcal{L}(\mathbb{R})$ and that $\tau(p, q) \cap S \subseteq \tau(v) \cap S$. Then for every $n \in \mathbb{N}$, we have $\tau(p, q) \cap S_n \subseteq \tau(v) \cap S_n$,

which follows that $\alpha|_{x_{a_n}}(p,q) \leq \alpha|_{x_{a_n}}(v)$. Here

$$\alpha(p,q) = \alpha(p,q) \wedge \bigvee_{n\in\mathbb{N}} x_{a_n} = \bigvee_{n\in\mathbb{N}} \alpha|_{x_{a_n}}(p,q) \leq \bigvee_{n\in\mathbb{N}} \alpha|_{x_{a_n}}(v) = \alpha(v) \wedge \bigvee_{n\in\mathbb{N}} x_{a_n} = \alpha(v)\,.$$

Therefore, $\alpha \in \mathcal{R}_c(L)$. □

Proposition 4.7. *Let L be a frame such that for every $a \in \Sigma L$, there exists an element $b \in L$ such that $a \leq b$ and $\mathcal{R}(\downarrow b) = \mathcal{R}_c(\downarrow b)$ and moreover $\bigvee \Sigma L = \top$. Then $\mathcal{R}_{\ell c}(L) = \mathcal{R}(L)$.*

Proof. Let $\alpha \in \mathcal{R}(L)$ and let $p \in \Sigma L$. Then there exists an element $a \in L$ such that $p \leq a$ and $\mathcal{R}(\downarrow a) = \mathcal{R}_c(\downarrow a)$. Since $\alpha|_a \in \mathcal{R}(\downarrow a) = \mathcal{R}_c(\downarrow a)$, then $a \in \mathscr{C}_\alpha$. Therefore, $\top = \bigvee_{p\in\Sigma L} p \leq \bigvee \mathscr{C}_\alpha$, and so $\alpha \in \mathcal{R}_{\ell c}(L)$. □

We finish this section with some results on ring homomorphisms on $\mathcal{R}_{lc}(L)$.

Definition 4.8. A frame L is said to be locally countably pseudocompact (briefly, lc-pseudocompact) if $\mathcal{R}^*_{\ell c}(L) = \mathcal{R}_{\ell c}(L)$, where $\mathcal{R}^*_{\ell c}(L) = \mathcal{R}_{\ell c}(L) \cap \mathcal{R}^*(L)$.

In what follows, by [9], for every $\alpha \in \mathcal{R}(L)$, we put

$$R_\alpha := \{r \in \mathbb{R} \colon \operatorname{coz}(\alpha - \mathbf{r}) \neq \top\}\,.$$

Proposition 4.9. [9] *If $\alpha \in \mathcal{R}_c(L)$, then R_α is a countable subset of $\mathbb{R}$.*

Let us remind the reader that although apparently $C_c(X)$ and $C^F(X)$ are not defined algebraically, but they are in fact algebraic objects, in the sense that if $C(X) \cong C(Y)$, then $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$. For this, it is easy to see that whenever $\varphi\colon C(X) \to C(Y)$ is a nonzero homomorphism, then $\varphi\big(C_c(X)\big) \subseteq C_c(Y)$.

Proposition 4.10. *If $\varphi\colon \mathcal{R}(L) \longrightarrow \mathcal{R}(M)$ is a ring homomorphism such that $\varphi(\mathbf{1}) = \mathbf{1}$ and $\alpha \in \mathcal{R}_c(L)$, then $R_{\varphi(\alpha)}$ is countable.*

Proof. Since φ preserves order and $\varphi(\mathbf{1}) = \mathbf{1}$, we conclude that $\varphi(\mathbf{r}) = \mathbf{r}$ for every $r \in \mathbb{R}$. By Proposition 4.9, it is enough to show, $R_{\varphi(\alpha)} \subseteq R_\alpha$. Let $r \in R_{\varphi(\alpha)} \setminus R_\alpha$ be given. Therefore, $\operatorname{coz}(\alpha - \mathbf{r}) = \top$, which follows that there exists an element $\beta \in \mathcal{R}(L)$ such that $(\alpha - \mathbf{r})\beta = 1$. Thus $\varphi(\alpha - \mathbf{r})\varphi(\beta) = \varphi(\mathbf{1}) = \mathbf{1}$ and hence $\operatorname{coz}\big(\varphi(\alpha - \mathbf{r})\big) = \top$, which is a contradiction. □

Remark 4.11. Note that the converse of Proposition 4.9 is not true, in general. For example, we consider the isomorphism $\varphi\colon \mathfrak{O}(\mathbb{Q}) \longrightarrow \mathfrak{O}(\mathbb{R})$ given by $\varphi\big(\tau(p,q)\cap\mathbb{Q}\big) = \tau(p,q)$. Then $\psi\colon \mathcal{R}(\mathfrak{O}(\mathbb{Q})) \longrightarrow \mathcal{R}(\mathfrak{O}(\mathbb{R}))$ given by $\psi(\alpha) = \varphi \circ \alpha$ is an isomorphism. We assume that $\alpha\colon \mathcal{L}\mathbb{R} \to \mathfrak{O}(\mathbb{Q})$ is given by $\alpha(p,q) = \tau(p,q) \cap \mathbb{Q}$. Then

$$\psi(\alpha)(p,q) = \varphi \circ \alpha(p,q) = \varphi(\tau(p,q) \cap \mathbb{Q}) = \tau(p,q)$$

for every $p, q \in \mathbb{Q}$. It is clear $\alpha \in \mathcal{R}_c(\mathfrak{O}(\mathbb{Q}))$. Indeed $\psi(\alpha) \notin \mathcal{R}_c(\mathfrak{O}(\mathbb{R}))$, because $\psi(\alpha)$ is not an overlap of S for every $S \subsetneq \mathbb{R}$.

In [3], Banaschewski showed that any $\mathbf{0} \leq \alpha \in \mathcal{R}(L)$ is a square. It is shown that this result holds for $\mathcal{R}_c(L)$, that is if $\mathbf{0} \leq \alpha \in \mathcal{R}_c(L)$, then there exists an element $\beta \in \mathcal{R}_c(L)$ such that $\alpha = \beta^2$. Here, we study this result for $\mathcal{R}_{\ell c}(L)$.

Proposition 4.12. *If $\mathbf{0} \leq \alpha \in \mathcal{R}_{\ell c}(L)$ and $\alpha = \beta^2$, then $\beta \in \mathcal{R}_{\ell c}(L)$.*

Proof. Since $\alpha|_a = \beta^2|_a = (\beta|_a)^2$ and $\alpha|_a \in \mathcal{R}_c(\downarrow a)$ for every $a \in \mathscr{C}_\alpha$, then $\beta|_a \in \mathcal{R}_c(\downarrow a)$. Therefore

$$C_\alpha = \bigvee\{a \in L\colon \alpha|_a \in \mathcal{R}_c(\downarrow a)\} \leq \bigvee\{a \in L\colon \beta|_a \in \mathcal{R}_c(\downarrow a)\} = C_\beta,$$

which implies that $(C_\beta)^* = \perp$, then $\beta \in \mathcal{R}_{\ell c}(L)$. $\square$

Now, an interesting function is introduced as below; see [2]. For a complemented element a of L, define the frame map $e_a\colon \mathcal{L}(\mathbb{R}) \to L$ given by

$$e_a(p,q) = \begin{cases} \top & \text{if } p < 0 < 1 < q\,, \\ a' & \text{if } p < 0 < q \leq 1\,, \\ a & \text{if } 0 \leq p < 1 < q\,, \\ \perp & \text{otherwise,} \end{cases}$$

for each p, $q \in \mathbb{Q}$.

Proposition 4.13. *Every homomorphism $\varphi\colon \mathcal{R}_{\ell c}(L) \longrightarrow \mathcal{R}_{\ell c}(M)$, takes $\mathcal{R}^*_{\ell c}(L)$ into $\mathcal{R}^*_{\ell c}(M)$.*

Proof. If $\varphi = \mathbf{0}$, then it is trivial. Let $\varphi \neq \mathbf{0}$; then $\varphi(\mathbf{1}) \neq \mathbf{0}$. Since $\varphi(\mathbf{1})$ is an idempotent element in $\mathcal{R}(M)$, then $\operatorname{coz}\big(\varphi(\mathbf{1})\big)$ is complemented and

$$\varphi(\mathbf{1})(p,q) = \begin{cases} \top & 0,1 \in \tau(p,q)\,, \\ \operatorname{coz}\big(\varphi(\mathbf{1})\big) & 0 \notin \tau(p,q), 1 \in \tau(p,q)\,, \\ \operatorname{coz}\big(\varphi(\mathbf{1})\big)' & 0 \in \tau(p,q), 1 \notin \tau(p,q)\,, \\ \perp & 0,1 \notin \tau(p,q)\,. \end{cases}$$

Therefore $\varphi(\mathbf{1}) \leq \mathbf{1}$, which implies that $\varphi(\mathbf{n}) \leq \mathbf{n}$. Let $\mathbf{0} \leq \alpha \in \mathcal{R}_{\ell c}(L)$ be given. Then, by Proposition 4.12, there exists an element $\beta \in \mathcal{R}_{\ell c}(L)$ such that $\alpha = \beta^2$. Therefore, $\varphi(\alpha) = \varphi(\beta)^2 \geq \mathbf{0}$. Now, if $\alpha \in \mathcal{R}^*_{\ell c}(L)$, then $|\alpha| \leq n$ for some $n \in \mathbb{N}$, which implies that $\varphi(|\alpha|) \leq \varphi(\mathbf{n})$, and so $|\varphi(\alpha)| \leq \mathbf{n}$. Therefore, $\varphi(\alpha) \in \mathcal{R}^*_{\ell c}(M)$. $\square$

Corollary 4.14. *If M is not an lc-pseudocompact frame, then $\mathcal{R}_{\ell c}(M)$ cannot be a homomorphic image of $\mathcal{R}^*_{\ell c}(L)$ for any frame L.*

Proof. Suppose that there is a frame map $\varphi\colon \mathcal{R}_{\ell c}(L) \longrightarrow \mathcal{R}_{\ell c}(M)$ such that $\varphi\big(\mathcal{R}^*_{\ell c}(L)\big) = \mathcal{R}_{\ell c}(M)$. By Proposition 4.13, we have

$$\varphi\big(\mathcal{R}^*_{\ell c}(L)\big) \subseteq \mathcal{R}^*_{\ell c}(M) \subseteq \mathcal{R}_{\ell c}(M) = \varphi\big(\mathcal{R}^*_{\ell c}(L)\big)\,,$$

which shows that $\mathcal{R}^*_{\ell c}(M) = \mathcal{R}_{\ell c}(M)$. That is a contradiction. $\square$

Corollary 4.15. *If φ is a homomorphism from $\mathcal{R}_{\ell c}(L)$ into $\mathcal{R}_{\ell c}(M)$ whose image contains $\mathcal{R}^*_{\ell c}(M)$, then $\varphi\big(\mathcal{R}^*_{\ell c}(L)\big) = \mathcal{R}^*_{\ell c}(M)$.*

5. Constant functions and sublocales

First, we recall some concepts of sublocales. For more information on locales, see [19]. If M is a sublocale of L, then the *associated frame surjection* is the surjective frame homomorphism $\nu_M \colon L \to M$ given by

$$\nu_M(a) = \bigwedge\{m \in M \colon a \leq m\} = \bigwedge \big(M \cap \mathfrak{c}_L(a)\big).$$

Let M be a sublocale of L and let $\alpha \in \mathcal{R}(L)$. We define the frame map $\alpha|^M \colon \mathcal{L}(\mathbb{R}) \longrightarrow M$ given by

$$\alpha|^M(p,q) = \upsilon_M\big(\alpha(p,q)\big) = \bigwedge\{m \in M \colon \alpha(p,q) \leq m\},$$

and we denote $\mathcal{R}_{(M,\mathrm{constant})}(L)$ to be the set of all $\alpha \in \mathcal{R}(L)$ such that $\alpha|^M \in \mathcal{R}^1(M)$.

Remark 5.1. Let $\nu_M \colon L \to M$ be the associated frame surjection to M and let $\alpha, \beta \in \mathcal{R}(L)$. Then $\upsilon_M \circ (\alpha \diamond \beta) = (\upsilon_M \circ \alpha) \diamond (\upsilon_M \circ \beta)$ for $\diamond \in \{+, \cdot, \wedge, \vee\}$. Therefore, $\mathcal{R}_{(M,\mathrm{constant})}(L)$ is a sub-f-ring and an $\mathbb{R}$-subalgebra of $\mathcal{R}(L)$. Moreover, note that for any frame L, it is clear that $\mathcal{R}_{(L,\mathrm{constant})}(L) = \mathcal{R}(L)$ if and only if every function in $\mathcal{R}(L)$ is constant.

Proposition 5.2. *Let L be a completely regular frame. Then $\mathcal{R}_{(L,\mathrm{constant})}(L) = \mathcal{R}(L)$ if and only if $L = \mathbf{2}$, where $\mathbf{2} = \{\perp, \top\}$.*

Proof. Suppose that there exists an element $a \in L$ such that $\top \neq a \neq \perp$. Since L is a completely regular frame, there exists a subset $\{\alpha_\gamma\}_{\gamma \in \Lambda} \subseteq \mathcal{R}(L)$ such that $a = \bigvee_{\gamma \in \Lambda} \mathrm{coz}(\alpha_\gamma)$. Hence for every $\gamma \in \lambda$, we have $\perp \leq coz(\alpha_\gamma) \leq a < \top$, which follows that $a = \perp$, a contradiction. The converse is evident. □

Proposition 5.3 ([9]). *If L is a connected frame, then $\mathcal{R}_c(L) \cong \mathbb{R}$. In fact, $|R_\alpha| = 1$ and $\alpha \blacktriangleleft R_\alpha$ for every $\alpha \in \mathcal{R}_c(L)$.*

Proposition 5.4 ([16, Proposition 3.19]). *Let $\alpha \colon \mathcal{L}(\mathbb{R}) \to L$ and $\beta \colon L \to M$ be frame maps.*

(1) *If $\alpha \blacktriangleleft S$, then $\beta \circ \alpha \blacktriangleleft S$.*

(2) *If β is monomorphism and $\beta \circ \alpha \blacktriangleleft S$, then $\alpha \blacktriangleleft S$.*

By these propositions, we have the next result. We conclude this section with the following fact.

Proposition 5.5. *Let M be a connected sublocale of L. Then $\mathcal{R}_c(L) \subseteq \mathcal{R}_{(M,\mathrm{constant})}(L)$. In particular, if M is a connected sublocale of L and $\upsilon_M \colon L \longrightarrow M$ is a monomorphism, then $\mathcal{R}_c(L) = \mathcal{R}_{(M,\mathrm{constant})}(L)$.*

Proof. Since M is connected, by Proposition 5.3, we have $\mathcal{R}_c(M) \cong \mathbb{R}$. Suppose that $\alpha \in \mathcal{R}_c(L)$. Then, there exists a countable subset $S \subseteq \mathbb{R}$ such that $\alpha \blacktriangleleft S$. Therefore, Proposition 5.4 implies $\alpha|^M = \upsilon_M \circ \alpha \blacktriangleleft S$, which follows $\alpha|^M \in \mathcal{R}_c(M) \cong \mathbb{R}$. Then $\alpha \in \mathcal{R}_{(M,\mathrm{constant})}(L)$, as desired. □

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