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A NOTE ON CERTAIN TANNAKIAN GROUP SCHEMES

Sanjay Amrutiya

Abstract. In this note, we prove that the F-fundamental group scheme is a birational invariant for smooth projective varieties. We prove that the F-fundamental group scheme is naturally a quotient of the Nori fundamental group scheme. For elliptic curves, it turns out that the F-fundamental group scheme and the Nori fundamental group scheme coincide. We also consider an extension of the Nori fundamental group scheme in positive characteristic using semi-essentially finite vector bundles, and prove that in this way, we do not get a non-trivial extension of the Nori fundamental group scheme for elliptic curves, unlike in characteristic zero.

1. Introduction

Madhav Nori (in [13]) has introduced the fundamental group scheme of a reduced and connected scheme X over a field k. If X is a proper integral scheme over a field k with $x \in X(k) \neq \emptyset$, then the Nori fundamental group scheme, denoted by $\pi^N(X, x)$, is Tannaka dual to a neutral Tannakian category of *essentially finite* vector bundles over X. It is proved in [13] that the fundamental group scheme is a birational invariant for smooth projective varieties. The notion of S-fundamental group scheme was introduced in [5] for curves and in [11] for higher dimensional varieties. The S-fundamental group scheme $\pi^S(X, x)$ for a pointed smooth projective variety is the affine group scheme corresponding to the Tannakian category of *Nori-semistable* bundles on X. The birational invariance of the S-fundamental group scheme is proved in [7].

In [1], the authors constructed a neutral Tannakian category over a perfect field of prime characteristic for any pointed projective variety (X, x). This neutral Tannakian category is a tensor generated category by the family of *Frobenius-finite* vector bundles over X. The corresponding affine group scheme is called the F-fundamental group scheme of X, and is denoted by $\pi_F(X, x)$. Here, the letter "F" stands for Frobenius.

In this note, we prove that the F-fundamental group scheme is a birational invariant following the techniques of [7]. We also observe that the F-fundamental group scheme is a quotient of the Nori fundamental group scheme, and for elliptic curves, they coincide using the results of Atiyah and Oda. In [15], Otabe introduced

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an extension of the Nori fundamental group scheme using semi-finite vector bundles, and studied it's properties in characteristic zero. We consider an analogue of this extension of the Nori fundamental group scheme in positive characteristic, and prove that we do not get a non-trivial extension of the Nori fundamental group scheme for elliptic curves. Over finite fields, all these Tannakian fundamental group schemes coincide.

2. Preliminaries

In this section, we recall the definition of certain fundamental group schemes associated to a proper connected scheme defined over a perfect field k.

Let X be a proper integral scheme defined over a perfect field k endowed with a rational point $x \in X(k)$.

A vector bundle E over X is called *étale trivializable* if there exists a finite étale covering $\psi\colon Y \longrightarrow X$ such that the pull-back ψ^*E is trivializable. The affine group scheme corresponding to the neutral Tannakian category $\mathcal{C}^{\text{ét}}(X)$ defined by all étale trivializable vector bundles E over X (the fiber functor for this neutral Tannakian category sends E to its fiber $E(x)$ over x) is called the *(geometric) étale fundamental group scheme* and is denoted by $\pi^{\text{ét}}(X, x)$. When k is an algebraically closed field of characteristic zero, the algebraic fundamental group $\pi_1(X, x)$ is canonically isomorphic to the group of k-valued points of the étale fundamental group scheme $\pi^{\text{ét}}(X, x)$.

A vector bundle E over X is said to be *Nori-semistable* if for every non-constant morphism $f\colon C \longrightarrow X$ with C a smooth projective curve, the pull-back f^*E on C is semi-stable of degree zero.

Let $\mathbf{Ns}(X)$ denote the category of all Nori-semistable vector bundles over X. With the usual notion of tensor product and the fibre functor defined by the k-rational point x, the category $\mathbf{Ns}(X)$ defines a neutral Tannakian category over k. The affine group scheme corresponding to the neutral Tannakian category $\mathbf{Ns}(X)$ is called the S-fundamental group scheme, and is denoted by $\pi^S(X, x)$ [5, 11].

For any vector bundle E over X and for any polynomial f with non-negative integer coefficients, we define

$$f(E) := \bigoplus_{i=0}^{n} (E^{\otimes i})^{\oplus a_i}\,,$$

where $f(t) = \sum_{i=0}^{n} a_i t^i$ with $a_i \in \mathbb{N} \cup \{0\}$, $\forall\, i \in \{0, 1, 2, \ldots, n\}$.

A vector bundle E over X is said to be *finite* if there are distinct polynomials f and g with non-negative integer coefficients such that $f(E)$ is isomorphic to $g(E)$.

For any vector bundle E, let $S(E)$ denote the collection of all the indecomposable components of $E^{\otimes n}$, for all non-negative integers n. Then, the vector bundle E is finite if and only if $S(E)$ is a finite set [13, Lemma 3.1]. A vector bundle E over X is said to be *essentially finite* if it is a Nori-semistable subquotient of a finite vector bundle over X.

Let $\mathcal{C}^N(X)$ be the full subcategory of $\mathbf{Ns}(X)$ of essentially finite vector bundles over X (see [13] for more about essentially finite vector bundles on X). With usual tensor product of vector bundles and the fibre functor defined by the k-rational point x, the category $\mathcal{C}^N(X)$ defines a neutral Tannakian category over k. The affine group scheme corresponding to the neutral Tannakian category $\mathcal{C}^N(X)$ is called *the Nori fundamental group scheme* of X over k with base point x, and is denoted by $\pi^N(X, x)$.

A vector bundle E over X is called F-trivial if $(F_X^i)^*E$ is trivial for some i. Note that the category of all F-trivial vector bundles form a Tannakian category; the corresponding affine group scheme is called *the local fundamental group scheme*, which is denoted by $\pi^{\text{loc}}(X, x)$.

For any polynomial $g(t) = \sum_{i=0}^m n_i t^i$ with $a_i \in \mathbb{N} \cup \{0\}$, $\forall\, i \in \{0, 1, \dots, n\}$, define

$$\widetilde{g}(E) := \bigoplus_{i=0}^{m} ((F_X^i)^*E)^{\oplus n_i},$$

where F_X^0 is the identity morphism of X.

A vector bundle E over X is called *Frobenius-finite* if there are two distinct polynomials f and g of the above type such that $\widetilde{f}(E)$ is isomorphic to $\widetilde{g}(E)$.

For any vector bundle E over X, let $\mathrm{I}(E)$ denote the set of all indecomposable components of $\{(F_X^n)^*E\}_{n \geq 0}$.

Let $\mathrm{TFF}(X)$ denote the collection of all finite tensor products of Frobenius-finite vector bundles over X. Consider the full subcategory, denoted by $\mathcal{C}_F(X)$, of the category $\mathbf{Ns}(X)$ whose objects are all Nori-semistable subquotients of finite direct sum of elements of $\mathrm{TFF}(X)$.

With the usual tensor product of vector bundles and the fibre functor defined by $x \in X(k)$, the category $\mathcal{C}_F(X)$ defines a neutral Tannakian category over k. The *F-fundamental group-scheme* of X with the base point x, denoted by $\pi_F(X, x)$, is the affine group scheme associated to the neutral Tannakian category $(\mathcal{C}_F(X), T_x)$, where T_x is the neutral fibre functor on $\mathcal{C}_F(X)$.

3. Some properties of Tannakian group schemes

We have the following diagram in which each arrow is a natural faithfully flat morphism of affine group schemes:

$$\begin{tikzcd}
\pi^S(X, x) \arrow[r] \arrow[dr] & \pi^N(X, x) \arrow[r] \arrow[d, dotted] \arrow[dr] & \pi^{\text{ét}}(X, x) \\
& \pi_F(X, x) \arrow[r] & \pi^{\text{loc}}(X, x)
\end{tikzcd}$$

The existence of the dotted arrow follows from Proposition 3.2. For this we will use the following theorem of Lange and Stuhler.

Proposition 3.1 ([9, Theorem 1.4]). *Let E be a vector bundle on X defined over k.*

(1) *If $(F_X^n)^*E$ is isomorphic to E for some $n > 0$, then E is étale trivializable.*

(2) *If E is stable étale trivializable, then there exist some $n > 0$ such that $(F_X^n)^*E$ is isomorphic to E (cf. [3]).*

(3) *If $k = \mathbb{F}_p$ or $k = \overline{\mathbb{F}_p}$ and E is étale trivializable, then there exist some $n > 0$ such that $(F_X^n)^*E$ is isomorphic to E.*

Proposition 3.2. *There is a faithfully flat morphism $\pi^N(X, x) \longrightarrow \pi_F(X, x)$ of affine group schemes.*

Proof. Let E be a Frobenius-finite vector bundle over X. Then $I(E)$ is a finite set. Since each Frobenius pullback $(F_X^n)^*E$ is of rank $r = \mathrm{rk}(E)$, we can conclude that there are integers $m > k > 0$ such that $(F_X^m)^*E \cong (F_X^k)^*E$. Hence, we have

$$(F_X^{m-k})^*(F_X^k)^*E = (F_X^m)^*E \cong (F_X^k)^*E\,.$$

By Proposition 3.1, there exists a finite étale covering $\psi\colon Y \longrightarrow X$ such that $\psi^*((F_X^k)^*E)$ is trivial. This proves that E is an essentially finite vector bundle on X (see [13, Chapter II, Proposition 7] or [4, Proposition 2.3]). □

Proposition 3.3. *If $k = \mathbb{F}_p$, then $\pi_F(X, x) = \pi^N(X, x) = \pi^S(X, x)$.*

Proof. We know that $\mathcal{C}^N(X) \subseteq \mathcal{C}_F(X) \subseteq \mathbf{Ns}(X)$ (cf. [1, Corollay 4.3]). Therefore, it is enough to show that every Nori-semistable bundle on X is essentially finite. To see this, let E be a Nori-semistable vector bundle on X. Then for each $n \geq 0$, the Frobenius pullback $(F_X^n)^*E$ is an H-semistable and $c_1(E)\cdot H^{n-1} = c_2(E)\cdot H^{n-2} = 0$ for any polarization H on X. Hence, the family $\{(F_X^n)^*E\}_{n\geq 0}$ is bounded family, i.e., there exists a k-scheme S of finite type and a coherent sheaf $\mathcal{E}$ on $X \times S$ such that each $(F_X^n)^*E$ is isomorphic to $\mathcal{E}|_{X\times\{s_n\}}$ for some k-point s_n of S (see [10, Theorem 4.4]). Since k is a finite field, it follows that the family $\{(F_X^n)^*E\}_{n\geq 0}$ contains only finitely many isomorphism classes. In other words, there are integers $n_1 > n_0 > 0$ such that $(F_X^{n_1})^*E \cong (F_X^{n_0})^*E$. Now, by Proposition 3.1, we conclude that certain Frobenius pullback of E is étale trivializable, and hence E is an essentially finite vector bundle [4, Proposition 2.3]. □

Birational Invariance. In [11, Lemma 8.3], Langer observed that $\pi^S(X, x)$ does not change while blowing up X along smooth centres, and raised the question of birational invariance of the S-fundamental group scheme. For smooth projective surfaces, this would imply the birational invariance of the S-fundamental group scheme using weak factorization theorem. Due to lack of weak factorization theorem for higher dimension in positive characteristic, this problem required a different techniques. In [7], Hogadi and Mehta proved the birational invariance of the S-fundamental group scheme using a vanishing theorem [7, Theorem 2.1].

In the same vein, we can see that the blow ups along smooth centres do not affect the F-fundamental group scheme. To see this, let $\varphi\colon \mathrm{Bl}_Z(Y) \longrightarrow Y$ be a blow-up of Y along a smooth center Z. By [1, Proposition 3.2], it follows that the induced homomorphism

$$\hat{\varphi}\colon \pi_F(\mathrm{Bl}_Z(Y), x) \longrightarrow \pi_F(Y, y)$$

is faithfully flat. To see that $\hat{\varphi}$ is closed immersion, by [6, Proposition 2.21], we need to show that for $E \in \mathcal{C}_F(\mathrm{Bl}_Z(Y))$ there exists $E' \in \mathcal{C}_F(Y)$ such that E is isomorphic to a subquotient of φ^*E'. By [8, Theorem 1], one can conclude that

$\varphi_* E$ is locally free and $E \simeq \varphi^* \varphi_* E$ (cf. [11, Lemma 8.3]). Take $E' = \varphi_* E$. Note that the following diagram

$$\begin{tikzcd} \mathrm{Bl}_Z(Y) \arrow[r, "F_{\mathrm{Bl}_Z(Y)}"] \arrow[d, "\varphi"'] & \mathrm{Bl}_Z(Y) \arrow[d, "\varphi"] \\ Y \arrow[r, "F_Y"'] & Y \end{tikzcd}$$

is Cartesian. From this, one can conclude that the vector bundle E' is an object of $\mathcal{C}_F(Y)$.

In the following, we will see that using the techniques of [7], one can easily deduce the birational invariance of the F-fundamental group scheme.

Theorem 3.4. *Let X and Y be smooth projective k-varieties, and let $\phi\colon X \dashrightarrow Y$ be a birational map. Let $x_0 \in X(k)$ be a point where ϕ is defined. Then $\pi_F(X, x)$ is isomorphic to $\pi_F(Y, \phi(x))$.*

Proof. Consider the diagram

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dotted, "\phi"'] & & Y \end{tikzcd}$$

where Z is the normalization of a closure of the graph of ϕ, the morphisms p and q are birational. It is proved in [7] that if $E \in \mathbf{Ns}(X)$, then $q_* p^* E \in \mathbf{Ns}(Y)$, and hence the functor

$$q_* p^* \colon \mathbf{Ns}(X) \longrightarrow \mathbf{Ns}(Y)$$

is an equivalence of Tannaka categories. To conclude that the F-fundamental group scheme is a birational invarience, we have to check that the functor $q_* p^*$ induces an equivalence between $\mathcal{C}_F(X)$ and $\mathcal{C}_F(Y)$. For this, let E be a Frobenius-finite vector bundle on X. Then, there exist two integers $m_1 > m_0 > 0$ such that we have an isomorphism

$$(F_X^{m_1})^* E \cong (F_X^{m_0})^* E \tag{3.1}$$

of vector bundles on X. Let $E' := p^* E$. Since $p \circ F_Z = F_X \circ p$, we have an isomorphism

$$(F_Z^{m_1})^* E' \cong (F_Z^{m_0})^* E' \tag{3.2}$$

of vector bundles on Z. By [7, Lemma 3.1], we have an isomorphism

$$(F_Y^{n})^* (q_* E') \cong q_*((F_Z^{n})^* E') \tag{3.3}$$

of vector bundles on Y, for all $n \geq 0$. Using isomorphisms (3.3) and (3.2), we can conclude that $q_* p^* E$ is a Frobenius-finite vector bundle on Y.

Let $E \in \mathcal{C}_F(X)$. Then, there are finitely many elements $E_1, E_2, \ldots, E_m$ in $\mathrm{TFF}(X)$ and Nori-semistable subbundles

$$V_1 \subseteq V_2 \subseteq \bigoplus_{i=1}^{m} E_i$$

such that $E \cong V_2/V_1$. Since the functor q_*p^* takes Frobenius-finite vector bundles on X to Frobenius-finite vector bundles on Y, it follows that $q_*p^*E_i \in \mathrm{TFF}(Y)$. It is easy to see that

$$q_*p^*E \cong q_*p^*(V_2/V_1) \cong q_*p^*V_2/q_*p^*V_1 .$$

Note that $q_*p^*V_2$ and $q_*p^*V_1$ are Nori-semistable. This proves that $q_*p^*E \in \mathcal{C}_F(Y)$. □

Unipotent group scheme. Recall that a vector bundle E on X is called unipotent if there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

such that $E_i/E_{i-1} \cong \mathcal{O}_X$ for $1 \le i \le n$. In [13, Chapter IV], Nori proved that the category $\mathcal{C}^{\mathrm{uni}}(X)$ of unipotent vector bundles on X is a Tannaka category. For $x \in X(k)$, the corresponding affine group scheme is called the unipotent group scheme of X and is denoted by $\pi^{\mathrm{uni}}(X, x)$.

If the characteristic of k is zero, then the category $\mathcal{C}^N(X)$ is semi-simple (cf. [13, p. 84]) and hence any finite unipotent vector bundle will be a trivial vector bundle. However, if characteristic of k is $p > 0$, then there is a natural faithfully flat morphism $\pi^N(X, x) \longrightarrow \pi^{\mathrm{uni}}(X, x)$ (see [13, Chapter IV; Proposition 3]).

The largest unipotent quotient of the F-fundamental group scheme of X is called the *unipotent part* of $\pi_F(X, x)$ and is denoted by $\pi_F^{\mathrm{uni}}(X, x)$. By Tannaka duality, the category of finite dimensional k-representations of $\pi_F^{\mathrm{uni}}(X, x)$ is equivalent (as a neutral Tannakian category) to the category ${\mathcal{C}_F}^{\mathrm{uni}}(X)$. Moreover, we have a natural faithfully flat morphism $\pi^{\mathrm{uni}}(X, x) \longrightarrow \pi_F^{\mathrm{uni}}(X, x)$ of affine group schemes.

If $k = \mathbb{F}_p$ or $k = \overline{\mathbb{F}_p}$, then by Proposition 3.3, we have $\pi^{\mathrm{uni}}(X, x) = \pi_F^{\mathrm{uni}}(X, x)$ (cf. [1, Corollary 4.3]).

An extension of the Nori fundamental group scheme. In [15], S. Otabe introduced an extension of the Nori fundamental group scheme using semi-finite vector bundles, and studied it's properties in characteristic zero.

To get the right analogue of an extension of the Nori fundamental group scheme in positive characteristic, we may consider the following:

Definition 3.5 ([15, cf. Definition 2.2])**.** A vector bundle E on X is called semi-essentially finite if there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

such that E_i/E_{i-1} is an essentially finite indecomposable bundle, for all $i = 1, 2, \ldots, n$.

Let $\mathcal{C}^{EN}(X)$ be the category of semi-essentially finite vector bundles on X. By following the similar argument as in [13, Chapter IV, Lemma 2], [15, Proposition 2.14], we have

Proposition 3.6. *With the usual tensor product of vector bundles and a neutral fibre functor T_x of $\mathcal{C}^{EN}(X)$ which sends a vector bundle $E \in \mathcal{C}^{EN}(X)$ to its fibre over x, the category $\mathcal{C}^{EN}(X)$ defines a neutral Tannakian category over k.*

The affine group scheme corresponding to $\mathcal{C}^{EN}(X)$ is denoted by $\pi^{EN}(X, x)$. From the definition, we have the following sequence of faithfully flat homomorphisms

$$\pi^S(X,\, x) \longrightarrow \pi^{EN}(X, x) \longrightarrow \pi^N(X,\, x)$$

of affine group schemes. To see this, let $E \in \mathcal{C}^N(X)$. Then there exists a finite principal G-bundle $\pi\colon P \longrightarrow X$ such that π^*E is trivial. We have a decomposition of a vector bundle E into direct sum of indecomposable bundles. Using induction, it is enough to assume that $E = E_1 \oplus E_2$. Since π^*E is trivial, it follows that π^*E_1 and π^*E_2 are degree zero subbundle of a trivial vector bundle. This implies that E_1 and E_2 are essentially finite vector bundles.

Remark 3.7. If X is an elliptic curve defined over an algebraically closed field of characteristic zero, then it is known that $\pi^{EN}(X, x) \cong \pi^N(X,\, x) \times \pi^{\text{uni}}(X,\, x)$ [15, Theorem 3.10]. The situation in positive characteristic is quite different (see Theorem 4.3).

4. Tannakian group schemes of elliptic curves

Let C be an elliptic curve defined over a perfect field k of positive characteristic. Let $\mathcal{E}_C(r, 0)$ be the set of all isomorphism classes of indecomposable vector bundles of rank r and degree 0 on C. It is proved in [2] that there is an unique element $F_r \in \mathcal{E}_C(r, 0)$ such that $H^0(C, F_r) \neq 0$. Moreover, if $E \in \mathcal{E}_C(r, 0)$, then $E \cong F_r \otimes L$, where L is a line bundle of degree 0 with $\det E \cong L$. Using Atiyah's classification of indecomposable vector bundles on elliptic curves, we have

Lemma 4.1. *Let E be a Nori-semistable vector bundle on C. Then*

$$E \cong \bigoplus_{i=1}^{k} F_{r_i} \otimes L_i \tag{4.1}$$

where each L_i is a line bundle of degree 0*.*

Proof. Since E is Nori-semistable, we have $E = \bigoplus_{i=1}^k E_i$, where $E_i \in \mathcal{E}_C(r, 0)$. By [2, Theorem 5], we have $E_i \cong F_{r_i} \otimes L_i$ where each L_i is a line bundle of degree 0. □

Theorem 4.2. *Let C be an elliptic curve defined over a perfect field k of positive characteristic p. For a k-rational point $x \in C(k)$, we have $\pi_F(C,\, x) = \pi^N(C,\, x)$.*

Proof. Let us first consider the case that C is an ordinary elliptic curve. Let E be an essentially finite vector bundle on C. Then by Lemma 4.1, it follows that $E \cong \bigoplus_{i=1}^k F_{r_i} \otimes L_i$ where each L_i is a line bundle of degree 0. By [14, Proposition

2.10], we have $F_C^* F_r \cong F_r$, and hence $F_{r_i} \in \mathcal{C}_F(C)$. Since $F_{r_i} \otimes L_i$ is an essentially finite vector bundle and $\det(F_{r_i} \otimes L_i) \cong L_i$, it follows that L_i is an essentially finite. By Theorem 3.1(2), we see that $L^{p^n-1} \cong \mathcal{O}_C$ for some $n > 0$. This proves that $E \in \mathcal{C}_F(C)$.

Now assume that C is a super-singular elliptic curve. Let E be an essentially finite vector bundle of rank r on C. By Lemma 4.1, it follows that the vector bundle E will be of the form (4.1). If $1 \leq r \leq p$, then by [14, Proposition 2.10], we have $F_C^* F_r \cong \mathcal{O}_C^r$, and hence $F_{r_i} \in \mathcal{C}_F(C)$. Now, using the argument as above, we can conclude that E is an object of $\mathcal{C}_F(X)$. If $r > p$, then using [14, Proposition 2.10], we can again conclude that $E \in \mathcal{C}_F(C)$. □

Theorem 4.3. *Let C be an elliptic curve defined over a perfect filed k of positive characteristic p. For a k-rational point $x \in C(k)$, we have $\pi^S(C, x) \longrightarrow \pi^{EN}(C,x) = \pi^N(C, x)$.*

Proof. First note that if $E \in \mathcal{C}^{EN}(C)$, then E is Nori-semistable. By Lemma 4.1, we have $E = \bigoplus_{i=1}^k F_{r_i} \otimes L_i$, where each L_i is a line bundle of degree 0. Since each $F_r \in \mathcal{C}^{\mathrm{uni}}(C) \subset \mathcal{C}^N(C)$ and $E \in \mathcal{C}^{EN}(C)$, it follows that all $F_{r_i} \otimes L_i$ are semi-essentially finite vector bundles and each L_i is a torsion line bundle. This completes the proof. □

Remark 4.4. In characteristic zero, the Tannakian group scheme corresponding to the category $\mathcal{C}(F_2)$ generated by $S(F_2)$ (in the sense of [13, §2.1]) is isomorphic to $\mathbb{G}_a$ [12, Proposition 3.3]. While in positive characteristic, we get a finite group scheme. To see this, let us recall that

$$F_2 \otimes F_r = \begin{cases} F_r \oplus F_r & \text{if } \ r = 0(\mathrm{mod})p\,; \\ F_{r-1} \oplus F_{r+1} & \text{otherwise.} \end{cases} \tag{4.2}$$

Using (4.2), it is easy to see that $S(F_2)$ is finite, and hence F_2 is a finite vector bundle [13, Lemma 3.1]. This implies that Tannakian group scheme corresponding to the category $\mathcal{C}(F_2)$ is a finite group scheme.

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References

[1] Amrutiya, S., Biswas, I., *On the F-fundamental group scheme*, Bull. Sci. Math. **34** (2010), 461–474.

[2] Atiyah, M.F., *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **3** (1957), 414–452.

[3] Biswas, I., Ducrohet, L., *An analog of a theorem of Lange and Stuhler for principal bundles*, C. R. Acad. Sci. Paris, Ser. I **345** (2007), 495–497.

[4] Biswas, I., Holla, Y., *Comparison of fundamental group schemes of a projective variety and an ample hypersurface*, J. Algebraic Geom. **16** (2007), 547–597.

[5] Biswas, I., Parameswaran, A.J., Subramanian, S., *Monodromy group for a strongly semistable principal bundle over a curve*, Duke Math. J. **132** (2006), 1–48.

[6] Deligne, P., Milne, J.S., *Tannakian Categories*, Hodge cycles, motives and Shimura varieties, Lecture Notes in Math, vol. 900, Springer–Verlag, Berlin–Heidelberg–New York, 1982.

[7] Hogadi, A., Mehta, V.B., *Birational invariance of the S-fundamental group scheme*, Pure Appl. Math. Q. **7** (4) (2011), 1361–1369, Special Issue: In memory of Eckart Viehweg.

[8] Ishimura, S., *A descent problem of vector bundles and its applications*, J. Math. Kyoto Univ. **23** (1983), 73–83.

[9] Lange, H., Stuhler, U., *Vektorbündel auf Kurven und Darstellungen der algebraischen Fudamentalgruppe*, Math. Z. **156** (1977), 73–83.

[10] Langer, A., *Semistable sheaves in positive characteristic*, Ann. of Math. **159** (2004), 251–276.

[11] Langer, A., *On S-fundamental group scheme*, Ann. Inst. Fourier (Grenoble) **61** (5) (2011), 2077–2119.

[12] Lekaus, S., *Vector bundles of degree zero over an elliptic curve*, C. R. Math. Acad. Sci. Paris **335** (2002), 351–354.

[13] Nori, M.V., *The fundamental group-scheme*, Proc. Indian Acad. Sci. Math. Sci. **91** (2) (1982), 73–122.

[14] Oda, T., *Vector bundles on an elliptic curve*, Nagoya Math. J. **43** (1971), 41–72.

[15] Otabe, S., *An extension of Nori fundamental group*, Comm. Algebra **45** (2017), 3422–3448.

Department of Mathematics, IIT Gandhinagar,
Near Village Palaj,
Gandhinagar – 382355, India
E-mail: samrutiya@iitgn.ac.in