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ON THE GEOMETRICAL PROPERTIES OF HEISENBERG GROUPS

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Abstract. In [20] the existence of major differences about totally geodesic two-dimensional foliations between Riemannian and Lorentzian geometry of the Heisenberg group H_3 is proved. Our aim in this paper is to obtain a comparison on some other geometrical properties of these spaces. Interesting behaviours are found. Also the non-existence of left-invariant Ricci and Yamabe solitons and the existence of algebraic Ricci soliton in both Riemannian and Lorentzian cases are proved. Moreover, all of the descriptions of their homogeneous Riemannian and Lorentzian structures and their types are obtained. Besides, all the left-invariant generalized Ricci solitons and unit time-like vector fields which are spatially harmonic are completely determined.

1. Introduction

Heisenberg groups play an important role in mathematical physics, geometric analysis and quantum mechanics. In particular, three-dimensional Heisenberg group H_3 has attracted a growing number of researchers in both Riemannian and Lorentzian cases, for example see [18, 19, 21]. Recently the existence of major differences about two-dimensional totally geodesic foliations and metrics between Riemannian and Lorentzian geometry of these spaces are found in [20] and [19]. These motivated us to obtain a comparison between Riemannian results and their Lorentzian analogous for some other geometrical properties of these spaces, which develop our understanding of which properties are more strictly related to the metric signature and which ones are more general. These geometrical properties are left-invariant generalized Ricci solitons, left-invariant Yamabe-solitons, algebraic Ricci solitons, harmonic maps, Einstein like metrics and homogeneous Riemannian and Lorentzian structures. As an application we prove that the existence of the special Einstein-Weyl equations (E-W) and the results concerning conformal flatness and locally symmetric, in the Lorentzian case cannot extend to the Riemannian case.

The structure of the paper is as follows. In Section 2 we remind some facts about the curvature and Ricci tensor components of these spaces which are given

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in [20] and [9] for the Lorentzian and Riemannian cases. In Section 3 we first fully describe left-invariant generalized Ricci solitons on these spaces. Then we prove that the existence of the special Einstein-Weyl equation (E-W) on these spaces in the Lorentzian case cannot extend to their Riemannian case. We also prove that in both Riemannian and Lorentzian cases algebraic Ricci solitons exist, whereas these spaces are neither left-invariant Yamabe solitons nor left-invariant Ricci solitons. Moreover, we show that contrary to the Riemannian case, there exist Lorentzian Heisenberg groups which are locally symmetric and conformally flat. We also completely determine Einstein-like metrics in both Riemannian and Lorentzian cases. In Section 4 we investigate the harmonicity properties of these spaces. In fact we first calculate the energy of an arbitrary left-invariant vector field V on these spaces. Then we show that there exist Lorentzian Heisenberg groups for which any left-invariant vector field V is a harmonic map. Moreover, we determine all the left-invariant unit time-like vector fields which are spatially harmonic. Finally in Section 5 we explicitly describe all homogeneous Riemannian and Lorentzian structures on these spaces and determine their types.

2. Preliminary

Let H_3 be the three-dimensional Heisenberg group, which is a two-step nilpotent Lie group of all the 3×3 real matrices with the following form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then by [20] and [9] this space can be equipped with one of the following Lorentzian metrics

$$g_1 = -\frac{1}{\lambda^2}dx^2 + dy^2 + (xdy + dz)^2 \,,$$
$$g_2 = \frac{1}{\lambda^2}dx^2 + dy^2 - (xdy + dz)^2 \,,$$
$$g_3 = dx^2 + (xdy + dz)^2 - \big((1-x)dy - dz\big)^2 \,,$$

and the Riemannian metric

$$g = dx^2 + dy^2 + \big(dz + \lambda(ydx - xdy)\big)^2 \,,\ \lambda > 0 \,. \tag{1}$$

Thus by the conventions $R_{ijkt} = \langle R(e_i, e_j)e_k, e_t\rangle$ and $R(e_i, e_j) = \nabla_{[e_i,e_j]} - [\nabla_{e_i}, \nabla_{e_j}]$ for the curvature tensor R and $\rho_{ii} = \sum \epsilon_i g(R(e_j, e_i)e_j, e_i)$ for Ricci tensor, we obtain the following cases.

For g_1: The Lie algebra $\mathcal{H}_3$ of H_3 has a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, where $e_1 = \frac{\partial}{\partial z}$ and $e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ are space-like and $e_3 = \lambda\frac{\partial}{\partial x}$ is time-like. Also the non-zero Lie bracket is $[e_2, e_3] = \lambda e_1$ and the non-zero curvature and Ricci-components are given by $R_{1212} = -R_{1313} = \frac{-\lambda^2}{4}$, $R_{2323} = -\frac{3\lambda^2}{4}$ and $\rho_{11} = \rho_{33} = -\rho_{22} = \frac{-\lambda^2}{2}$.

For g_2: The Lie algebra $\mathcal{H}_3$ of H_3 has a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, where $e_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ and $e_2 = \lambda\frac{\partial}{\partial x}$ are space-like and $e_3 = \frac{\partial}{\partial z}$ is time-like.

Also the non-zero Lie bracket is $[e_1, e_2] = \lambda e_3$ and the non-zero curvature and Ricci tensor components are given by $R_{1212} = \frac{3\lambda^2}{4}, R_{1313} = -R_{1331} = \frac{\lambda^2}{4}$ and $\rho_{11} = \rho_{22} = \rho_{33} = \frac{\lambda^2}{2}$.
For g_3: This metric is flat and the Lie algebra $\mathcal{H}_3$ of H_3 has a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, where $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}$ are space-like and $e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ is time-like. Moreover, we have the non-zero Lie brackets $[e_3, e_1] = e_2 - e_3$ and $[e_2, e_1] = e_2 - e_3$.
For g: The Lie algebra $\mathcal{H}_3$ of H_3 has an orthonormal basis $\{e_1, e_2, e_3\}$, where $e_1 = \frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y} + \lambda x\frac{\partial}{\partial z}$ and $e_3 = \frac{\partial}{\partial z}$. Also, the non-zero Lie bracket is given by $[e_1, e_2] = 2\lambda e_3$ and the non-zero curvature and Ricci tensor components are given by $R_{1212} = -3\lambda^2, R_{1313} = R_{2323} = \lambda^2$ and $\rho_{11} = \rho_{22} = -\rho_{33} = -2\lambda^2$, respectively.

3. Left-invariant generalized Ricci solitons on H_3

A left-invariant generalized Ricci soliton is a Lie group G with a left-invariant metric $\bar{g}$ admitting a left-invariant vector field V, such that

$$\mathcal{L}_V\bar{g} + 2\alpha V^b \odot V^b - 2\beta\rho = 2\delta\bar{g}\,, \tag{2}$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V, α, β, δ are real constants, V^b is a left-invariant 1-form which is defined by $V^b(Y) = \bar{g}(V, Y)$ and ρ is the Ricci tensor (for more details see [17]). Recently left-invariant generalized Ricci solitons in three-dimensional Lie groups have been determined in [7]. Also in [16, 14] and [15] we completely determined these structures on some solvable extensions of the Heisenberg groups and four-dimensional hypercomplex and para-hypercomplex Lie groups. Here we extend this result to any Riemannian and Lorentzian Heisenberg group H_3 as follows:

Theorem 3.1. *Consider the Lie algebra $\mathcal{H}_3$ of the Heisenberg group H_3. Then the non-trivial left-invariant generalized Ricci solitons on $\mathcal{H}_3$ are given as follows*

- *for g_1: Either $\alpha, \beta, \delta = 0$, $V = k_1e_1$ or $\alpha \neq 0$, $\delta = -\frac{\beta}{2}\lambda^2$, $V = \pm\frac{\sqrt{-\alpha\beta}\lambda}{\alpha}e_1$.*
- *for g_2: Either $\alpha, \beta, \delta = 0$, $V = k_3e_3$ or $\alpha \neq 0, \delta = -\frac{\beta}{2}\lambda^2, V = \pm\frac{\sqrt{\alpha\beta}\lambda}{\alpha}e_3$.*
- *for g_3: $\alpha, \delta = 0$, $V = k_2e_2 - k_2e_3$.*
- *for g: Either $\alpha, \beta, \delta = 0$, $V = k_3e_3$ or $\alpha \neq 0, \delta = \beta\lambda^2$, $V = \pm 2\,\frac{\sqrt{\alpha\beta}\lambda}{\alpha}e_3$.*

Proof. Assume that $V = k_1e_1 + k_2e_2 + k_3e_3$ is an arbitrary left-invariant vector field on H_3. Then for g_1 we obtain

$$\nabla_{e_1}V = \frac{\lambda}{2}(k_2e_3 + k_3e_2)\,, \quad \nabla_{e_2}V = \frac{\lambda}{2}(k_1e_3 + k_3e_1\,,$$
$$\nabla_{e_3}V = \frac{\lambda}{2}(k_1e_2 - k_2e_1)\,.$$

Thus by using $L_Vg(Y, Z) = g(\nabla_Y V, Z) + g(V, \nabla_Y Z)$ we get $L_Vg_1(e_1, e_2) = \lambda k_3$, $L_Vg_1(e_1, e_3) = -\lambda k_2$, with the rest vanishing. Then by the equation (2), where

$V^b \odot V^b(e_i, e_j) = \varepsilon_i \varepsilon_j k_i k_j$ we obtain

$$\begin{cases} 2\,\alpha\, {k_1}^2 + \beta\,\lambda^2 = 2\,\delta, \ k_3\lambda + 2\,\alpha\, k_1 k_2 = k_2\lambda + 2\,\alpha\, k_1 k_3 = 0\,, \\ 2\,\alpha\, {k_2}^2 - \beta\,\lambda^2 = 2\,\delta\,, \ -2\,\alpha\, k_2 k_3 = 0\,, \ 2\,\alpha\, {k_3}^2 + \beta\,\lambda^2 = -2\,\delta\,. \end{cases}$$

Thus by considering two cases $\alpha = 0$ and $\alpha \neq 0$ we get the result. For the remaining metrics we have similar computations. □

As it is mentioned in [17] by considering different values for α, β and δ we obtain several important equations which are: (H) the *homothetic vector field equation*, when $\alpha, \beta = 0$; (K) the *Killing vector field equation*, when $\alpha, \delta, \beta = 0$; (RS) the *Ricci soliton equation*, when $\alpha = 0$ and $\beta = 1$; (PS) the equation for a *metric projective structure* with a skew-symmetric Ricci tensor representative in the projective class, when $\alpha = 1, \beta = \frac{1}{1-n}$ and $\delta = 0$; (VN-H) the *vacuum near-horizon geometry equation* of a space-time, when $\alpha = 1$, $\beta = 1/2$ and δ plays the role of the cosmological constant; (E-W) a special case of the *Einstein-Weyl equation* in conformal geometry, when $\alpha = 1$ and $\beta = \frac{1}{2-n}$, $(n > 2)$. Thus these formulae and Theorem 3.1 give us the following result.

Theorem 3.2. *All Lorentzian and Riemannian Heisenberg groups (H_3, g_i) and (H_3, g) give left-invariant solutions to the Killing vector field equation (K). However, only (H_3, g_1) gives left-invariant solutions to the special Einstein-Weyl equation (E-W). Also only (H_3, g_2) and (H_3, g) give left-invariant solutions to the vacuum near-horizon geometry equation (VN-H).*

Recall that a pseudo-Riemannian manifold $(M, \bar{g})$ is a Yamabe soliton if it admits a vector field V such that $\mathfrak{L}_V \bar{g} = (\tau - \lambda)\bar{g}$, where τ is the scalar curvature tensor, λ is a real number [8]. Also, in the case that $\tau = \lambda$, i.e. $L_V g = 0$, the vector field V is a Killing vector field. Using $L_V \bar{g} = 0$, for $\bar{g} = g_1, g_2, g_3, g$, we respectively obtain Killing vector fields $V = k_1 e_1, V = k_3 e_3, V = k_2 e_2 - k_2 e_3$ and $V = k_3 e_3$. Thus by Theorem 3.2 we get that there is no left-invariant Ricci and Yamabe soliton.

Recall that a left-invariant pseudo-Riemannian metric $\bar{g}$ on a simply connected Lie group G with the Lie algebra $\mathcal{G}$ is called algebraic Ricci soliton if it satisfies $\mathrm{Ric} = c\,\mathrm{Id} + D$, where Ric denotes the Ricci operator and is given by $\rho(X, Y) = \bar{g}(\mathrm{Ric}(X), Y)$, c is a real constant and $D \in \mathrm{Der}(\mathcal{G})$, that is

$$D[X, Y] = [DX, Y] + [X, DY]\,, \qquad \forall \ \ X, Y \in \mathcal{G}\,. \tag{3}$$

For more details see [1, 3]. Here we prove the existence of these structures on Heisenberg group H_3 in both Riemannian and Lorentzian cases as follows.

Theorem 3.3. *Among all Riemannian and Lorentzian Heisenberg groups (H_3, g), (H_3, g_1), (H_3, g_2), (H_3, g_3), only (H_3, g_3) does not admit any algebraic Ricci soliton.*

Proof. We put $De_j = \sum_{i=1}^{3} \lambda_j^i e_i$, where $j = 1, \ldots, 3$. Then for the type (H_3, g_1) by replacing (X, Y) in (3) by (e_2, e_2) and (e_3, e_3) we respectively obtain $\lambda_2^3 = 0$ and $\lambda_3^2 = 0$. Also if we replace (X, Y) in (3) by (e_2, e_3) we obtain $\lambda_1^3 = \lambda_1^2 = 0$ and $\lambda_1^1 = \lambda_2^2 + \lambda_3^3$. Moreover, the condition $\mathrm{Ric} = c\,\mathrm{Id} + D$, satisfies if and only if we have $\lambda_3^1 = \lambda_2^1 = 0$ and $\lambda_3^3 = \lambda_2^2 = -\frac{2}{3}c$. Using these relations we obtain

$D = \begin{pmatrix} -2\lambda^2 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \end{pmatrix}$ and $c = \frac{3\lambda^2}{2}$. For the types (H_3, g_2) and (H_3, g) by similar computations we obtain $D = \begin{pmatrix} -\lambda^2 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -2\lambda^2 \end{pmatrix}$, $c = \frac{3\lambda^2}{2}$ and $D = \begin{pmatrix} 4\lambda^2 & 0 & 0 \\ 0 & 4\lambda^2 & 0 \\ 0 & 0 & 8\lambda^2 \end{pmatrix}$, $c = -6\lambda^2$, and for (H_3, g_3) we obtain $c = 0$ and $\lambda_i^j = 0$, where i, $j = 1, 2, 3$. □

Thus Theorem 3.3 gives us the following existence result.

Corollary 3.4. *Riemannian and Lorentzian Heisenberg groups $(H_3, g), (H_3, g_1)$ and (H_3, g_2) are algebraic Ricci solitons, whereas these spaces are neither left-invariant Yamabe solitons nor left-invariant Ricci solitons.*

The existence of Einstein metrics on the Lorentzian Heisenberg groups (H_3, g_2) and (H_3, g_3) is proved in [19]. On the other hand natural generalizations of Einstein metrics are Einstein-like metrics which were introduced by Gray in [13]. Thus a natural question is that whether there exist Einstein-like metrics on these spaces. Recall that Einstein-like metrics are defined through conditions on the Ricci tensor as follows. A pseudo-Riemannian manifold $(M, \bar{g})$ respectively belongs to one of the classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{P}$ if and only if its Ricci tensor is cyclic-parallel, i.e., $\nabla_i \rho_{jk} + \nabla_j \rho_{ki} + \nabla_k \rho_{ij} = 0$, is parallel i.e., $\nabla_i \rho_{jk} = 0$ and is Codazzi tensor i.e., $\nabla_i \rho_{jk} = \nabla_j \rho_{ik}$, where $\nabla_i \rho_{jk} = -\sum_t (\varepsilon_j B_{ijt} \rho_{tk} + \varepsilon_k B_{ikt} \rho_{tj})$ and $\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k$ [4]. Considering the Lorentzian Heisenberg group (H_3, g_1), we have $B_{123} = B_{213} = B_{312} = -\frac{\lambda}{2}$. Thus we get $\frac{\lambda^3}{2} = \nabla_2 \rho_{31} = \nabla_3 \rho_{21} = -\frac{\lambda^3}{2}$ which implies that (H_3, g_1) is of class $\mathcal{A}$, but not of class $\mathcal{B}$. Using similar computations for g_2, g_3 and g we obtain the following result.

Theorem 3.5. *All Lorentzian and Riemannian Heisenberg groups (H_3, g_i) and (H_3, g) which are equipped with Einstein-like metrics, are*
(a) (H_3, g), (H_3, g_1), (H_3, g_2) whose Ricci tensors are cyclic parallel.
(b) (H_3, g_3) whose Ricci tensor is parallel.

Recall that a (pseudo)-Riemannian manifold $(M, \bar{g})$ with dimension $n \geq 3$, is conformally flat if and only if its Schouten tensor c vanishes, i.e., $0 = c(x, y, z) = (\nabla_x \rho)(y, z) - (\nabla_y \rho)(x, z) - 1/2(\bar{g}((\nabla_x \tau)y, z) - \bar{g}((\nabla_y \tau)x, z))$, where x, y, z are vector fields tangent to M and τ is the scalar curvature. Also a pseudo-Riemannian manifold $(M, \bar{g})$ is locally symmetric if and only if we have $\nabla R = 0$. Using these relations we obtain the following result.

Theorem 3.6. *For the Riemannian and Lorentzian metrics $\bar{g} = g_1, g_2, g_3, g$ of the Heisenberg group H_3, the following properties are equivalent.*
(a) *$\bar{g}$ is of the type g_3;* (b) *$\bar{g}$ is locally symmetric;* (c) *$\bar{g}$ is Ricci-parallel;* (d) *$\bar{g}$ is conformally flat.*

By Theorem 3.6 we obtain the following result witch has no Riemannian counterpart.

Corollary 3.7. *Any Lorentzian Heisenberg group* (H_3, g_3) *is conformally flat, Ricci-parallel and locally symmetric. However, none of these properties exists on the Riemannian Heisenberg Lie group* (H_3, g).

4. Harmonicity of invariant vector fields on H_3

Let $(M, \bar{g})$ be a (smooth, connected, oriented) n-dimensional pseudo Riemannian manifold and TM be its tangent bundle which is equipped with the Sasakian metric $\bar{g}^s$. Then the energy of the smooth vector field $V : (M, \bar{g}) \to (TM, \bar{g}^s)$ is defined by $E(V) = \frac{n}{2}\operatorname{vol}(M, \bar{g}) + \frac{1}{2}\int_M \parallel \nabla V \parallel^2 dv$. For more details see [6, 5, 12]. Here we assume that D is a relatively compact domain for the Heisenberg group H_3 and calculate the energy of $V|_D$, with respect to different metrics. Thus we obtain the the following result.

Theorem 4.1. *The energy* $E_D(V)$ *of* $V|_D$ *on the Heisenberg group* H_3 *with the Lorentzian and Riemannian metrics* g_1, g_2, g_3, g *is given respectively by*

$$(a) \quad E_D(V) = \{\tfrac{3}{2} - \frac{1}{4}\lambda^2\|V\|^2\}\operatorname{vol} D\,, \quad (b) \quad E_D(V) = \{\tfrac{3}{2} - \tfrac{1}{4}\lambda^2\|V\|^2\}\operatorname{vol} D\,,$$
$$(c) \quad E_D(V) = \tfrac{3}{2}\operatorname{vol} D\,, \qquad (d) \quad E_D(V) = \{\tfrac{3}{2} + \|V\|^2\}\operatorname{vol} D\,,$$

where for the Lorentzian and Riemanian cases cases we have $\|V\|^2 = k_1^2 + k_2^2 - k_3^2$ *and* $\|V\|^2 = k_1^2 + k_2^2 + k_3^2$, *respectively.*

The critical points for the energy functional are harmonic maps. These vector fields are characterized by Euler-Lagrange equations. In fact a vector field V defines a harmonic map from $(M, \bar{g})$ to $(TM, \bar{g}^s)$ if and only if V satisfies the conditions $\nabla^*\nabla V = 0$ and $\operatorname{tr}[R(\nabla_. V, V).] = 0$, where $\nabla^*\nabla V = \sum_i \epsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V)$ and $\operatorname{tr}[R(\nabla_. V, V).] = \sum_i \epsilon_i R(\nabla_{e_i}V, V)e_i$. Also if we put $\chi^\rho(M) = \{w \in \chi(M) : \|w\|^2 = \rho^2\}$, where $\rho \neq 0$ is a real constant, then by the Euler-Lagrange equations, V is a critical point for $E|_{\chi^\rho(M)}$ if and only if $\nabla^*\nabla V$ is collinear to V. Thus for the Lorentzian and Riemannian metrics g_1, g_2, g_3, g we obtain $\nabla^*\nabla V = \frac{\lambda^2}{2}V, \nabla^*\nabla V = \frac{\lambda^2}{2}V$, $\nabla^*\nabla V = 0$, $\nabla^*\nabla V = -2\lambda^2 V$, respectively which give us the following result.

Theorem 4.2. (a) *Each vector field V on the Lorentzian Heisenberg group* (H_3, g_3) *is a harmonic map. However, none of these vector fields is a critical point for the energy functional restricted to vector fields of the same length.*
(b) *Each vector field* V *on the Lorentzian and Riemannian Heisenberg groups* (H_3, g_1), (H_3, g_2) *and* (H_3, g) *is a critical point for the energy functional restricted to vector fields of the same length. However, none of these vector fields is a harmonic map.*

Let $(M, \tilde{g})$ be a Lorentzian manifold and let V be a unit time-like vector field on M. Then by the Euler-Lagrange equations V is spatially harmonic if and only if we have $\widehat{V}_V = \delta V$, where $\delta \in \mathbb{R}$ and for $\operatorname{div} V = \sum \varepsilon_i \tilde{g}(\nabla_{e_i}V, e_i)$ and

$(\nabla V)^t\nabla_V V = \sum_i \varepsilon_i \tilde{g}(\nabla_V V, \nabla_{e_i} V)e_i$, $\widehat{V}_V$ is defined by $\widehat{V}_V = -\nabla^*\nabla V - \nabla_V\nabla_V V - \operatorname{div} V.\nabla_V V + (\nabla V)^t\nabla_V V$. Assume that $V = k_1e_1 + k_2e_2 + k_3e_3$ is an arbitrary left-invariant vector field on H_3. Then for the case that the Lorentzian metric is g_1 we obtain $(\nabla V)^t\nabla_V V = \frac{\lambda^2}{2}\{(k_3^2 - k_2^2)k_1e_1 - k_1^2(k_2e_2 + k_3e_3)\} = -\nabla_V\nabla_V V$, and $\operatorname{div} V = 0$. Thus the condition $\widehat{V}_V = \delta X$ gives us two cases $k_1 = 0$ and $k_1 \neq 0$. If $k_1 = 0$, then we obtain a contradiction and if $k_1 = 0$, then we obtain $k_3^2 = 1 + k_2^2$ which gives us $V = k_2e_2 + k_3e_3$. Using similar computations for the remaining cases we obtain the following result.

Theorem 4.3. *A time-like unit left invariant vector field V on the Lorentzian Heisenberg groups (H_3, g_1) and (H_3, g_2) is a spatially harmonic vector field if and only if there exist real numbers k_2 and k_3 such that respectively $V = k_2e_2 + k_3e_3$ and $V = \pm e_3$ where $k_3^2 = 1+k_2^2$. However, the Lorentzian Heisenberg group (H_3, g_3) never admits any spatially harmonic vector field.*

Notice that critical points for the space-like energy are spatially harmonic. Thus by Theorem 4.4 we obtain the following result.

Corollary 4.4. *No left-invariant unit time-like vector fields of the Lorentzian Heisenberg groups (H_3, g_3) are critical points for the space-like energy.*

5. Homogeneous Riemannian and Lorentzian structures on H_3

A homogeneous pseudo-Riemannian structure on a pseudo-Riemannian manifold $(M, \bar{g})$ is a tensor field S of type $(1, 2)$ on M such that the connection $\widehat{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations $\widehat{\nabla}\bar{g} = 0, \widehat{\nabla}R = 0$ and $\widehat{\nabla}S = 0$, where the first and second equations of this system are equivalent with $S_{xyz} = -S_{xzy}$ and

$$(\nabla_u R)(x, y, z, w) = -R(S_u x, y, z, w) - R(x, S_u y, z, w) \\ -R(x, y, S_u z, w) - R(x, y, z, S_u w)\,. \tag{4}$$

where $S_{xyz} = \bar{g}(S_x y, z)$, [11]. All the homogeneous Riemannian and Lorentzian structures for the Heisenberg group H_3 with the left-invariant Riemannian metric $\tilde{g} = dx^2 + dz^2 + (dy - xdz)^2$ and the left-invariant Lorentzian metrics g_1 and g_2, when $\lambda = 1$ are given in [21], [18] and [2], respectively. Here we extend these results for the Riemannian metric (1) and the metrics g_1, g_2 and g_3, where λ is any non-zero real constant.

Theorem 5.1. *All the homogeneous Lorentzian and Riemannian structures on Heisenberg group H_3 with the metrics g_1, g_2, g_3, g are given by*

$$S = \alpha_2 \otimes (e^1 \wedge e^2) + \alpha_3 \otimes (e^1 \wedge e^3) + \beta_3 \otimes (e^2 \wedge e^3)\,, \tag{5}$$

where for metrics g_1, g_2, g, respectively $(\alpha_2, \alpha_3, \beta_3)$ is equal to $(\frac{\lambda}{2}e^3, -\frac{\lambda}{2}e^2, ae^1)$, $(be^3, \frac{\lambda}{2}e^2, -\frac{\lambda}{2}e^1)$ and $(ce^3, -\lambda e^2, \lambda e^1)$, such that a, b and c are real constants. Also, *for the metric g_3, differential 1-forms α_2, α_3 and β_3 satisfying the following relations*

$$\begin{cases} \widehat{\nabla}\alpha_2 = (e^2 + e^3) \otimes \beta_3 + \beta_3 \wedge \alpha_3 \,, \\ \widehat{\nabla}\alpha_3 = (e^2 + e^3) \otimes \beta_3 + \beta_3 \wedge \alpha_2 \,, \\ \widehat{\nabla}\beta_3 = (e^2 + e^3) \otimes (\alpha_2 - \alpha_3) + \alpha_2 \wedge \alpha_3 \,. \end{cases}$$

where $\{e^1, e^2, e^3\}$ is dual to the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$.

Proof. For the metric g_1, if we replace (x, y, z, w) in the equation (4) by (e_3, e_2, e_1, e_2), (e_2, e_3, e_1, e_3), then we get $S_{ue_3e_1} = \frac{\lambda}{2}e_2^*(u)$, $S_{ue_2e_1} = -\frac{\lambda}{2}e_3^*(u)$ which give us the structure (5). Thus the relation $\widehat{\nabla} = \nabla - S$ implies that

$$\widehat{\nabla}e^1 = 0\,, \quad \widehat{\nabla}e^2 = (\frac{\lambda}{2}e^1 + \beta_3) \otimes e^3\,, \quad \widehat{\nabla}e^3 = (\frac{\lambda}{2}e^1 + \beta_3) \otimes e^2\,. \tag{6}$$

Then if we put $\beta_3 = f_1e^1 + f_2e^2 + f_3e^3$, where $f_1, \ldots, f_4$ are differentiable function on H_3. Then by (6) we obtain $z(f_2) = f_3(\frac{\lambda}{2}e^1 + \beta_3)(z)$, $z(f_3) = f_2(\frac{\lambda}{2}e^1 + \beta_3)(z)$, $z(f_1) = 0$. Hence if we replace z in these equations by e^1, e^2, e^3 and use the Lie bracket relations, then we obtain that f_1 is a real constant and $f_2 = f_3 = 0$. For the metrics g_2, g_3 and g we have similar computations. □

Here to obtain the types of the homogeneous Riemannian and Lorentzian structures given in Theorem 5.1, we use the classification of homogeneous Riemannian and pseudo-Riemannian structures in [21] and [10]. Thus we obtain the following result.

Corollary 5.2. *All the homogeneous Lorentzian and Rimannian structures on the Heisenberg groups (H_3, g_1), (H_3, g_2) and (H_3, g) are of the type $S_2 \oplus S_3$. Also these structures are of the types $S_1 \oplus S_2$ and $S_1 \oplus S_3$ if and only if we have $2a = -2b = c = -2\lambda$ and $2a = -2b = c = \lambda$, respectively.*

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