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UNIT-REGULARITY AND REPRESENTABILITY FOR SEMIARTINIAN $*$-REGULAR RINGS

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Abstract. We show that any semiartinian $*$-regular ring R is unit-regular; if, in addition, R is subdirectly irreducible then it admits a representation within some inner product space.

1. Introduction

The motivating examples of $*$-regular rings, due to Murray and von Neumann, were the $*$-rings of unbounded operators affiliated with finite von Neumann algebra factors; to be subsumed, later, as $*$-rings of quotients of finite Rickart C^*-algebras. All the latter have been shown to be $*$-regular and unit-regular (Handelman [5]). Representations of these as $*$-rings of endomorphisms of suitable inner product spaces have been obtained first, in the von Neumann case, by Luca Giudici (cf. [7]), in general in joint work with Marina Semenova [9]. The existence of such representations implies direct finiteness [8]. In the present note we show that every semiartinian $*$-regular ring is unit-regular and a subdirect product of representables. This might be a contribution to the question, asked by Handelman (cf. [3, Problem 48]), whether all $*$-regular rings are unit-regular. We rely heavily on the result of Baccella and Spinosa [1] that a semiartinian regular ring is unit-regular provided that all its homomorphic images are directly finite. Also, we rely on the theory of representations of $*$-regular rings developed by Florence Micol [12] (cf. [9, 10]). Thanks are due to the referee for a timely, concise, and helpful report.

2. Preliminaries: Regular and $*$-regular rings

We refer to Berberian [2] and Goodearl [3]. Unless stated otherwise, rings will be associative, with unit 1 as constant. A (von Neumann) *regular* ring R is such that for each $a \in R$ there is $x \in R$ such that $axa = a$; equivalently, every right (left) principal ideal is generated by an idempotent. The *socle* $\mathrm{Soc}(R)$ is the sum of all minimal right ideals. A regular ring R is *semiartinian* if each of its homomorphic images has non-zero socle; that is, R has Loewy length $\xi + 1$ for some ordinal ξ. A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all x, $y \in R$. A ring R is

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unit-regular if for any $a \in R$ there is a unit u of R such that $aua = a$. The crucial fact to be used, here, is the following result of Baccella and Spinosa [1].

Theorem 1. *A semiartinian regular ring is unit-regular provided all its homomorphic images are directly finite.*

A $*$-*ring* is a ring R endowed with an involution $r \mapsto r^*$. Such R is $*$-*regular* if it is regular and $rr^* = 0$ only for $r = 0$. A *projection* is an idempotent e such that $e = e^*$; we write $e \in P(I)$ if $e \in I$. A $*$-ring is $*$-regular if and only if for any $a \in R$ there is is a projection e with $aR = eR$; such e is unique and obtained as aa^+ where a^+ is the pseudo-inverse of a. In particular, for $*$-regular R, each ideal I is a $*$-ideal, that is, closed under the involution. Thus, R/I is a $*$-ring with involution $a + I \mapsto a^* + I$ and a homomorphic image of the $*$-ring R. In particular, R/I is regular; and $*$-regular since $aa^+ + I$ is a projection generating $(a + I)(R/I)$.

If R is a $*$-regular ring and $e \in P(R)$ then the *corner* eRe is a $*$-regular ring with unit e, operations inherited from R, otherwise. For a $*$-regular ring, $P(R)$ is a modular lattice, with partial order given by $e \leq f \Leftrightarrow fe = e$, which is isomorphic to the lattice $L(R)$ of principal right ideals of R via $e \mapsto eR$. In particular, eRe is artinian if and only if e is contained in the sum of finitely many minimal right ideals.

A $*$-ring is *subdirectly irreducible* if it has a unique minimal ideal, denoted by $M(R)$. Observe that $Soc(R) \neq 0$ implies $M(R) \subseteq Soc(R)$ since $Soc(R)$ is an ideal. For the following see Lemma 2 and Theorem 3 in [6].

Fact 2. *If R is a subdirectly irreducible $*$-regular ring then eRe is simple for all $e \in P(M(R))$ and R a homomorphic image of a $*$-regular sub-$*$-ring of some ultraproduct of the eRe, $e \in P(M(R))$.*

3. Preliminaries: Representations

We refer to Gross [4] and Sections 1 of [9], 2–4 of [10]. By an *inner product space* V_F we will mean a right vector space (also denoted by V_F) over a division $*$-ring F, endowed with a sesqui-linear form $\langle . \mid . \rangle$ which is *anisotropic* ($\langle v \mid v \rangle = 0$ only for $v = 0$) and *orthosymmetric*, that is, $\langle v \mid w \rangle = 0$ if and only if $\langle w \mid v \rangle = 0$. Let $\mathsf{End}^*(V_F)$ denote the $*$-ring consisting of those endomorphisms φ of the vector space V_F which have an adjoint φ^* w.r.t. $\langle . \mid . \rangle$.

A *representation* of a $*$-ring R within V_F is an embedding of R into $\mathsf{End}^*(V)$. R is *representable* if such exists. The following is well known, cf. [11, Chapter IV.12]

Fact 3. *Each simple artinian $*$-regular ring is representable.*

The following two facts are consequences of Propositions 13 and 25 in [9] (cf. Micol [12, Corollary 3.9]) and, respectively, [8, Theorem 3.1] (cf. [6, Theorem 4]).

Fact 4. *A $*$-regular ring is representable provided it is a homomorphic image of a $*$-regular sub-$*$-ring of an ultraproduct of representable $*$-regular rings.*

Fact 5. *Every representable $*$-regular ring is directly finite.*

4. Main results

Theorem 6. *If R is a subdirectly irreducible $*$-regular ring such that $Soc(R) \neq 0$, then $Soc(R) = M(R)$, each eRe with $e \in P(M(R))$ is artinian, and R is representable.*

Proof. Consider a minimal right ideal aR. As R is subdirectly irreducible, $M(R)$ is contained in the ideal generated by a; that is, for any $0 \neq e \in P(M(R))$ one has $e = \sum_i r_i a s_i$ for suitable r_i, $s_i \in R$, $r_i a s_i \neq 0$. By minimality of aR, one has $a s_i R = aR$ and $r_i a s_i R = r_i aR$ is minimal, too. Thus, $e \in \sum_i r_i aR$ means that eRe is artinian. By Facts 3, 2, and 4, R is representable.

It remains to show that $\mathrm{Soc}(R) \subseteq M(R)$. Recall that the congruence lattice of $L(R)$ is isomorphic to the ideal lattice of R ([13, Theorem 4.3] with an isomorphism $\theta \mapsto I$ such that $aR/0 \in \theta$ if and only if $a \in I$. In particular, since R is subdirectly irreducible so is $L(R)$. Choose $e \in M(R)$ with eR minimal. Then for each minimal aR one has $eR/0$ in the lattice congruence θ generated by $aR/0$. Since both quotients are prime, by modularity this means that they are projective to each other. Thus, $aR/0$ is in the lattice congruence generated by $eR/0$ whence a is in the ideal generated by e, that is, in $M(R)$. □

Theorem 7. *Every semiartinian $*$-regular ring R is unit-regular and a subdirect product of representable homomorphic images.*

Proof. Consider an ideal I of R. Then $I = \bigcap_{x \in X} I_x$ with completely meet irreducible I_x, that is, subdirectly irreducible R/I_x. Since R is semiartinian one has $\mathrm{Soc}(R/I_x) \neq 0$, whence R/I_x is representable by Theorem 6 and directly finite by Fact 5. Then R/I is directly finite, too, being a subdirect product of the R/I_x. By Theorem 1 it follows that R is unit-regular. □

5. Examples

It appears that semiartinian $*$-regular rings form a very special subclass of the class of unit-regular $*$-regular rings, even within the class of those which are subdirect products of representables. E.g. the $*$-ring of unbounded operators affiliated to the hyperfinite von Neumann algebra factor is representable, unit-regular, and $*$-regular with zero socle. On the other hand, due to the following, for every simple artinian $*$-regular ring R and any natural number $n > 0$ there is a semiartinian $*$-regular ring having ideal lattice an n-element chain and R as a homomorphic image.

Proposition 8. *Every representable $*$-regular ring R embeds into some subdirectly irreducible representable $*$-regular ring $\hat{R}$ such that $R \cong \hat{R}/M(\hat{R})$. In particular, $\hat{R}$ is semiartinian if and only if so is R.*

The proof needs some preparation. Call a representation $\iota : R \to \mathsf{End}^*(V_F)$ *large* if for all $a, b \in R$ with $\mathsf{im}\,\iota(b) \subseteq \mathsf{im}\,\iota(a)$ and finite $\dim(\mathsf{im}\,\iota(a)/\,\mathsf{im}\,\iota(b))_F$ one has $\mathsf{im}\,\iota(a) = \mathsf{im}\,\iota(b)$.

Lemma 9. *Any representable $*$-regular ring admits some large representation.*

Proof. Inner product spaces can be considered as 2-sorted structures V_F with sorts V and F. In particular, the class of inner product spaces is closed under formation of ultraproducts. Representations of $*$-rings R can be viewed as R-F-bimodules ${}_RV_F$, that is as 3-sorted structures, with R acting faithfully on V. It is easily verified that the class of representations of $*$-rings is closed under ultraproducts cf. [9, Proposition 13].

Now, given a representation η of R in W_F, form an ultrapower ι, that is ${}_SV_{F'}$, such that $\dim F'_F$ is infinite (recall that F' is an ultrapower of F). Observe that $\mathsf{End}^*(V_{F'})$ is a sub-$*$-ring of $\mathsf{End}^*(V_F)$ and $\dim(U/W)_F$ is infinite for any subspaces $U \supseteq W$ of $V_{F'}$. Also, S is an ultrapower of R with canonical embedding $\varepsilon\colon R \to S$. Thus, $\varepsilon \circ \iota$ is a large representation of R in V_F. □

Proof of Proposition 8. In view of Lemma 9 we may assume a large representation ι of R in V_F. Identifying R via ι with its image, we have R a $*$-regular sub-$*$-ring of $\mathsf{End}^*(V_F)$. Let I denote the set of all $\varphi \in \mathsf{End}(V_F)$ such that $\dim(\mathsf{im}\,\varphi)_F$ is finite. According to Micol [12, Proposition 3.12] (cf. Propositions 4.4(i), (iii) and 4.5 in [10]) $R+I$ is a $*$-regular sub-$*$-ring of $\mathsf{End}^*(V_F)$, with unique minimal ideal I. By Theorem 6 one has $I = Soc(R+I)$. Moreover, $R \cap I = \{0\}$ since the representation ι of R in V_F is large. Hence, $R \cong (R+I)/I$. □

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