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METRICALLY REGULAR SQUARE OF METRICALLY REGULAR BIPARTITE GRAPHS OF DIAMETER $D = 7$

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Abstract. The present paper deals with the spectra of powers of metrically regular graphs. We prove that there is only two tables of the parameters of an association scheme so that the corresponding metrically regular bipartite graph of diameter $D = 7$ (8 distinct eigenvalues of the adjacency matrix) has the metrically regular square. The results deal with the graphs of the diameter $D < 7$ see [8], [9] and [10].

1. Introduction and Notation

The theory of *metrically regular graphs* originates from the theory of *association schemes* first introduced by R.C. Bose and Shimamoto [1]. All graphs will be undirected, without loops and multiple edges.

Definition 1. Let X be a finite set, $\operatorname{card} X \geq 2$. For an arbitrary natural number D let $\boldsymbol{R} = \{R_0, R_1, \dots, R_D\}$ be a system of binary relations on X. A pair $(X, \boldsymbol{R})$ will be called *an association scheme* with D classes if and only if it satisfies the axioms $A1$ - $A4$:

- **A1.** The system $\boldsymbol{R}$ is a partition of the set X^2 and R_0 is the diagonal relation, $R_0 = \{(x, x);\, x \in X\}$.
- **A2.** For each $i \in \{0, 1, \dots, D\}$, it holds $R_i^{-1} \in \boldsymbol{R}$.
- **A3.** For each $i, j, k \in \{0, 1, \dots, D\}$ it holds $(x, y) \in R_k \wedge (x_1, y_1) \in R_k$ then $p_{ij}(x, y) = p_{ij}(x_1, y_1)$, where $p_{ij}(x, y) = |\{z; (x, z) \in R_i \wedge (z, y) \in R_j\}|$. Then define $p_{ij}^k := p_{ij}(x, y)\,,$ where $(x, y) \in R_k$.
- **A4.** For each $i,\ j,\ k \in \{0, 1, \dots, D\}$ it holds $p_{ij}^k = p_{ji}^k$.

The set X will be called the *carrier* of the association scheme $(X, \mathbf{R})$. Especially, $p_{i0}^k = \delta_{ik}$, $p_{ij}^0 = v_i \delta_{ij}$, where δ_{ij} is the Kronecker-Symbol and $v_i := p_{ii}^0$, and define $P_j := (p_{ij}^k)$, $0 \leq i, j, k \leq D$.

Given a graph $G = (X, E)$ of diameter D we may define $R_k = \{(x, y); d(x, y) = k\}$, where $d(x, y)$ is the distance from the vertex x to the vertex y in the standard

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graph metric. If $(X, \boldsymbol{R})$, $\boldsymbol{R} = \{R_0, R_1, \dots, R_D\}$, gives rise to an association scheme, the graph is called *metrically regular* and the p_{ij}^k are said to be its *parameters* or its *structural constants*. Especially, metrically regular graphs with the diameter $D = 2$ are called *strongly regular*.

Let $G = (X, Y)$ be an undirected graph without loops and multiple edges. *The second power* (or *square* of G) is the graph $G^2 = (X, E')$ with the same vertex set X and in which mutually different vertices are adjacent if and only if there is at least one path of length 1 or 2 in G between them.

The characteristic polynomial of the adjacency matrix A of a graph G is called the *characteristic polynomial* of G and the eigenvalues and the spectrum of A are called the *eigenvalues* and the *spectrum of G*. The greatest eigenvalue ρ of G is called the *index* of G.

Define $(0,1)$-matrices $A_0, \dots, A_D$ by $A_0 = I$ and $(A_i)_{jk} = 1$ if and only if the distance from the vertex j to the vertex k in G is $\mathrm{d}\,(j,k) = i$. Using these notations it follows:

Theorem 1 ([4]). *For a metrically regular graph G with diameter D and for any real numbers $r_1, \dots, r_D$ the distinct eigenvalues of $\sum_{i=1}^{D} r_i A_i$ and $\sum_{i=1}^{D} r_i P_i$ are the same. In particular the distinct eigenvalues of a metrically regular graph are the same as those of P_1.*

Theorem 2 ([7]). *A metrically regular graph with diameter D has $D+1$ distinct eigenvalues.*

Theorem 3 ([6]). *The number of components of a regular graph G is equal to the multiplicity of its index.*

Theorem 4 ([5]). *A graph containing at least one edge is bipartite if and only if its spectrum, considered as a set of points on the real axis, is symmetric with respect to the zero point.*

Theorem 5 ([5]). *A strongly connected digraph G with the greatest eigenvalue r has no odd cycles if and only if $-r$ is also an eigenvalue of G.*

Theorem 6 ([8]). *For every $k \in \mathbb{N}$, $k \geq 2$ there is one and only one metrically regular bipartite graph $G = (X, E)$ with diameter $D = 3$, $n = |X| = 2k + 2$, so that G^2 is a strongly regular graph.*

Theorem 7 ([8]). *There is only one table of the parameters of an association scheme so that the corresponding metrically regular bipartite graph with 5 distinct eigenvalues has the strongly regular square. The realization of this table is the 4-dimensional unit cube.*

Theorem 8 ([9]). *There are only four tables of the parameters of association schemes for $k \in \{1, 2, 4, 10\}$ so that the corresponding metrically regular bipartite graphs with 6 distinct eigenvalues have the metrically regular square.*

Theorem 9 ([10]). *There is only one table of the parameters of an association scheme with 6 classes so that the corresponding metrically regular bipartite graph of diameter $D = 6$ (7 distinct eigenvalues of the adjacency matrix) has the metrically regular square. The realization of this table is the 6-dimensional unit cube.*

Further, we use some of the known relations from the theory of associations schemes (see eg. [3]).

$$v_i = \sum_j p_{ij}^k \tag{1.1}$$

$$v_i p_{jk}^i = v_j p_{ik}^j \tag{1.2}$$

2. Main result

Let $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6 > \lambda_7 > \lambda_8$ are the eigenvalues of MRG G with respective multiplicities $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8$. As G is a bipartite graph we obtain from Theorem 4:

$$\begin{aligned} \lambda_1 &= -\lambda_8, & m_1 &= m_8 = 1\,, \\ \lambda_2 &= -\lambda_7, & m_2 &= m_7\,, \\ \lambda_3 &= -\lambda_6, & m_3 &= m_4\,, \\ \lambda_4 &= -\lambda_5, & m_4 &= m_5\,. \end{aligned} \tag{2.1}$$

and it holds for the structural constants of G:

$$\begin{gathered} p_{ij}^k = 0 \quad \text{for} \quad i,j,k \in \{0,1,\dots,7\}\ , \\ i+j+k \equiv 0 \,(\mathrm{mod}\ 2) \quad \text{and also for} \quad i+j<k \quad \text{and} \quad |i-j|>k\,. \end{gathered} \tag{2.2}$$

According to Theorem 1. λ_i $(i = 1, 2, \dots, 8)$ is the solution of the equation $|\lambda I - P_1| = 0$ and we get

$$\begin{aligned} \lambda^8 &- \lambda^6 \big[\lambda_1 + p_{12}^1 p_{11}^2 + p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7\big] \\ &+ \lambda^4 \big[p_{12}^1 p_{11}^2 (p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7) \\ &+ p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7) + + p_{14}^3 p_{13}^4 (p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7) \\ &+ p_{15}^4 p_{14}^5 p_{17}^6 p_{16}^7 + \lambda_1 (p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7)\big] \\ &- \lambda_2 \big\{p_{11}^2 p_{14}^3 p_{13}^4 (p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7) + (p_{12}^1 p_{11}^2 + p_{13}^2 p_{12}^3) p_{15}^4 p_{14}^5 p_{17}^6 p_{16}^7 \\ &+ \lambda_1 \big[p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7) \\ &+ p_{14}^3 p_{13}^4 (p_{16}^5 p_{15}^6 + p_{17}^6 p_{16}^7)\big] + p_{15}^4 p_{14}^5 p_{17}^6 p_{16}^7 \big\} \\ &+ \lambda_1 p_{13}^2 p_{12}^3 p_{15}^4 p_{14}^5 p_{17}^6 p_{16}^7 = 0 \end{aligned} \tag{2.3}$$

The condition for G to have the square G^2 metrically regular gives the following relations for the structural constants ${}^2p_{ij}^k$ of G^2:

$$
\begin{aligned}
&(2.4) && {}^2p_{11}^1 = 2p_{12}^1 = p_{11}^2 + p_{22}^2\\
&(2.5) && {}^2p_{12}^1 = p_{23}^1 = p_{13}^2 + p_{24}^2\\
&(2.6) && {}^2p_{22}^1 = 2p_{34}^1 = p_{33}^2 + p_{44}^2\\
&(2.7) && {}^2p_{23}^1 = p_{45}^1 = p_{35}^2 + p_{46}^2\\
&(2.8) && {}^2p_{33}^1 = 2p_{56}^1 = p_{55}^2 + p_{66}^2\\
&(2.9) && {}^2p_{34}^1 = p_{67}^1 = p_{57}^2\\
&(2.10) && {}^2p_{44}^1 = p_{77}^1 = p_{77}^2 = 0\\
&(2.11) && {}^2p_{11}^2 = 2p_{12}^3 = p_{22}^4\\
&(2.12) && {}^2p_{12}^2 = p_{14}^3 + p_{23}^3 = p_{13}^4 + p_{24}^4\\
&(2.13) && {}^2p_{13}^2 = p_{25}^3 = p_{15}^4 + p_{26}^4\\
&(2.14) && {}^2p_{22}^2 = 2p_{34}^3 = p_{33}^4 + p_{44}^4\\
&(2.15) && {}^2p_{23}^2 = p_{36}^3 + p_{45}^3 = p_{35}^4 + p_{46}^4\\
&(2.16) && {}^2p_{24}^2 = p_{47}^3 = p_{37}^4\\
&(2.17) && {}^2p_{33}^2 = 2p_{56}^3 = p_{55}^4 + p_{66}^4\\
&(2.18) && {}^2p_{34}^2 = p_{67}^3 = p_{57}^4\\
&(2.19) && {}^2p_{44}^2 = p_{77}^3 = p_{77}^4 = 0\\
&(2.20) && {}^2p_{12}^3 = p_{14}^5 + p_{23}^5 = p_{24}^6\\
&(2.21) && {}^2p_{13}^3 = p_{16}^5 + p_{25}^5 = p_{15}^6 + p_{26}^6\\
&(2.22) && {}^2p_{14}^3 = p_{27}^5 = p_{17}^6\\
&(2.23) && {}^2p_{22}^3 = 2p_{34}^5 = p_{33}^6 + p_{44}^6\\
&(2.24) && {}^2p_{23}^3 = p_{36}^5 + p_{45}^5 = p_{35}^6 + p_{46}^6\\
&(2.25) && {}^2p_{24}^3 = p_{47}^5 = p_{37}^6\\
&(2.26) && {}^2p_{33}^3 = 2p_{56}^5 = p_{55}^6 + p_{66}^6\\
&(2.27) && {}^2p_{34}^3 = p_{67}^5 = p_{57}^6\\
&(2.28) && {}^2p_{44}^3 = p_{77}^5 = p_{77}^6 = 0\\
&(2.29) && {}^2p_{13}^4 = p_{16}^7 + p_{25}^7\\
&(2.30) && {}^2p_{14}^4 = p_{27}^7\\
&(2.31) && {}^2p_{22}^4 = 2p_{34}^7\\
&(2.32) && {}^2p_{23}^4 = p_{36}^7 + p_{45}^7\\
&(2.33) && {}^2p_{24}^4 = p_{47}^7\\
&(2.34) && {}^2p_{33}^4 = 2p_{56}^7\\
&(2.35) && {}^2p_{34}^4 = p_{67}^7\\
&(2.36) && {}^2p_{44}^4 = p_{77}^7 = 0
\end{aligned}
$$

If A denotes the adjacency matrix of G and A_2 is the adjacency matrix of G^2 it is easy to see

$$A_2 = \frac{1}{p_{11}^2} A^2 + \frac{p_{11}^2 - p_{11}^1}{p_{11}^2} A - \frac{\lambda_1}{p_{11}^2} I \, .$$

The eigenvalues of G^2 are in regard of (2.2) in the form

$$\mu_i = \frac{\lambda_i^2 + p_{11}^2 \lambda_i - \lambda_1}{p_{11}^2} \, , \quad i \in \{1, 2, \ldots, 8\} \, . \tag{2.37}$$

As G^2 is a metrically regular graph with diameter 4 it must have just 5 distinct numbers as its eigenvalues with regard of Theorem 2. So it must hold $\mu_i = \mu_j = \mu_k = \mu_l$ or $\mu_i = \mu_j = \mu_k$ and $\mu_l = \mu_m$ or $\mu_i = \mu_j$, $\mu_k = \mu_l$ and $\mu_m = \mu_n$ (for distinct numbers i, j, k, l, m, n; $i, j, k, l, m, n \neq 1$ because G^2 is connected and therefore its index μ_1 has the multiplicity 1).

A. $\mu_i = \mu_j = \mu_k = \mu_l$
According to (2.37) we obtain

$$\lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_i + \lambda_l = \lambda_j + \lambda_k = \lambda_j + \lambda_l = \lambda_k + \lambda_l = -p_{11}^2$$

and we get the contradiction with $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_l \neq \lambda_i$.

B. $\mu_i = \mu_j = \mu_k, \mu_l = \mu_m$.
According to (2.37) we obtain

$$\lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_j + \lambda_k = -p_{11}^2 \, , \quad \lambda_l + \lambda_m = -p_{11}^2 \, .$$

and we get the contradiction with $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i$.

C. $\mu_i = \mu_j, \ \mu_k = \mu_l, \ \mu_m = \mu_n$.
As $\lambda_2 > \lambda_3 > \lambda_4 > 0$, $p_{11}^2 > 0$, $\lambda_2 \geq |\lambda_t|$, $t \in \{3, 4, 5, 6, 7\}$,
$\lambda_3 \geq |\lambda_s|$, $s \in \{4, 5, 6\}$ and $\lambda_4 = |\lambda_5|$ we get:

$$\begin{array}{lllll}
\mu_2 = \mu_j & \Longrightarrow & \lambda_2 + \lambda_j = -p_{11}^2 & \Longrightarrow & j \in \{8\} \, , \\
\mu_3 = \mu_k & \Longrightarrow & \lambda_3 + \lambda_k = -p_{11}^2 & \Longrightarrow & k \in \{7, 8\} \, , \\
\mu_4 = \mu_n & \Longrightarrow & \lambda_4 + \lambda_n = -p_{11}^2 & \Longrightarrow & n \in \{6, 7, 8\} \, , \\
\mu_5 = \mu_s & \Longrightarrow & \lambda_5 + \lambda_s = -p_{11}^2 & \Longrightarrow & s \in \{6, 7, 8\} \, , \\
\mu_6 = \mu_t & \Longrightarrow & \lambda_6 + \lambda_t = -p_{11}^2 & \Longrightarrow & t \in \{4, 5, 7, 8\} \, , \\
\mu_7 = \mu_u & \Longrightarrow & \lambda_7 + \lambda_u = -p_{11}^2 & \Longrightarrow & u \in \{3, 4, 5, 6, 8\} \, , \\
\mu_8 = \mu_v & \Longrightarrow & \lambda_8 + \lambda_v = -p_{11}^2 & \Longrightarrow & v \in \{2, 3, 4, 5, 6, 7\} \, ,
\end{array}$$

and we obtain

$$\lambda_2 = \lambda_1 - p^2_{11}\,,$$
$$\lambda_3 = \lambda_2 - p^2_{11} = \lambda_1 - 2p^2_{11}\,,$$
$$\lambda_4 = \lambda_3 - p^2_{11} = \lambda_1 - 3p^2_{11}\,,$$
$$\lambda_5 = -\lambda_4 = -\lambda_1 + 3p^2_{11}\,, \tag{2.38}$$
$$\lambda_6 = -\lambda_4 - p^2_{11} = -\lambda_1 + 2p^2_{11}\,,$$
$$\lambda_7 = -\lambda_3 - p^2_{11} = -\lambda_1 + p^2_{11}\,,$$
$$\lambda_8 = -\lambda_1\,. \tag{2.39}$$

From (2.10), (2.19), (2.28) and (2.36) it follows

$$p^7_{27} = p^7_{47} = p^7_{67} = p^7_{77} = 0$$

and we get from (1.1) $(i = 7;\ k = 7)$

$$v_7 = \sum_{j=0}^{7} p^7_{7j} = 1\,. \tag{2.40}$$

As diameter of G $D = 7$ we have

$$p^k_{j,7} = \begin{cases} 1, & \text{for } j+k = 7, j,k \in \{0,1,\dots,7\}\,,\\ 0, & \text{for } j+k \neq 7, j,k \in \{0,1,\dots,7\}\end{cases} \tag{2.41}$$

and with respect of (1.2) for $(i = 1,\ j = 6,\ k = 7)$, $(i = 2,\ j = 5,\ k = 7)$ and $(i = 3,\ j = 4,\ k = 7)$ we obtain

$$v_0 = v_7 = 1\,; \quad \lambda_1 = v_1 = v_6\,; \quad v_2 = v_5\,; \quad v_3 = v_4\,. \tag{2.42}$$

From (1.1) $(i = 1;\ k = 1, 2, 7)$ it follows

$$p^1_{12} = \lambda_1 - 1\,, \quad p^2_{13} = \lambda_1 - p^2_{11}, \quad p^7_{16} = \lambda_1\,. \tag{2.43}$$

With regard of (1.1) $(i = 6,\ k = 1)$ and (2.42) it follows

$$p^1_{56} = \lambda_1 - 1 \tag{2.44}$$

and from (1.2) we get

$$v_1 p^1_{56} = v_6 p^6_{15}\,, \quad \text{so} \quad p^6_{15} = \lambda_1 - 1\,. \tag{2.45}$$

With respect to (1.1) $(i = 2, 6;\ k = 7)$, (2.43), (2.41) it follows

$$p^7_{25} = v_2,\ p^7_{36} = 0\,. \tag{2.46}$$

Relations (1.2) and (2.42) $(v_3 = v_4)$ gives

$$p^3_{14} = p^4_{13}\,, \tag{2.47}$$
$$p^3_{44} = p^4_{34}\,. \tag{2.48}$$

$$\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 = \lambda_1 p^2_{13} p^3_{12} p^4_{15} p^5_{14} p^6_{17} p^7_{16} \tag{2.49}$$

So, from (2.3), (2.38), (2.41) ($p^6_{17} = 1$), (2.43), (1.2) ($i = 2$, $j = 3$, $k = 1$); ($i = 4$, $j = 5$, $k = 1$) and with respect to $\lambda_4 = \lambda_1 - 3p^2_{11} > 0$ we obtain

$$\lambda_1 > 3p^2_{11} \tag{2.50}$$

$$\lambda_1 p^2_{13}\Big(\frac{v_2}{v_3}p^2_{13}\Big)p^4_{15}\Big(\frac{v_4}{v_5}p^4_{15}\Big)p^6_{17}p^7_{16} = \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2$$

$$\lambda_1^2(\lambda_1 - p_{11})^2\left(p^4_{15}\right)^2 = \lambda_1^2(\lambda_1 - p^2_{11})^2(\lambda_1 - 2p^2_{11})^2(\lambda_1 - 3p^2_{11})^2$$

$$\text{so} \qquad p^4_{15} = (\lambda_1 - 2p^2_{11})(\lambda_1 - 3p^2_{11})\,. \tag{2.51}$$

With regard of $D = 7$ we get $p^4_{13} > 0$ so from (1.1) ($i = 1, k = 4$) we obtain

$$1 \leqslant p^4_{15} \leqslant \lambda_1 - 1\,, \tag{2.52}$$

$$\lambda_1^2 - (5p^2_{11} + 1)\lambda_1 + (6p^2_{11}p^2_{11} + 1) \leqq 0\,.$$

$$3p^2_{11} < \lambda_1 \leqslant \frac{5p^2_{11} + 1 + \sqrt{(p^2_{11} + 5)^2 - 28}}{2} < 3\left(p^2_{11} + 1\right)$$

$$\lambda_1 \in \left\{3p^2_{11}\ +\ 1,\ 3p^2_{11}\ +\ 2\right\} \tag{2.53}$$

A. $\boldsymbol{\lambda_1 = v_1 = 3p^2_{11}\ +\ 2}$

From (2.51) we get $p^4_{15} = 2(p^2_{11} + 2)$ and from (1.1) ($i = 1, k = 1$) we obtain $p^1_{12} = \lambda_1 - 1 = 3p^2_{11} + 1$. As $1 \leqslant p^4_{15} \leqslant \lambda_1 - 1$ we get

$$p^2_{11} \geqslant 3 \tag{2.54}$$

From (1.2) ($i = 1, j = 1, k = 2$) we obtain

$$v_2 = \frac{(3p^2_{11} + 2)(3p^2_{11} + 1)}{p^2_{11}} = 9p^2_{11} + 9 + \frac{2}{p^2_{11}}$$

so $p^2_{11} \in \{1,\ 2\}$ and we get the contradiction with (2.54).

B. $\boldsymbol{\lambda_1 = v_1 = 3p^2_{11}\ +\ 1}$

From (2.51), (1.1) ($i = 1, k = 1$; $i = 1, k = 4$) and (2.42) $v_3 = v_4$ we obtain

$$p^4_{15} = p^2_{11} + 1 \tag{2.55}$$

$$p^1_{12} = \lambda_1 - 1 = 3p^2_{11} \tag{2.56}$$

$$p^4_{13} = 2p^2_{11} = p^3_{14} \tag{2.57}$$

From (1.2) ($i = 1, j = 1, k = 2$), (2.55) and (2.42) it follows

$$v_2 = 3\left(3p^2_{11} + 1\right) = v_5 \tag{2.58}$$

From (1.1) ($i = 2, k = 1$), (2.44) and (1.1) ($i = 5, k = 1$) we obtain

$$p^1_{23} = 3(2p^2_{11} + 1) \tag{2.59}$$

$$p^1_{56} = 3p^2_{11} \tag{2.60}$$

$$p^1_{54} = 3(2p^2_{11} + 1) \tag{2.61}$$

(2.26), (2.42), (2.58) and (1.2) $(i = 5, j = 6, k = 5)$ give

$$\begin{aligned} 2p^5_{56} &= (p^6_{55} + p^6_{66})\,, \\ 2\frac{v_6}{v_5}p^6_{55} &= (p^6_{55} + p^6_{66})\,, \\ p^6_{55} + 3p^6_{66} &= 0 \end{aligned}$$

and we obtain

$$p^6_{55} = p^6_{66} \tag{2.62}$$

(1.2) $(i = 1, j = 4, k = 5)$, (2.55) and (2.61) give

$$v_4 = \frac{(3p^2_{11} + 1)3(2p^2_{11} + 1)}{p^2_{11} + 1} = 18p^2_{11} - 3 + \frac{6}{p^2_{11} + 1}\,. \tag{2.63}$$

So, we get the spectrum of G in the form

$$S_p(G) = \left\{ \begin{matrix} \pm(3p^2_{11} + 1); & \pm(2p^2_{11} + 1); & \pm(p^2_{11} + 1); & \pm 1 \\ 1 & m_2 & m_3 & m_4 \end{matrix} \right\}. \tag{2.64}$$

From (2.58), (2.63) and (2.64) we obtain for the number of the considered graph

$$n = 4\frac{(5p^2_{11} + 2)(3p^2_{11} + 2)}{p^2_{11} + 1}\,.$$

For the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_8$ and their corresponding multiplicities $m_1, m_2, \dots, m_8$ of the considered graphs it holds

$$\begin{aligned} m_1 &= 1, && \text{since the graph is connected,} \\ \sum_{i=1}^{8} m_i &= \sum_{j=0}^{7} v_j = n, && \text{the number of vertices,} \\ \sum_{i=1}^{8} m_i\lambda_i &= 0, && \text{since the graph has no loops,} \\ \sum_{i=1}^{8} m_i\lambda_i^2 &= n\lambda_1, && \text{since the graph is regular,} \\ \sum_{i=1}^{8} m_i\lambda_i^3 &= n\lambda_1 p^1_{11}, && \text{the number of the chains of the length 3,} \\ \sum_{i=1}^{8} m_i\lambda_i^4 &= n\lambda_1[2\lambda_1 - 1 + p^1_{11}(p^1_{11} - 1) + p^1_{12}(p^2_{11} - 1)]\,, && \\ & && \text{the number of the chains of the length 4.} \end{aligned}$$

So, with respect to (2.1), (2.2) and (2.61) we obtain

$$\begin{aligned}
&m_1 = 1\\
&2m_1 + 2m_2 + 2m_3 + 2m_4 = 4\frac{(5p_{11}^2+2)(3p_{11}^2+2)}{p_{11}^2+1}\,,\\
&2m_1(3p_{11}^2+1)^2 + 2m_2(2p_{11}^2+1)^2 + 2m_3(p_{11}^2+1)^2 + 2m_4\\
&\qquad = 4\frac{(5p_{11}^2+2)(3p_{11}^2+2)(3p_{11}^2+1)}{p_{11}^2+1}\\
&2m_1(3p_{11}^2+1)^4 + 2m_2(2p_{11}^2+1)^4 + 2m_3(p_{11}^2+1)^4 + 2m_4\\
&\qquad = 4\frac{(5p_{11}^2+2)(3p_{11}^2+2)(3p_{11}^2+1)}{p_{11}^2+1}\left[3(p_{11}^2)^2 + 3p_{11}^2 + 1\right]\,.
\end{aligned}$$

These equations imply

$$m_1 = 1\,, \tag{2.65}$$

$$m_2 = \frac{3(3p_{11}^2+1)}{p_{11}^2+1}\,, \tag{2.66}$$

$$m_3 = \frac{3(5p_{11}^2+2)(3p_{11}^2+1)}{(p_{11}^2+2)(p_{11}^2+1)}\,, \tag{2.67}$$

$$m_4 = \frac{(5p_{11}^2+2)(3p_{11}^2+1)(2p_{11}^2+1)}{(p_{11}^2+2)(p_{11}^2+1)}\,. \tag{2.68}$$

As m_2, m_3, $m_4 \in \mathbb{N}$ it must hold

$$p_{11}^2 \in \{1, 2\}\,. \tag{2.69}$$

The relations (1.1), (1.2), (2.1) – (2.69) give the following tables of the association schemes:

1. $p_{11}^2 = 1$

$p_{10}^1 = 1$	$p_{20}^2 = 1$	$p_{30}^3 = 1$	$p_{40}^4 = 1$	$p_{50}^5 = 1$	$p_{60}^6 = 1$	$p_{70}^7 = 1$
$p_{12}^1 = 3$	$p_{11}^2 = 1$	$p_{12}^3 = 2$	$p_{13}^4 = 2$	$p_{14}^5 = 3$	$p_{15}^6 = 3$	$p_{16}^7 = 4$
$p_{23}^1 = 9$	$p_{13}^2 = 3$	$p_{14}^3 = 2$	$p_{15}^4 = 2$	$p_{16}^5 = 1$	$p_{17}^6 = 1$	$p_{25}^7 = 12$
$p_{34}^1 = 9$	$p_{22}^2 = 5$	$p_{23}^3 = 6$	$p_{22}^4 = 4$	$p_{23}^5 = 6$	$p_{24}^6 = 9$	$p_{34}^7 = 18$
$p_{45}^1 = 9$	$p_{24}^2 = 6$	$p_{25}^3 = 4$	$p_{24}^4 = 6$	$p_{25}^5 = 5$	$p_{26}^6 = 3$	$\lambda_7 = -3$
$p_{56}^1 = 3$	$p_{33}^2 = 9$	$p_{34}^3 = 9$	$p_{26}^4 = 2$	$p_{27}^5 = 1$	$p_{33}^6 = 9$	$m_7 = 6$
$p_{67}^1 = 1$	$p_{35}^2 = 6$	$p_{36}^3 = 2$	$p_{33}^4 = 9$	$p_{34}^5 = 9$	$p_{35}^6 = 9$	$\lambda_8 = -4$
$\lambda_1 = 4$	$p_{44}^2 = 9$	$p_{45}^3 = 6$	$p_{35}^4 = 6$	$p_{36}^5 = 3$	$p_{44}^6 = 9$	$m_8 = 1$

$m_1 = 1$	$p^2_{46} = 3$	$p^3_{47} = 1$	$p^4_{37} = 1$	$p^5_{45} = 6$	$\lambda_5 = -1$
$\lambda_2 = 3$	$p^2_{55} = 5$	$p^3_{56} = 2$	$p^4_{44} = 9$	$\lambda_4 = 1$	$m_5 = 14$
$m_2 = 6$	$p^2_{57} = 1$	$\lambda_3 = 2$	$p^4_{46} = 2$	$m_4 = 14$	$\lambda_6 = -2$
	$p^2_{66} = 1$	$m_3 = 14$	$p^4_{55} = 4$		$m_6 = 14$

2. $p^2_{11} = 2$

$p^1_{10} = 1$	$p^2_{20} = 1$	$p^3_{30} = 1$	$p^4_{40} = 1$	$p^5_{50} = 1$	$p^6_{60} = 1$	$p^7_{70} = 1$
$p^1_{12} = 6$	$p^2_{11} = 2$	$p^3_{12} = 3$	$p^4_{13} = 4$	$p^5_{14} = 5$	$p^6_{15} = 6$	$p^7_{16} = 7$
$p^1_{23} = 15$	$p^2_{13} = 5$	$p^3_{14} = 4$	$p^4_{15} = 3$	$p^5_{16} = 2$	$p^6_{17} = 1$	$p^7_{25} = 21$
$p^1_{34} = 20$	$p^2_{22} = 10$	$p^3_{23} = 12$	$p^4_{22} = 6$	$p^5_{23} = 10$	$p^6_{24} = 15$	$p^7_{34} = 35$
$p^1_{45} = 15$	$p^2_{24} = 10$	$p^3_{25} = 6$	$p^4_{24} = 12$	$p^5_{25} = 10$	$p^6_{26} = 6$	$\lambda_7 = -5$
$p^1_{56} = 6$	$p^2_{33} = 20$	$p^3_{34} = 18$	$p^4_{26} = 3$	$p^5_{27} = 1$	$p^6_{33} = 20$	$m_7 = 7$
$p^1_{67} = 1$	$p^2_{35} = 10$	$p^3_{36} = 4$	$p^4_{33} = 18$	$p^5_{34} = 20$	$p^6_{35} = 15$	$\lambda_8 = -1$
$\lambda_1 = 7$	$p^2_{44} = 20$	$p^3_{45} = 12$	$p^4_{35} = 12$	$p^5_{36} = 5$	$p^6_{44} = 20$	$m_8 = 1$
$m_1 = 1$	$p^2_{46} = 5$	$p^3_{47} = 1$	$p^4_{37} = 1$	$p^5_{45} = 10$	$\lambda_5 = -1$	
$\lambda_2 = 5$	$p^2_{55} = 10$	$p^3_{56} = 3$	$p^4_{44} = 18$	$\lambda_4 = 1$	$m_5 = 35$	
$m_2 = 7$	$p^2_{57} = 1$	$\lambda_3 = 3$	$p^4_{46} = 4$	$m_4 = 35$	$\lambda_6 = -3$	
	$p^2_{66} = 2$	$m_3 = 21$	$p^4_{55} = 6$		$m_6 = 21$	

So we have proved the following theorem:

Theorem 10. *There are only two tables of the parameters of association schemes for $p^2_{11} \in \{1, 2\}$ so that the corresponding metrically regular bipartite graphs with 8 distinct eigenvalues (diameter $D = 7$) have the metrically regular square.*

The realization of the case $p^2_{11} = 1$ is (7,3)-bipartite Kneser graph [2] with the intersection array

$$\{4,3,3,2,2,1,1;1,1,2,2,3,3,4\} = \\ \{p^0_{11}, p^1_{21}, p^2_{31}, p^3_{41}, p^4_{51}, p^5_{61}, p^6_{71}; p^1_{01}, p^2_{11}, p^3_{21}, p^4_{31}, p^5_{41}, p^6_{51}, p^7_{61}\}\,.$$

In the case $p^2_{11} = 2$ it is the 7-dimensional unit cube.

According to Theorems 7 and 9 we can see, that there is only one table of parameters of an association scheme with $2k$ classes ($k \in \{2, 3\}$) so that the corresponding metrically regular bipartite graph of diameter $D \in \{4, 6\}$ has a metrically regular square.

So, with respect to Theorems 7 – 10 it would be reasonable to conjecture:

Conjecture 11. *There is only one table of parameters of an association scheme with $2k$ classes ($k \geq 2$) so that the corresponding metrically regular bipartite graph of diameter $D = 2k$ has a metrically regular square. The realization of this table is the $2k$-dimensional unit cube.*

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