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TOPOLOGICAL DEGREE THEORY IN FUZZY METRIC SPACES

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Abstract. The aim of this paper is to modify the theory to fuzzy metric spaces, a natural extension of probabilistic ones. More precisely, the modification concerns fuzzily normed linear spaces, and, after defining a fuzzy concept of completeness, fuzzy Banach spaces. After discussing some properties of mappings with compact images, we define the (Leray-Schauder) degree by a sort of colimit extension of (already assumed) finite dimensional ones. Then, several properties of thus defined concept are proved. As an application, a fixed point theorem in the given context is presented.

1. Introduction and preliminaries

Topological degree theory is a generalization of the winding number of a curve in the complex plane. It can be used to estimate the number of solutions of an equation, and is closely connected to *fixed-point theory*. When one solution of an equation is easily found, degree theory can often be used to prove existence of a second, nontrivial, solution. There are different types of degree for different types of maps: e.g. for maps between Banach spaces there is the Brouwer degree in $\mathbb{R}^n$, the *Leray-Schauder degree* for compact mappings in normed spaces, the coincidence degree and various other types. There is also a degree for continuous maps between manifolds. Topological degree theory has applications in complementarity problems, differential equations, differential inclusions and dynamical systems [10].

Many problems in science lead to the equation $x = y$ in infinite dimensional spaces rather than to the finite dimensional case. In particular, ordinary and partial differential equations, and integral equations can be formulated as abstract equations on infinite dimensional spaces of functions. In 1934, Leray and Schauder [18] generalized Brouwer degree theory to a finite Banach space and established the so-called the Leray Schauder degree. It turns out that the Leray Schauder degree is a very powerful tool in proving various existence results for nonlinear partial differential equations (see [15], [18], [19], [21], etc.).

The Leray Schauder degree theory is very useful in solving an operator equation of the type $(I - S)x = y$, where S is compact. In many applications S is not

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compact, so one may ask it is possible to give an analogue of the Leray Schauder theory in the noncompact case. In 1936, Leray [17] constructed an example to show that it is impossible to define a degree theory for mappings with only a continuity condition.

To solve an infinite dimensional equation $Sx = y$, a very natural method is to approximate the original equation by finite dimensional equations, as we have seen in the Leray Schauder theory. The well-known Galerkin method has proved to be a very efficient tool in finite dimensional approximation. In the 1960s, Browder and Petryshyn systematically studied the finite dimensional method for a large class of mappings, which they called A-proper mappings, and they developed a similar theory to the Brouwer degree.

The question of stability in optimization deals with what happens to an optimization problem when the elements of the problem are in some way deformed. As being expressed by Felix E. Browder, the concept of degree of a mapping, in all its different forms, is one of the most effective tools for studying the properties of the existence and multiplicity of solutions of nonlinear equations. Historically, the well known topological degree is a useful tool in applied mathematics, for example to prove that some nonlinear equations have solutions and to investigate the stability by using the continuation method. The notion of the degree was first introduced explicitly by Brouwer in 1912 in the case of finite dimensional spaces. Leray and Schauder extended this theme in 1934 to the context of Banach spaces and mappings of the form $f = I - g$, with I the identity and g a compact mapping (we refer to [6], [12] and [18] for a wide bibliography on the subject.) Afterwards many authors defined and developed the topological degree theory for various classes of non-compact nonlinear mappings between Banach spaces. For references on these notions see [1, 2, 3], [5, 6, 7, 8, 9, 11], [13, 14, 16] and [12].

In recent years, many great developments has been made in the theory and applications of fuzzy metric spaces. In 1960, B. Schweizer and A. Sklar [23] gave a description of the topological structure for a special class of probabilistic metric spaces. In 1983, B. Schweizer and A. Sklar [24] summarized and presented the generally developing situation in this field up-to-date. In H. Sherwood [25] has pointed out the *ordinary probability space* is a special case of *probabilistic metric space* and as known that the *probabilistic metric space* is a special case of the *fuzzy metric space* [22]. This implies that the study of theory and applications relevant to fuzzy metric space has important practical significant.

As is known to the researchers in this subject, the *Leray-Schauder topological degree theory* is a forceful tool in the research of operator theory in normed spaces. This motivates us to establish and study the Leray-Schauder topological degree in fuzzy metric spaces.

Definition 1.1 ([24])**.** A binary operation $T\colon [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1], T)$ is a topological monoid with unit 1 such that $T(a,b) \leq T(c,d)$ whenever $a \leq c$, $b \leq d$ for all a, b, c, $d \in [0,1]$.

Some typical examples of t-norm are the following:

$$T(a,b) = ab\,, \qquad \text{(product)}$$
$$T(a,b) = \min\{a,b\}\,, \qquad \text{(minimum)}$$
$$T(a,b) = \max\{a+b-1,0\}\,, \qquad \text{(Lukasiewicz)}$$
$$T(a,b) = \frac{ab}{a+b-ab}\,, \qquad \text{(Hamacher)}$$

Definition 1.2. [4] Let X be a linear space over $\mathbb{K}$ (field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$, the set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in \mathbb{K}$.

(FN1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$,

(FN2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x,t) = 1$, if and only if $x = 0$,

(FN3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$, if $c \neq 0$,

(FN4) for all s, $t \in \mathbb{R}$, x, $u \in X$, $N(x+u, s+t) \geq T\{N(x,s), N(u,t)\}$,

(FN5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X,N) will be referred to as a fuzzy normed linear space (breifly FNLS).

Theorem 1.3 ([20])**.** *Let (X,N,T) be a FNLS. For $x \in X$, $r \in (0,1)$, $t > 0$, we define the open ball*

$$B_x(r,t) := \{y \in X : N(x-y,t) > r\}\,.$$

Then

$$\tau_A := \{A \subset X : x \in A \iff \exists t > 0, r \in (0,1) : B_x(r,t) \subset A\}$$

is a topology on X. Moreover, if the t-norm T satisfies $\sup_{t\in(0,1)} T(t,t) = 1$, then (X,τ_N) is Hausdorff.

Theorem 1.4 ([20])**.** *Let (X,N,T) be a FNLS. Then (X,τ_N) is a metrizable topological vector space.*

Definition 1.5 ([20])**.** Let (X,N,T) be a FNLS and $\{x_n\}$ be the sequence in X.

(1) The sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that
$$\lim_{t\to\infty} N(x_n - x, t) = 1, \text{ for all } t > 0.$$
In this case x is called the limit of the sequence $\{x_n\}$ and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) The sequence $\{x_n\}$ is called Cauchy sequence if
$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$$
for all $t > 0$ and all $p \in \mathbb{N}$.

(3) (X,N,T) is said to be complete if every Cauchy sequence in X is convergent to a point in X. A complete FNLS will be called a fuzzy Banach space.

Definition 1.6 ([24])**.** Let (X, N, T) be a fuzzy normed linear space.

(a) A sequence $\{x_n\}$ in X is τ-convergent to $x \in X$ if for any $\epsilon > 0$, $\lambda > 0$, there exists a positive integer $k = k(\epsilon, \lambda)$ such that

$$N(x_n - x, \epsilon) > 1 - \lambda$$

whenever $n \geq k$. In this case, we write $x_n \xrightarrow{\tau} x$.

(b) A sequence $\{x_n\}$ in X is a τ-Cauchy sequence if for any $\epsilon > 0$, $\lambda > 0$, there exist a positive integer $k = k(\epsilon, \lambda)$ such that

$$N(x_n - x_m, \epsilon) > 1 - \lambda$$

whenever $n, m \geq k$.

(c) (X, N, T) is said to be τ-complete if every τ-Cauchy sequence in X is τ-convergent to some point in X.

2. Results

Definition 2.1. Let (X, N, T) be a a fuzzy normed space and D be a subset of X. A mapping $A\colon D \to X$ is said to be compact if $\overline{A(D)}$ is a compact subset of X.

Lemma 2.2. *Let (X, N, T) be a fuzzy normed space, T is a t-norm satisfying $T(t, t) \geq t$ for all $t \in [0, 1]$, Ω be a nonempty subset of X, $S\colon \Omega \to X$ be a compact continuous mapping. Then for any neighborhood of θ, $u(\epsilon, \lambda)$, $\epsilon > 0$, $\lambda > 0$, there exists a finite dimension-valued compact mapping $S_{\epsilon,\lambda}$ such that*

$$Sx - S_{\epsilon,\lambda} \in u(\epsilon, \lambda)\,, \qquad x \in \Omega\,.$$

Proof. Since $S\colon \Omega \to X$ is compact, $\overline{S(\Omega)}$ is compact subset of X. For any neighborhood of θ, $u(\epsilon, \lambda)$, $\epsilon > 0$, $\lambda > 0$, there exist $y_1, y_2, \ldots, y_m \in \overline{S(\Omega)}$ such that $\overline{S(\Omega)} \subset \cup_{i=1}^m (y_i + u(\epsilon, \lambda))$. Letting

$$\lambda_i(x) = \max\{0, \epsilon - \{t, N(Sx - y_i, t) > 1 - \lambda\}\}\,, \quad x \in \Omega, i = 1, \ldots, m\,,$$

we prove that for each $x \in \Omega$, there exists some i_0 such that $\lambda_{i_0}(x) > 0$, $1 \leq i_0 \leq m$. In fact, since $Sx \in \overline{S(\Omega)} \subset \cup_{i=1}^m (y_i + u(\epsilon, \lambda))$, there exists an i_0, $1 \leq i_0 \leq m$, such that $Sx \in y_{i_0} + u(\epsilon, \lambda)$, i.e., $N(Sx - y_{i_0}, t) > 1 - \lambda$. By the left continuity of N there exist $i_0 < \epsilon$ such that $N(Sx - y_{i_0}, t) > 1 - \lambda$. Hence we have $\lambda_{i_0}(x) > 0$. Denote

$$\phi(x) = \sum_{i=1}^m \lambda_i(x)\,.$$

Then for any $x \in \Omega$, we have $\phi(x) \neq 0$. Now we define a mapping $S_{\epsilon,\lambda} : \Omega \to X$ as follows:

$$S_{\epsilon,\lambda}(x) = \sum_{i=1}^m \frac{\lambda_i(x)}{\phi(x)} y_i.$$

Now, we prove that $S_{\epsilon,\lambda}$ satisfies the requirements of the lemma. For this purpose, it suffices to prove that λ_i, $i = 1, \cdots, m$, is a continuous function, i.e., we show that

$$p_i(x) = \inf\{t : N(Sx - y, t) > 1 - \lambda\}$$

is a continuous function. If $x_n \xrightarrow{\tau} x$, it is easy to see that

$$p_i(x_n) \le p_i(x_0) + \inf\{t : N(Sx_n - Sx_0, t) > 1 - \lambda\},$$
$$p_i(x_0) \le p_i(x_n) + \inf\{t : N(Sx_0 - Sx_n, t) > 1 - \lambda\}.$$

Hence we have

$$|p_i(x_n) - p_i(x_0)| \le \inf\{t : N(Sx_0 - Sx_n, t) > 1 - \lambda\}.$$

If the right side of the preceding expression were not convergent to 0 as $n \to \infty$, then there would exist an $\epsilon_0 > 0$ such that given positive integer N, there exists an $n_0 > N$ such that

$$\inf\{t : N(Sx_{n_0} - Sx_0, t) > 1 - \lambda\} > \epsilon_0$$

and consequently, we have

$$N(Sx_{n_0} - Sx_0, \epsilon_0) \le 1 - \lambda. \tag{2.1}$$

Since S is continuous, $Sx_n \xrightarrow{\tau} Sx_0$, and so we have

$$\lim_{n\to\infty} N(Sx_{n_0} - Sx_0, \epsilon_0) = 1,$$

which contradicts (2.1). Thus, it follows that it gets $p_i(x_n) \to p_i(x_0)$ as $n \to \infty$, $i = 1, \ldots, m$. By (FN4), we have

$$N(Sx - S_{\epsilon,\lambda}x, \epsilon) \ge \min_{1\le i\le m, \lambda_i(x)\ne 0} \{N(Sx - y_i, \epsilon)\} > 1 - \lambda.$$

This implies that $Sx - S_{\epsilon,\lambda}x \in u(\epsilon, \lambda)$ for all $x \in \Omega$. Moreover, obviously, $S_{\epsilon,\lambda}$ is compact. This achieves the proof. □

Lemma 2.3. *Let (X, N, T) satisfy all the conditions of Lemma 2.2. Let Ω be a nonempty open subset of X and $S\colon \overline{\Omega} \to X$ be a compact continuous mapping. Then $R = I - S$ is a closed mapping.*

Proof. The conclusion can be proved immediately. The details are omitted here. □

Definition 2.4. Let (X, N, T) be a fuzzy normed space, T is a t-norm satisfying $T(t, t) \ge t$ for all $t \in [0, 1]$. Let Ω be a nonempty open subset of X and $S\colon \overline{\Omega} \to X$ be a compact continuous mapping. Let $R = I - S$ and $p \in X \setminus R(\partial\Omega)$. By Lemma 2.3, R is a closed mapping, $R(\partial\Omega)$ is a closed subset of X, and, consequently, there exists a neighborhood of θ, $u(\epsilon, \lambda)$, such that

$$(p + u(\epsilon, \lambda)) \cap R(\partial\Omega) = \emptyset.$$

By Lemma 2.2, there exists a finite dimension subspace $X^{(n)}$ of X with $p \in X^{(n)}$ and a continuous compact mapping $S_n\colon \overline{\Omega} \to X^{(n)}$ such that $N(Sx - S_nx, \epsilon) > 1 - \lambda$ for all $x \in \overline{\Omega}$. Letting $\Omega_n = \Omega \cap X^{(n)}$ and $R_n = I - S_n$, we are going to prove $p \notin R_n(\partial\Omega)$.

In fact, if there exists some $x_0 \in \partial\Omega$ such that $p = R_nx_0$, then we have

$$N(Rx_0 - p, \epsilon) = N(Sx_0 - R_nx_0, \epsilon) = N(Sx_0 - S_nx_0, \epsilon) > 1 - \lambda.$$

This contradicts $(p + u(\epsilon,\lambda)) \cap R(\partial\Omega) = \emptyset$. Beside, since $\overline{(I-(I-S_n))(\Omega_n)}$ is a compact set, the topological degree $\deg_n(R_n, \Omega_n, p)$ in finite dimensional space $X^{(n)}$ is significant. We define the *Leray-Schauder topological degree* of R as follows:

$$\operatorname{Deg}(R,\Omega,p) = \deg_n(R_n,\Omega_n,p)\,. \tag{2.2}$$

In order to explain the topological degree defined by (2.2) is significant, it suffices to show that it is independent of the choice of the neighborhood of θ, $u(\epsilon,\lambda)$, the space $X^{(n)}$ and the mapping S_n.

First, we prove that, when $u(\epsilon,\lambda)$ is given, $\operatorname{Deg}(R,\Omega,p)$ is independent of the choice of $X^{(n)}$ and S_n. In fact, if $X^{(m)}$ and R_m also satisfy the requirements in Definition 2.4, now we prove the following expression holds:

$$\deg_n(R_n,\Omega_n,p) = \deg_m(R_m,\Omega_m,p). \tag{2.3}$$

Letting $X^{(l)}$ be the linear sum of $X^{(n)}$ and $X^{(m)}$, $\Omega_l = X^{(l)} \cap \Omega$ and noting that S_n can be seen as a mapping from $\overline{\Omega} \to X^{(l)}$, we know that R_n is a mapping from $\overline{\Omega_l}$ into $X^{(l)}$. By the reduced theorem of topological degree, we have

$$\deg_l(R_n,\Omega_l,p) = \deg_n(R_n,\Omega_n,p)\,.$$

Similarly, we can prove that

$$\deg_l(R_n,\Omega_l,p) = \deg_m(R_m,\Omega_m,p)\,.$$

Next, we prove that

$$\deg_l(R_n,\Omega_l,p) = \deg_l(R_m,\Omega_l,p)\,.$$

Write

$$h_t(x) = tR_n(x) + (1-t)S_m(x)\,.$$

If there exists a $t_0 \in [0,1]$, $x_0 \in \partial\Omega$ such that $p = h_{t_0}(x_0)$, then we have

$$\begin{aligned}
N(Rx_0 - p,\epsilon) &= N(Rx_0 - t_0R_n(x_0) - (1-t_0)R_m(x_0),\epsilon) \\
&= N(t_0S_nx_0 + (1-t_0)S_mx_0 - Sx_0,\epsilon) \\
&\geq T(N(t_0(S_nx_0 - Sx_0), t_0\epsilon), N((1-t_0)(S_mx_0 - Sx_0),(1-t_0)\epsilon)) \\
&> 1-\lambda\,,
\end{aligned}$$

which is a contradiction. This implies that $p \notin h_t(\partial\Omega)$ for all $t \in [0,1]$. By the homotopy inveriance of topological degree in finite dimensional spaces, we have

$$\deg_l(R_n,\Omega_l,p) = \deg_l(R_m,\Omega_l,p).$$

This shows that (2.3) is true.

Next, we prove that $\operatorname{Deg}(R,\Omega,p)$ is independent of the choice of $u(\epsilon,\lambda)$. Suppose that there exists neighborhood of θ, $u_1(\epsilon_1,\lambda_1)$, satisfying all the conditions of Definition 2.4. Taking

$$0 < \epsilon_0 \leq \min\{\epsilon,\epsilon_1\}, \quad 0 < \lambda_0 \leq \min\{\lambda,\lambda_1\}\,,$$

it follows that $u(\epsilon_0,\lambda_0)$ also satisfies all the conditions of Definition 2.4 for $u(\epsilon,\lambda)$, $u(\epsilon_1,\lambda_1)$ and $u(\epsilon_0,\lambda_0)$, respectively, by the choice of ϵ_0, λ_0, it is obvious that R_l,

Ω_l satisfy all the conditions of Definition 2.4 for both $u(\epsilon,\lambda)$ and $u_1(\epsilon_1,\lambda_1)$, too. It follows from (2.3) that

$$\deg_n(R_n,\Omega_n,p)=\deg_l(R_l,\Omega_l,p)\,,$$
$$\deg_m(R_m,\Omega_m,p)=\deg_l(R_l,\Omega_l,p)\,.$$

Hence we have

$$\deg_m(R_m,\Omega_m,p)=\deg_n(R_n,\Omega_n,p)\,.$$

Thus, summing up the above explanation, we know that the topological degree defined by (2.2) is significant.

In the sequel of this section, we study the properties of topological degree defined by (2.2).

Theorem 2.5. *The topological degree defined by* (2.2) *has the following properties:*

(a) $\mathrm{Deg}\,(I,\Omega,p)=1$ *for all* $p\in\Omega$,

(b) *If* $\mathrm{Deg}\,(R,\Omega,p)\neq 0$, *then the equation* $R(x)=p$ *has a solution in* Ω,

(c) *If* $H(t,x)$ *is a continuous compact mapping defined on* $[0,1]\times\overline{\Omega}$ *and* $p\notin(I-H(t,\cdot))(\partial\Omega)$ *for all* $t\in[0,1]$, *then* $\mathrm{Deg}\,(I-H(t,\cdot),\Omega,p)$ *is independent of* $t\in[0,1]$,

(d) *If* Ω_1, Ω_2 *are two disjoint open subsets of* Ω *and* $p\notin R(\overline{\Omega}\setminus(\Omega_1\cup\Omega_2))$, *then*

$$\mathrm{Deg}\,(R,\Omega,p)=\mathrm{Deg}\,(R,\Omega_1,p)+\mathrm{Deg}\,(R,\Omega_2,p),$$

(e) *If* Ω_0 *is an open subset of* Ω *and* $p\notin R(\overline{\Omega}\setminus\Omega_0)$, *then*

$$\mathrm{Deg}\,(R,\Omega,p)=\mathrm{Deg}\,(R,\Omega_0,p)\,,$$

(f) *If* $p\notin R(\partial\Omega)$, *then*

$$\mathrm{Deg}\,(R,\Omega,p)=\mathrm{Deg}\,(R-p,\Omega,\theta)\,.$$

Proof. (a) and (f) can be obtained from Definition 2.4 immediately.
(b) Suppose that the equation $R(x)=p$ has no solution in Ω. Then $p\notin R(\overline{\Omega})$. In view of Lemma 2.3, $R(\overline{\Omega})$ is a closed subset and hence there exists a neighborhood of θ, $u(\epsilon,\lambda)$, such that $(p+u(\epsilon,\lambda))\cap R(\overline{\Omega})=\emptyset$. Take a finite dimension subspace $X^{(n)}$ of X and a finite dimension-valued continuous compact mapping $S_n\colon\overline{\Omega}\to X^{(n)}$ such that

$$Sx-S_nx\in u(\epsilon,\lambda)\,,\quad x\in\overline{\Omega}\,.$$

Letting $R_n=I-S_n$ and $\Omega_n=X^{(n)}\cap\Omega$, by Definition 2.4, we have

$$\mathrm{Deg}\,(R,\Omega,p)=\deg_n(R_n,\Omega_n,p)\,.$$

If there exist an $x_0\in\overline{\Omega_n}\subset\overline{\Omega}$ such that $R_nx_0=p$, then we have

$$N(Rx_0-p,\epsilon)=N(Rx_0-R_nx_0,\epsilon)=N(Sx_0-S_nx_0,\epsilon)>1-\lambda\,.$$

This contradicts $(p+u(\epsilon,\lambda))\cap R(\overline{\Omega})=\emptyset$. Thus we have $p\in R_n(\overline{\Omega_n})$, hence we have

$$\mathrm{Deg}\,(S,\Omega,p)=\deg(R_n,\Omega_n,p)=0\,,$$

which is a contradiction. This achieves the proof of (b).
(c) First we prove that there exists a neighborhood of θ, $u(\epsilon,\lambda)$, such that the following expression uniformly holds in $t\in[0,1]$:

$$(p+u(\epsilon,\lambda))\cap(I-H(t,\cdot))(\partial\Omega)=\emptyset\,.$$

Otherwise, there exist $\epsilon_n>0$, $\lambda_n>0$, $n=1,2,\dots$, with $\lambda_n\to 0,\epsilon_n\to 0$ as $n\to\infty$ and $x_n\in\partial\Omega$, $t_n\in[0,1]$, $n=1,2,\dots$, such that

$$N(p-x_n+H(x_n,x_n),\epsilon_n)>1-\lambda_n\,.$$

Since both $\{t_n\}$ and $\{H(t_n,x_n)\}$ have convergent subsequences, without loss of generality, we still denote these subsequences by $\{t_n\}$ and $\{H(t_n,x_n)\}$ and $t_n\to t_0$, $H(t_n,x_n)\to q$ as $n\to\infty$. By (FN5), we have

$$N(p-x_n+q,\epsilon)\geq T\Big(N\Big(p-x_n+H(t_n,x_n),\frac{\epsilon}{2}\Big),N(q-H(t_n,x_n),\frac{\epsilon}{2})\Big)\,,$$

it follows that $x_n\to p+q\in\partial\Omega$ as $n\to\infty$. Thus we have

$$p=(I-H(t_0,\cdot))(p+q)\,,$$

which is a contradiction. Therefore the assertion is true.

Besides, by virtue of Lemma 2.2, there exist a finite dimension subspace $X^{(n)}\subset X$ and a finite dimension-valued compact continuous mapping $Q_n\colon[0,1]\times\overline{\Omega}\to X^{(n)}$ such that

$$H(t,x)-Q_n(t,x)\in u(\epsilon,\lambda)\,,\quad (t,x)\in[0,1]\times\overline{\Omega}\,.$$

Letting $q_t(x)=x-Q_n(t,x)$ and $\Omega_n=X^{(n)}\cap\Omega$, then we have

$$\operatorname{Deg}(I-H(t,\cdot),\Omega,p)=\deg_n(q_t,\Omega_n,p),\quad t\in[0,1]\,.$$

If there exist $x_0\in\partial\Omega_n$, $t_0\in[0,1]$ such that $q_{t_0}(x_0)=p$, then we have

$$N(x_0-H(t_0,x_0)-p,\epsilon)=N(x_0-H(t_0,x_0)-x_0+Q_n(t_0,x_0),\epsilon)>1-\lambda\,,$$

which is a contradiction. Therefore, we know that $p\notin q_t(\partial\Omega)$ for all $t\in[0,1]$ and hence we have

$$\operatorname{Deg}(I-H(t,\cdot),\Omega,p)=\deg_n(q_t,\Omega_n,p)=\text{a constant}\,.$$

(d) Since $\overline{\Omega}\setminus(\Omega_1\cup\Omega_2)$ is a closed subset, $R(\overline{\Omega}\setminus(\Omega_1\cup\Omega_2))$ is also a closed subset. Hence there exists a neighborhood of θ, $u(\epsilon,\lambda)$, such that

$$(p+u(\epsilon,\lambda))\cap R(\overline{\Omega}\setminus(\Omega_1\cup\Omega_2))=\emptyset\,.$$

Consequently, we can a finite dimension subspace $X^{(n)}$ of X and a finite dimension-valued continuous compact $R_n\colon\overline{\Omega}\to X^{(n)}$ such that for any $x\in\overline{\Omega}$, $Sx-S_nx\in u(\epsilon,\lambda)$. Letting

$$R_n=I-S_n\,,\quad \Omega_n=X^{(n)}\cap\Omega_1\,,\quad \Omega_n^{(1)}=X^{(n)}\cap\Omega_1\,,\quad \Omega_n^{(2)}=X^{(n)}\cap\Omega_2\,,$$

it follows from Definition 2.4 that

$$\operatorname{Deg}(R,\Omega,p)=\deg_n(R_n,\Omega_n,p)\,,$$
$$\operatorname{Deg}(R,\Omega_1,p)=\deg_n(R_n,\Omega_n^{(1)},p)\,,$$
$$\operatorname{Deg}(R,\Omega_2,p)=\deg_n(R_n,\Omega_n^{(2)},p)\,.$$

It is obvious that $\Omega_n^{(1)} \cap \Omega_n^{(2)} = \emptyset$. If $p \in R_n(\overline{\Omega_n} \setminus (\Omega_n^{(1)} \cup \Omega_n^{(2)}))$, then there exists an x_0 such that $R_n(x_0) = p$. However, since we have

$$N(x_0 - Sx_0 - p, \epsilon) = N(x_0 - Sx_0 - x_0 + S_n x_0, \epsilon) > 1 - \lambda\,,$$

this contradicts

$$(p + u(\epsilon, \lambda)) \cap R(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)) = \emptyset\,.$$

Hence we have $p \notin R_n(\overline{\Omega_n} \setminus (\Omega_n^{(1)} \cup \Omega_n^{(2)}))$ and so

$$\deg_n(R_n, \Omega_n, p) = \deg_n(R_n, \Omega_n^{(1)}, p) + \deg_n(R_n, \Omega_n^{(2)}, p)\,,$$

that is to say,

$$\mathrm{Deg}\,(R, \Omega, p) = \mathrm{Deg}\,(R, \Omega_1, p) + \mathrm{Deg}\,(R, \Omega_2, p).$$

The conclusion (f) Can be obtained from (d) immediately. This achieves the proof. □

Theorem 2.6. *The topological degree defined by* (2.2) *has the following properties:*

(i) *If there exist the degrees of R_1 and R_2 such that $p \in X \setminus R_1(\partial\Omega)$ and $R_1(x) = R_2(x)$ for all $x \in \partial\Omega$, then*

$$\mathrm{Deg}\,(R_1, \Omega, p) = \mathrm{Deg}\,(R_2, \Omega, p),$$

(ii) *If p varies on every connected component of $X \setminus R(\partial\Omega)$, then the degree* $\mathrm{Deg}\,(R, \Omega, p)$ *is a constant.*

Proof. (i) can be obtained immediately.
(ii) Let V be a connected component of $X \setminus R(\partial\Omega)$ and $p \in V$. Then there exists a neighborhood of θ, $u(\epsilon_0, \lambda_0)$, such that $(p + u(\epsilon_0, \lambda_0)) \cap R(\partial\Omega) = \emptyset$. Take positive numbers ϵ_1, λ_1 with $\epsilon_1 < \epsilon_0$, $\lambda_1 < \lambda_0$, $q \in V(p + u(\epsilon_1, \lambda_1))$, and write

$$q_t(x) = R(x) - t(q - p), \quad 0 \le t \le 1\,, \quad x \in \overline{\Omega}\,.$$

If there exist $t_0 \in [0, 1]$, $x_0 \in \partial\Omega$ such that $R(x_0) - t_0(q - p) = p$, then we have

$$N(R(x_0) - p, \epsilon_0) = N(t_0(q - p), \epsilon_0) > 1 - \lambda_0\,,$$

which is a contradiction. Thus it follows that $p \notin q_t(\partial\Omega)$ for all $t \in [0, 1]$. Therefore we have

$$\begin{aligned}\mathrm{Deg}\,(R, \Omega, p) &= \mathrm{Deg}\,(R - (q - p), \Omega, p)\\ &= \mathrm{Deg}\,(S - q, \Omega, \theta) = \mathrm{Deg}\,(R, \Omega, q)\,.\end{aligned}$$

This implies that the mapping $\Theta\colon p \to \mathrm{Deg}\,(R, \Omega, p)$ is a continuous mapping on V. By a well-known result of general topology, we know that $\Theta(V)$ is a connected component. Since Θ is an integer-valued function, for any $p \in V$, $\mathrm{Deg}\,(R, \Omega, p)$ has the same value. This achieves the proof. □

Theorem 2.7. *Let S and S_1 be two compact continuous mappings from Ω into X. If $p \notin R_1(\partial\Omega)$, $p \notin R(\partial\Omega)$, $R_1 = I - S_1$, $R = I - S$ and the following condition is satisfied:*

$$N(S_1 x - Sx, t) \ge N(x - Sx - p, t)\,, \qquad t > 0, x \in \partial\Omega\,, \tag{2.4}$$

then

$$\mathrm{Deg}\,(R-1,\Omega,p)=\mathrm{Deg}\,(R,\Omega,p)\,.$$

Proof. Letting

$$q_t(x)=x-Sx-t(S_1x-Sx)\,,\quad t\in[0,1]\,,\,x\in\overline{\Omega}\,,$$

we are going to prove $p\notin q_t(\partial\Omega)$ for all $t\in[0,1]$. Suppose that this is not the case. Then there exist some $t_0\in[0,1]$ and an $x_0\in\partial\Omega$ such that $q_{t_0}(x_0)=p$. It follows from the assumptions of theorem that $t_0\neq 0$ and $t_0\neq 1$. In view of $x_0-Sx_0-p=t_0(S_1x_0-Sx_0)$, we have

$$N(x_0-Sx_0-p,t)=N\Big(S-1x_0-Sx_0,\frac{t}{t_0}\Big)\,,\quad t>0\,. \tag{2.5}$$

It follows from (2.5) and the conditions of this theorem that

$$N(S_1x_0-Sx_0,t)=N\Big(S_1x_0-Sx_0,\frac{t}{t_0}\Big)=\cdots=N\Big(S_1x_0-Sx_0,\frac{t}{t_0^n}\Big),\quad n=1,2,\dots\,.$$

This implies that $N(S_1x_0-Sx_0,t)=1$ for all $t>0$. By (2.5), we have

$$N(x_0-Sx_0-p,t)=1\,,\quad t>0\,,$$

which shows that $p=x_0-Sx_0$, i.e., $p\in R(\partial\Omega)$. This contradicts $p\notin R(\partial\Omega)$. Thus $p\notin q_t(\partial\Omega)$ for all $t\in[0,1]$ and so we have

$$\mathrm{Deg}\,(R_1,\Omega,p)=\mathrm{Deg}\,(R,\Omega,p)\,.$$

This achieves the proof. □

Corollary 2.8. *If $\theta\in\Omega$, $S_1:\overline{\Omega}\to X$ is a continuous compact mapping satisfying the conditions:*

$$x\neq S_1x\,,\;\;N(S_1x,t)\geq N(x,t)\,,\quad t>0,\;x\in\partial\Omega\,.$$

Then

$$\mathrm{Deg}\,(I-S_1,\Omega,\theta)=1\,.$$

Theorem 2.9. *Let Ω be an open set with $\theta\in\Omega$ and let Ω be symmetric with respect to θ. Suppose that $S\colon\overline{\Omega}\to X$ is a continuous compact mapping and $R=I-S$. If*

$$S(-x)=-S(x)\,,\quad Sx\neq x\,,\quad x\in\partial\Omega\,,$$

then $\mathrm{Deg}\,(R,\Omega,\theta)$ *is an odd number.*

Proof. Imitating the proof of Lemma 2.2, for any neighborhood of θ, $u(\epsilon,\lambda)$, $\epsilon>0$, $\lambda>0$, we can make a finite dimension-valued continuous compact mapping S_n satisfying the following conditions:

(a) $S_n(-x)=-S_n(x)$ for all $x\in\partial(\Omega\cap X^{(n)})$,

(b) $Sx-S_nx\in u(\epsilon,\lambda)$ for all $x\in\overline{\Omega}$.

Since the value of degree $\deg(R_n,\Omega,\theta)$ is odd, the value of degree $\mathrm{Deg}\,(R,\Omega,\theta)$ is also odd, where $\Omega_n=X^{(n)}\cap\Omega$. □

Now, we shall utilize the theory of topological degree to study some fixed point theorems for mappings in fuzzy normed spaces. Let us assume that the t-norm T satisfies the condition $T(t,t)\geq t$ for all $t\in[0,1]$.

Theorem 2.10. *Let Ω be an open convex subset of X and $S\colon \overline{\Omega} \to X$ be a compact continuous mapping such that $S(\partial\Omega) \subset \overline{\Omega}$. Then S has a fixed point in $\overline{\Omega}$.*

Proof. Without loss of generality, we can assume that $Sx \neq x$ for all $x \in \partial\Omega$ (otherwise the conclusion of Theorem has been proved). Taking $x_0 \in \Omega$ and letting $H(t,x) = tSx + (1-t)x_0$, we know that $H\colon [0,1] \times \overline{\Omega} \to X$ is a continuous compact mapping. Letting $q_t(x) = x - H(t,x)$, we prove that

$$\theta \notin q_t(\partial\Omega)\,, \qquad t \in [0,1]\,.$$

Suppose this is not the case. Then there exist an $t_1 \in [0,1]$ and an $x_1 \in \partial\Omega$ such that $q_{t_1}(x_1) = \theta$, i.e.,

$$x_1 = t_1 S x_1 + (1-t_1)x_0\,.$$

It is obvious that $t_1 \neq 0$ and $t_1 \neq 1$. Since Ω is an open set, there exist $\epsilon_0 > 0$, $\lambda_0 > 0$ such that $x_0 + u(\epsilon,\lambda) \subset \Omega$. Because $Sx_1 \in \overline{\Omega}$, we have $x_0 \in \Omega$ and

$$Sx_1 - z_0 \in \frac{1-t_1}{t_1} u(\epsilon_0,\lambda_0)\,. \tag{2.6}$$

Next we prove that

$$t_1 z_0 + (1-t_1)x_0 + (1-t_1)u(\epsilon_0,\lambda_0) \subset \Omega\,. \tag{2.7}$$

In fact, if $x \in t_1 z_0 + (1-t_1)x_0 + (1-t_1)u(\epsilon_0,\lambda_0)$, then there exists some $z \in u(\epsilon_0,\lambda_0)$ such that

$$x = t_1 z_0 + (1-t_1)x_0 + (1-t_1)z = t_1 z_0 + (1-t_1)(x_0 + z)\,.$$

Since $x_0 + u(\epsilon_0,\lambda_0) \subset \Omega$, $x_0 + z \in \Omega$. Next since $z_0 \in \Omega$ and Ω is a convex set, we have $x \in \Omega$. This shows that (2.7) is true. Hence we have

$$x_1 = t_1 S x_1 + (1-t_1)x_0 = t_1 z_0 + (1-t_1)x_0 + t_1(Sx_1 - z_0)\,.$$

It follows from (2.6) that $t_1(Sx_1 - z_0) \in (1-t_1)u(\epsilon_0,\lambda_0)$. By (2.7), it follows that $x_1 \in \Omega$. This contradicts $x_1 \in \partial\Omega$ and hence $\theta \notin q_t(\partial\Omega)$ for all $t \in [0,1]$. Therefore we have

$$\operatorname{Deg}(I - S, \Omega, \theta) = \operatorname{Deg}(I - x_0, \Omega, \theta) = 1\,,$$

which implies that S has a fixed point in Ω. This achieves the proof. □

Theorem 2.11. *Let Ω be an open subset of X with $\theta \in \Omega$. If Ω is symmetric with respect to θ and if $S\colon \partial\Omega \to X$ is a compact continuous mapping satisfying the following condition:*

$$S(-x) = -Sx, \quad x \in \partial\Omega\,.$$

Then S has a fixed point in $\overline{\Omega}$.

Proof. The assertion follows from Theorem 2.9 immediately. □

Moreover, from Corollary 2.8, we can obtain the following:

Theorem 2.12. *Let Ω be an open subset of X with $\theta \in \Omega$. If $S\colon \partial\Omega \to X$ is a compact continuous mapping satisfying the following condition:*

$$N(Sx,t) \geq N(x,t)\,, \quad x \in \partial\Omega,\ t > 0\,.$$

Then S has a fixed point in $\overline{\Omega}$.

Theorem 2.13. *Let Ω_1, Ω_2 be two open subsets of an infinite dimension fuzzy normed space (X, N, T), $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, where the t-norm T satisfies the condition: $T(t,t) \geq t$ for all $t \in [0,1]$. Suppose that $S\colon \overline{\Omega_2} \to X$ is a continuous compact mapping. If one the following conditions holds:*

(i) *for any $x \in \partial\Omega_1$, $N(Sx,t) \geq N(x,t)$ for all $t \geq 0$, and for any $x \in \partial\Omega_2$, $N(Sx,t) \leq N(x,t)$ for all $t \geq 0$,*

(ii) *for any $x \in \partial\Omega_1$, $N(Sx,t) \leq N(x,t)$ for all $t \geq 0$, and for any $x \in \partial\Omega_2$, $N(Sx,t) \geq N(x,t)$ for all $t \geq 0$.*

Then S has at least a fixed point in $\overline{\Omega_2} \setminus \Omega_1$.

In order to give the proof of Theorem 2.13, we need the following lemma:

Lemma 2.14. *Let Ω be an open subset of an infinite dimension fuzzy normed space (X, N, T) with $T(t,t) \geq t$ for all $t \in [0,1]$. Suppose that $S\colon \overline{\Omega} \to X$ is a continuous compact mapping satisfying the following conditions:*

(i) *$\theta \notin \overline{S(\partial\Omega)}$,*

(ii) *$Sx \neq \mu x$ for all $\mu \in [0,1]$ and $x \in \partial\Omega$.*

Then $\operatorname{Deg}(I - S, \Omega, \theta) = 0$.

Proof. First we prove that $\theta \notin \overline{\cup_{\mu\in[0,1]}(\mu I - S)(\partial\Omega)}$. Suppose this is not the case. Then there exist $x_n \in \partial\Omega$, $\mu_n \in [0,1]$ such that $\mu_n x_n - Sx_n \to \theta$ as $n \to \infty$. Since S is a compact continuous mapping, there exist subsequences $\{\mu_{n_k}\} \subset \{\mu_n\}$ and $\{x_{n_k}\} \subset \{x_n\}$ such that $\mu_{n_k} \to \mu_0 \in [0,1]$, $Sx_{n_k} \to y_0 \in X$.

(a) If $\mu_0 = 0$, then $Sx_{n_k} \to \theta$, which contradicts condition (i).

(b) If $\mu_0 \neq 0$, then $x_{n_k} \to y_0/\mu_0 \in \partial\Omega$ and hence we have

$$S\Big(\frac{y_0}{\mu_0}\Big) = y_0 = \mu_0 \cdot \frac{y_0}{\mu_0}\,.$$

This contradicts the condition (ii) and so the the assertion holds.

Therefore there exists some neighborhood of θ, $u(\epsilon, \lambda)$, $\epsilon > 0$, $\lambda > 0$, such that

$$u(\epsilon,\lambda) \cap \overline{\cup_{\mu\in[0,1]}(\mu I - S)(\partial\Omega)} = \emptyset. \tag{2.8}$$

By Lemma 2.2 and Definition 2.4, there exists a finite dimension-valued compact continuous mapping $S_n\colon \overline{\Omega} \to X^{(n)}$ such that

$$Sx - S_n x \in u(\epsilon,\lambda)\,, \qquad x \in \overline{\partial\Omega}\,,$$
$$\operatorname{Deg}(I - S, \Omega, \theta) = \deg(I - S_n, \Omega_n, \theta)\,,$$

where $\Omega_n = \Omega \cap X^{(n)}$. By assumption, (X, N, T) is infinitely dimensional and hence there exists an $e_1 \neq \theta$ and $e_1 \notin X^{(n)}$. Letting $X^{(n+1)} = span\{e_1, X^{(n)}\}$, we can assume that S_n is a mapping from $\overline{\Omega}$ into $X^{(n+1)}$. Put $\Omega_{n+1} = \Omega \cap X^{(n+1)}$. By Definition 2.4, it follows that

$$\operatorname{Deg}(I - S, \Omega, \theta) = \deg(I - S_n, \Omega_{n+1}, \theta)\,. \tag{2.9}$$

Next, we prove that for any $x \in \partial\Omega_{n+1} \subset \partial\Omega$, $\theta \neq \mu x - S_n x$ for all $\mu \in [0,1]$. In fact, if there exist some $\mu_0 \in [0,1]$ and an $x_0 \in \partial\Omega_{n+1}$ such that $\mu_0 x_0 - S_n x_0 = \theta$, then we have $\mu_0 x_0 = S_n x_0$. Since $Sx - S_n x \in u(\epsilon, \lambda)$ for all $x \in \overline{\Omega}$, we have $Sx_0 - \mu_0 x_0 \in u(\epsilon, \lambda)$. This contradicts (2.8). Thus the assertion is true. Therefore, on $\overline{\Omega_{n+1}}$, we have

$$\deg(I - S_n, \Omega_{n+1}, \theta) = \deg(-S_n, \Omega_{n+1}, \theta)\,. \tag{2.10}$$

However, since S_n is a mapping from $\overline{\Omega_{n+1}}$ into $X^{(n)}$, we have $\deg(-S_n, \Omega_{n+1}, \theta) = 0$. It follows from (2.9) and (2.10) that

$$\mathrm{Deg}\,(I - S, \Omega, \theta) = 0\,.$$

This achieves the proof. □

Proof of Theorem 2.13. Suppose that the condition (i) is satisfied and S has no fixed point in $\partial\Omega_1 \cup \partial\Omega_2$ (otherwise, the conclusion of theorem has been shown). It follows from Corollary 2.8 that

$$\mathrm{Deg}\,(I - S, \Omega_1, \theta) = 1\,.$$

By the assumption, for any $x \in \partial\Omega_2$, $N(Sx, t) \leq N(x, t)$ for all $t \geq 0$ and hence we have

$$\theta \notin \overline{S(\partial\Omega_2)} \quad \text{and} \quad Sx \neq \mu x, \quad \mu \in (0, 1]\,.$$

From Lemma 2.14, it follows that $\mathrm{Deg}\,(I - S, \Omega_2, \theta) = 0$. Besides, since

$$\begin{aligned}\mathrm{Deg}\,(I - S, \Omega_2 \setminus \overline{\Omega_1}, \theta) &= \mathrm{Deg}\,(I - S, \Omega_2, \theta) - \mathrm{Deg}\,(I - S, \Omega_1, \theta)\\ &= 0 - 1 = -1\,,\end{aligned}$$

S has a fixed point in $\Omega_2 \setminus \Omega_1$.

If the condition (ii) is satisfied, in the same way, we can prove the assertion holds too. This achieves the proof. □

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