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EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS IN A HIGHER ORDER DIFFERENCE EQUATION

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Abstract. Consider the following higher order difference equation

$$x(n+1) = f\Big(n, x(n)\Big) + g\Big(n, x(n-k)\Big)\,, \quad n = 0, 1, \dots$$

where $f(n,x)$ and $g(n,x)\colon \{0,1,\dots\}\times[0,\infty) \to [0,\infty)$ are continuous functions in x and periodic functions in n with period p, and k is a nonnegative integer. We show the existence of a periodic solution $\{\tilde{x}(n)\}$ under certain conditions, and then establish a sufficient condition for $\{\tilde{x}(n)\}$ to be a global attractor of all nonnegative solutions of the equation. Applications to Riccati difference equation and some other difference equations derived from mathematical biology are also given.

1. Introduction

Our aim in this paper is to study the existence and global attractivity of periodic solutions of the following higher order nonlinear difference equation

$$x(n+1) = f\big(n, x(n)\big) + g\big(n, x(n-k)\big)\,, \quad n = 0, 1, \dots \tag{1.1}$$

where $f(n,x)$ and $g(n,x)\colon \{0,1,\dots\} \times [0,\infty) \to [0,\infty)$ are continuous functions in x and periodic functions with period p in n, and k is a nonnegative integer.

A solution $\{x(n)\}_{n\geq -k}$ is said to be eventually periodic with period $p \in \{1,2,\dots\}$ if there is an $n_0 \geq -k$ such that $x(n) = x(n+p)$ for every $n \geq n_0$. For the case that $p = 1$, such kind solutions are eventually constant, see for example, [6] and the references therein. If $n_0 = -k$, it is said that the solution is periodic. However, as usual, both type of solutions will be simply called periodic [7]. The existence of periodic solutions of difference equations has been studied by numerous authors and many interesting results have been obtained, see, for example, [1], [2], [5]–[10], [17]–[22], [24], [25] and the references cited therein. However, to the best of our knowledge, no much work has been done for the equations of form (1.1) on this topic. In addition, the study of global attractivity of periodic solutions of difference

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equations is also relatively scarce. Recently, the global attractivity of periodic solutions of the following difference equation

$$x(n+1) = a(n)x(n) + g\big(n, x(n-k)\big)\,, \quad n = 0, 1, \ldots \tag{1.2}$$

which is a special case of Eq. (1.1) with $f(n,x) = a(n)x$, has been studied in [15]. Applications to some difference equations derived from mathematical biology have been shown in [15] also. However, the results obtained there can not be applied to the cases when $f(n,x)$ is a nonlinear function in x such as the equation

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}}\,, \quad n = 0, 1, \ldots \tag{1.3}$$

where $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$, $\{q(n)\}$ and $\{r(n)\}$ are nonnegative periodic sequences with period p, and k is a nonnegative integer. When $a(n) \equiv a$, $b(n) \equiv b$, $c(n) \equiv c$, $q(n) \equiv q$ and $r(n) \equiv r$ are all positive constants and $k = 0$, Eq. (1.3) reduces to

$$x(n+1) = \frac{ax^2(n)}{b+x(n)} + c\frac{e^{r-qx(n)}}{1+e^{r-qx(n)}}\,, \quad n = 0, 1, \ldots. \tag{1.4}$$

Eq. (1.4) is a biological model derived from the evaluation of a perennial grass [23]. The boundedness and the persistence of positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions of Eq. (1.4) have been studied in [10] and [15]. For some other recent work on the attractivity of periodic solutions of difference equations, one may see [21], [22] and the references cited therein.

In the next section, we first show that under certain conditions, Eq. (1.1) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p by employing Schauder Fixed Point Theorem. Then we establish a sufficient condition for $\{\tilde{x}(n)\}$ to be a global attractor of all nonnegative solutions of the equation. For the proof of the global attractivity, we develop the method used in [15]. The related methods have been used in a recent paper [16] for the global attractivity of equilibrium of a nonlinear difference equation. In addition, some related equations have been studied by the first author of the present paper and collaborators for some time [3, 4, 11, 14].

In Section 3, we apply the results obtained in Section 2 to Eq. (1.3) to establish some sufficient conditions for the existence of a periodic solution $\{\tilde{x}(n)\}$ and for $\{\tilde{x}(n)\}$ to be a global attractor of all nonnegative solutions of the equation. Some interesting special cases of Eq. (1.3) are also discussed. In addition, we show that our results can be applied to the following Riccati difference equation

$$x(n+1) = \frac{\alpha(n)x(n)+\beta(n)}{\gamma(n)x(n)+\delta(n)}\,, \quad n = 0, 1, \ldots \tag{1.5}$$

where $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$ and $\{\delta(n)\}$ are nonnegative periodic sequences with period p. Riccati difference equations appear in mathematical biology. For instance, the discrete logistic model without delay is a Riccati difference equation. See [2], [12], [13] and [18]. Various properties and applications of Riccati difference equations have been studied by numerous authors, see for example, [7], [8] and the references cited therein. However, it seems that there are no many results on the existence and global attractivity of periodic solutions of these kinds of equations.

2. Main results

The following theorem provides a sufficient condition for the existence of nonnegative periodic solutions of Eq. (1.1). For the sake of convenience, we adopt the notation $\prod_{i=m}^{n} s(i) = 1$ and $\sum_{i=m}^{n} s(i) = 0$ whenever $\{s(n)\}$ is a real sequence and $m > n$ in the following discussion.

Theorem 1. *Assume that there is a nonnegative periodic sequence $\{a(n)\}$ with period p such that*

$$\hat{a} = \prod_{i=0}^{p-1} a(i) < 1\,, \quad \text{and} \quad f(n,x) \le a(n)x \quad \text{for} \quad n = 0,1,\dots,p-1 \ \text{and}\ x \ge 0 \tag{2.1}$$

and that $f(n,x) - a(n)x$ is nonincreasing in x. Suppose also that $g(n,x)$ is nonincreasing in x and that there is a positive constant B such that

$$\sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) \big[f(j,B) - a(j)B + g(j,B) \big] \ge 0\,, \quad n = 0,1,\dots,p-1 \tag{2.2}$$

and

$$\frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) g(j,0) \le B\,, \quad n = 0,1,\dots,p-1\,. \tag{2.3}$$

Then Eq. (1.1) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p.*

Proof. Let $\mathbf{x} = \{x(n)\}_{n=-k}^{\infty}$ be a real sequence and let

$$X = \{\mathbf{x} : \mathbf{x} \text{ satisfies } x(n+p) = x(n),\ n \ge -k\}\,.$$

Then X is a normed vector space with the usual linear operations and the norm $\|\mathbf{x}\| = \sup_{0 \le n \le p-1} |x(n)|$. Let Λ be a subset of X defined by

$$\Lambda = \{\mathbf{x} : \mathbf{x} \in X \text{ with } 0 \le x(n) \le B,\ n \ge -k\}\,.$$

It is easy to see that Λ is a compact and convex subset of X.

Now, define a mapping T on Λ as the following: for each $\mathbf{x} = \{x(n)\} \in \Lambda$,

$$Tx(n) = \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) \big[f\big(j, x(j)\big) - a(j)x(j) + g\big(j, x(j-k)\big) \big]\,. \tag{2.4}$$

Clearly T is continuous since f and g are periodic in the first variable and continuous in the second one. We show that $T\colon \Lambda \to \Lambda$. In fact, by noting $f(n,x) - a(n)x$ and $g(n,x)$ are nonincreasing in x, and (2.2) and (2.3) hold, it is easy to see that

$$Tx(n) \ge \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) \big[f(j,B) - a(j)B + g(j,B) \big] \ge 0$$

and

$$Tx(n) \le \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) g(j,0) \le B\,.$$

Next, observe that

$$\begin{aligned}
Tx(n+p) &= \frac{1}{1-\hat{a}} \sum_{j=n+p}^{n+2p-1} \Big(\prod_{i=j+1}^{n+2p-1} a(i) \Big) \big[f\big(j, x(j)\big) - a(j)x(j) + g\big(j, x(j-k)\big) \big] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i+p) \Big) \big[f\big(j+p, x(j+p)\big) \\
&\quad - a(j+p)x(j+p) + g\big(j+p, x(j+p-k)\big) \big] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) \big[f\big(j, x(j)\big) - a(j)x(j) + g\big(j, x(j-k)\big) \big] \\
&= Tx(n)\,.
\end{aligned}$$

Hence, $\{Tx(n)\}$ is periodic with period p and so $T\mathbf{x} \in \Lambda$. Then by Schauder Fixed Point Theorem, T has a fixed point $\tilde{\mathbf{x}} = \{\tilde{x}(n)\} \in \Lambda$. We claim that $\tilde{\mathbf{x}}$ is a solution of Eq. (1.1). In fact, by noting $\prod_{i=n+1}^{n+p} a(i) = \prod_{i=0}^{p-1} a(i) = \hat{a}$, $\prod_{i=n+p+1}^{n+p} a(i) = 1$, $f(n+p, \tilde{x}(n+p)) = f(n, \tilde{x}(n))$, $g(n+p, \tilde{x}(n+p-k)) = g(n, \tilde{x}(n-k))$ and $T\tilde{x}(n) = \tilde{x}(n)$, we see that

$$\begin{aligned}
T\tilde{x}(n+1) &= \frac{1}{1-\hat{a}} \sum_{j=n+1}^{n+p} \Big(\prod_{i=j+1}^{n+p} a(i) \Big) \big[f\big(j, \tilde{x}(j)\big) - a(j)\tilde{x}(j) + g\big(j, \tilde{x}(j-k)\big) \big] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p} a(i) \Big) \big[f\big(j, \tilde{x}(j)\big) - a(j)\tilde{x}(j) + g\big(j, \tilde{x}(j-k)\big) \big] \\
&\quad - \frac{1}{1-\hat{a}} \Big(\prod_{i=n+1}^{n+p} a(i) \Big) \big[f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n) + g\big(n, \tilde{x}(n-k)\big) \big] \\
&\quad + \frac{1}{1-\hat{a}} \Big(\prod_{i=n+p+1}^{n+p} a(i) \Big) \big[f\big(n+p, \tilde{x}(n+p)\big) - a(n+p)\tilde{x}(n+p) \\
&\quad + g\big(n+p, \tilde{x}(n+p-k)\big) \big] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p} a(i) \Big) \big[f\big(j, \tilde{x}(j)\big) - a(j)\tilde{x}(j) + g\big(j, \tilde{x}(j-k)\big) \big] \\
&\quad - \frac{\hat{a}}{1-\hat{a}} \big[f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n) + g\big(n, \tilde{x}(n-k)\big) \big] \\
&\quad + \frac{1}{1-\hat{a}} \big[f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n) + g\big(n, \tilde{x}(n-k)\big) \big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a(n+p)}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i) \Big) \big[f\big(j, \tilde{x}(j)\big) - a(j)\tilde{x}(j) + g\big(j, \tilde{x}(j-k)\big) \big] \\
&\quad + f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n) + g\big(n, \tilde{x}(n-k)\big) \\
&= a(n)T\tilde{x}(n) + f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n) + g\big(n, \tilde{x}(n-k)\big) \\
&= f\big(n, \tilde{x}(n)\big) + g\big(n, \tilde{x}(n-k)\big) = f\big(n, T\tilde{x}(n)\big) + g\big(n, T\tilde{x}(n-k)\big) .
\end{aligned}
$$

Hence, $\{T\tilde{x}(n)\}$ satisfies Eq. (1.1) and so $\{T\tilde{x}(n)\}$, that is $\{\tilde{x}(n)\}$, is a periodic solution of Eq. (1.1) with period p. The proof is complete. □

The following theorem provides a sufficient condition for a periodic solution of Eq. (1.1) to be a global attractor of all nonnegative solutions of Eq. (1.1). This theorem is an extension and improvement of the corresponding result obtained in [15] for Eq. (1.2). We relax the condition

$$0 < a(n) \le 1 \quad \text{and} \quad a(n) \not\equiv 1\,,\ n = 0, 1, \dots, p-1$$

assumed in [15] to the more general condition

$$\prod_{i=0}^{p-1} a(i) < 1 \quad \text{and} \quad a(n) \ge 0\,,\ n = 0, 1, \dots, p-1$$

which has been assumed above in the hypotheses of Theorem 1 for the existence of periodic solutions. This relaxation of the condition makes the result more applicable.

Theorem 2. *Assume that $f(n,x)$ is nondecreasing in x and there is a nonnegative sequence $\{a(n)\}$ with period p such that* (2.1) *holds and $f(n,x) - a(n)x$ is nonincreasing in x. Suppose also that $g(n,x)$ is nonincreasing in x and there is a nonnegative periodic sequence $\{L(n)\}$ with period p such that*

$$|g(n,x) - g(n,y)| \le L(n)|x-y|\,, \quad n = 0, 1, \dots, p-1 \tag{2.5}$$

and that either

$$a(n) \le 1 \quad \textit{and} \quad \sum_{j=n}^{n+k} \Big(\prod_{i=j+1}^{n+k} a(i) \Big) L(j) < 1\,, \quad n = 0, 1, \dots, p-1 \tag{2.6}$$

or

$$\sum_{j=n}^{n+k+p-1} \Big(\prod_{i=j+1}^{n+k+p-1} a(i) \Big) L(j) < 1\,, \quad n = 0, 1, \dots, p-1\,. \tag{2.7}$$

If Eq. (1.1) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p, then $\{\tilde{x}(n)\}$ is the only periodic solution of Eq.* (1.1) *and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (1.1) *in the sense that every nonnegative solution $\{x(n)\}$ of Eq.* (1.1) *satisfies*

$$\lim_{n\to\infty} \big(x(n) - \tilde{x}(n) \big) = 0\,. \tag{2.8}$$

Proof. Clearly, if we can show that every nonnegative solution of Eq. (1.1) converges to $\{\tilde{x}(n)\}$, then $\{\tilde{x}(n)\}$ is the unique periodic solution. Let $y(n) = x(n) - \tilde{x}(n)$. Then $\{y(n)\}$ satisfies

$$y(n+1) + \tilde{x}(n+1) = f\big(n, y(n) + \tilde{x}(n)\big) + g\big(n, y(n-k) + \tilde{x}(n-k)\big)\,.$$

Since $\{\tilde{x}(n)\}$ is a solution of Eq. (1.1), $\tilde{x}(n+1) = f(n, \tilde{x}(n)) + g(n, \tilde{x}(n-k))$. Hence, it follows that

$$\begin{aligned} y(n+1) = {} & f\big(n, y(n) + \tilde{x}(n)\big) - f\big(n, \tilde{x}(n)\big) \\ & + g\big(n, y(n-k) + \tilde{x}(n-k)\big) - g\big(n, \tilde{x}(n-k)\big)\,. \end{aligned} \tag{2.9}$$

First, assume that $\{x(n)\}$ does not oscillate about $\{\tilde{x}(n)\}$. Then, $\{y(n)\}$ is either eventually positive or eventually negative. We assume that $\{y(n)\}$ is eventually positive. The proof for the case that $\{y(n)\}$ is eventually negative is similar and will be omitted. Hence there is a positive integer n_0 such that $y(n) > 0$, $n \geq n_0$. Then by noting that g is nonincreasing in x, it follows from (2.9) that

$$y(n+1) \leq f\big(n, y(n) + \tilde{x}(n)\big) - f\big(n, \tilde{x}(n)\big)\,, \quad n \geq n_0 + k\,. \tag{2.10}$$

Since $f(n, x) - a(n)x$ is nonincreasing in x also,

$$f\big(n, y(n) + \tilde{x}(n)\big) - a(n)\big(y(n) + \tilde{x}(n)\big) \leq f\big(n, \tilde{x}(n)\big) - a(n)\tilde{x}(n)\,, \quad n \geq n_0\,. \tag{2.11}$$

Hence, (2.10) and (2.11) yield

$$y(n+1) \leq a(n)\big(y(n) + \tilde{x}(n)\big) - a(n)\tilde{x}(n) = a(n)y(n)\,, \quad n \geq n_0 + k$$

and so it follows that

$$y(n) \leq \Big(\prod_{i=n_0+k}^{n-1} a(i)\Big) y(n_0 + k)\,, \quad n \geq n_0 + k\,.$$

By noting $a(n) \geq 0$, $a(n)$ is p-periodic and $\prod_{i=0}^{p-1} a(i) < 1$, we see that $\prod_{i=n_0+k}^{n-1} a(i) \to 0$ as $n \to \infty$. Hence $y(n) \to 0$ as $n \to \infty$ and so (2.8) holds.

Next, assume that $\{x(n)\}$ oscillates about $\{\tilde{x}(n)\}$ and so $\{y(n)\}$ oscillates about zero. Then there is an increasing sequence $\{n_t\}$ of positive integers with $n_1 \geq k$ such that $y(n_1) \leq 0$ and for $t = 1, 2, \ldots$

$$y(n) > 0 \quad \text{for} \quad n_{2t-1} < n \leq n_{2t} \tag{2.12}$$

and

$$y(n) \leq 0 \quad \text{for} \quad n_{2t} < n \leq n_{2t+1}\,. \tag{2.13}$$

We claim that when $n_1 < n \leq n_2$,

$$y(n) \leq \sum_{j=n_1}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i)\Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big)\big]\,. \tag{2.14}$$

In fact, from (2.9) we see that

$$\begin{aligned} y(n_1+1) = {} & f\big(n_1, y(n_1) + \tilde{x}(n_1)\big) - f\big(n_1, \tilde{x}(n_1)\big) \\ & + g\big(n_1, y(n_1-k) + \tilde{x}(n_1-k)\big) - g\big(n_1, \tilde{x}(n_1-k)\big)\,. \end{aligned} \tag{2.15}$$

Since $y(n_1) \le 0$ and $f(n,x)$ is nondecreasing in x, $f(n_1, y(n_1)+\tilde{x}(n_1)) \le f(n_1, \tilde{x}(n_1))$. Hence, it follows from (2.15) that

$$y(n_1+1) \le g\big(n_1, y(n_1-k) + \tilde{x}(n_1-k)\big) - g\big(n_1, \tilde{x}(n_1-k)\big)\,,$$

that is, (2.14) holds when $n = n_1+1$. Next, assume that (2.14) holds when $n = m$ with $n_1 < m < n_2$,

$$y(m) \le \sum_{j=n_1}^{m-1} \Big(\prod_{i=j+1}^{m-1} a(i) \Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big]\,. \tag{2.16}$$

From (2.9) we see that

$$\begin{aligned} y(m+1) - a(m)y(m) &= f\big(m, y(m) + \tilde{x}(m)\big) - f\big(m, \tilde{x}(m)\big) - a(m)y(m) \\ &\quad + g\big(m, y(m-k) + \tilde{x}(m-k)\big) - g\big(m, \tilde{x}(m-k)\big)\,. \end{aligned} \tag{2.17}$$

Since $f(n,x) - a(n)x$ is nonincreasing in x and $y(m) > 0$,

$$f\big(m, y(m) + \tilde{x}(m)\big) - a(m)\big(y(m) + \tilde{x}(m)\big) \le f\big(m, \tilde{x}(m)\big) - a(m)\tilde{x}(m)$$

which yields

$$f\big(m, y(m) + \tilde{x}(m)\big) - f\big(m, \tilde{x}(m)\big) - a(m)y(m) \le 0\,.$$

Hence, it follows from (2.17) that

$$y(m+1) \le a(m)y(m) + g\big(m, y(m-k) + \tilde{x}(m-k)\big) - g\big(m, \tilde{x}(m-k)\big)\,.$$

Then by noting (2.16), we see that

$$\begin{aligned} y(m+1) &\le a(m) \sum_{j=n_1}^{m-1} \Big(\prod_{i=j+1}^{m-1} a(i) \Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big] \\ &\quad + g\big(m, y(m-k) + \tilde{x}(m-k)\big) - g\big(m, \tilde{x}(m-k)\big) \\ &= \sum_{j=n_1}^{m} \Big(\prod_{i=j+1}^{m} a(i) \Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big]\,, \end{aligned}$$

that is, (2.14) holds when $n = m+1$. Therefore, by mathematical induction, (2.14) holds when $n_1 < n \le n_2$.

Since $g(n,x)$ is periodic in n with period p and (2.5) holds, we see that

$$|g(n,x) - g(n,y)| \le L(n)|x-y|\,, \quad n = 0, 1, \dots\,.$$

Then it follows from (2.14) that when $n_1 < n \le n_2$

$$\begin{aligned} y(n) &\le \sum_{j=n_1}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i) \Big) \big| g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big| \\ &\le \sum_{j=n_1}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i) \Big) L(j) \big| y(j-k) \big|\,. \end{aligned} \tag{2.18}$$

First assume that (2.6) holds. By noting the periodic property of $\{a(n)\}$ and $\{L(n)\}$, we see that there is a positive constant $c < 1$ such that

$$\sum_{j=n}^{n+k} \Big(\prod_{i=j+1}^{n+k} a(i) \Big) L(j) \le c \,, \quad n = 0, 1, \dots \,. \tag{2.19}$$

Then by an argument similar to that used in the proof of Theorem 1 in [15], we may show that there is a subsequence $\{n_{t_m}\}$ of $\{n_t\}$ with $n_{t_{m+1}} \ge n_{t_m} + k,\ m = 1, 2, \dots$ such that

$$\big|y(n)\big| \le c^m \max_{n_1-k \le s \le n_1} \big|y(s)\big| \quad \text{for} \quad n > n_{t_m} \,. \tag{2.20}$$

For the sake of completeness, we give detail of the proof in the following. First, we claim that

$$y(n) \le c \max_{n_1-k \le s \le n_1} \big|y(s)\big| \,, \quad n_1 < n \le n_2 \,. \tag{2.21}$$

To this end, consider the two cases $n_2 \le n_1 + k + 1$ and $n_2 > n_1 + k + 1$, respectively. When $n_2 \le n_1 + k + 1$, then for any $n_1 < n \le n_2$, $n - k - 1 \le n_1$ and so (2.18) yields

$$\begin{aligned} y(n) &\le \sum_{j=n_1}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i) \Big) L(j) \max_{n_1-k \le s \le n_1} |y(s)| \\ &\le \sum_{j=n-k-1}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i) \Big) L(j) \max_{n_1-k \le s \le n_1} |y(s)| \,. \end{aligned} \tag{2.22}$$

Then by noting (2.19), we see that (2.21) holds. Next, consider the case that $n_2 > n_1 + k + 1$. When $n_1 < n \le n_1 + k + 1$, as we have shown above, (2.21) holds. Hence, we only need to show that (2.21) holds also when $n_1 + k + 1 < n \le n_2$. To this end, first we show that

$$y(n) \le a(n-1)y(n-1) \,, \quad n_1 + k + 1 < n \le n_2 \,. \tag{2.23}$$

In fact, by noting that when $n_1 + k + 1 < n \le n_2$, $y(n-1) > 0$ and $f(n,x) - a(n)x$ is nonincreasing in x, we see that

$$\begin{aligned} f\big(n-1, y(n-1) + \tilde{x}(n-1)\big) &- a(n-1)\big(y(n-1) + \tilde{x}(n-1)\big) \\ &\le f\big(n-1, \tilde{x}(n-1)\big) - a(n-1)\tilde{x}(n-1) \end{aligned}$$

which yields

$$f\big(n-1, y(n-1) + \tilde{x}(n-1)\big) - f\big(n-1, \tilde{x}(n-1)\big) \le a(n-1)y(n-1) \,.$$

In addition, by noting that when $n_1 + k + 1 < n \le n_2$, $y(n-1-k) > 0$ and $g(\cdot, x)$ is nonincreasing in x, we see that

$$g\big(n-1, y(n-1-k) + \tilde{x}(n-1-k)\big) - g\big(n-1, \tilde{x}(n-1-k)\big) \le 0 \,.$$

Then it follows from (2.9) that

$$\begin{aligned} y(n) &= f(n-1, y(n-1) + \tilde{x}(n-1)) - f(n-1, \tilde{x}(n-1)) \\ &\quad + \ g(n-1, y(n-1-k) + \tilde{x}(n-1-k)) - g(n-1, \tilde{x}(n-1-k)) \\ &\le a(n-1)y(n-1) \end{aligned}$$

and so (2.23) holds. Then by noting $a(n) \in [0,1]$, it follows that $y(n_2) \le y(n_2-1) \le \cdots \le y(n_1+1+k)$, which implies that (2.21) holds when $n_1+k+1 < n \le n_2$. Hence for any $n_1 < n \le n_2$, (2.21) holds. Then by a similar argument, we may show that

$$y(n) \ge -c \max_{n_2-k \le s \le n_2} |y(s)|\,, \quad n_2 < n \le n_3\,. \tag{2.24}$$

To this end, we first claim that when $n_2 < n \le n_3$,

$$y(n) \ge \sum_{j=n_2}^{n-1} \Big(\prod_{i=j+1}^{n-1} a(i) \Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big]\,. \tag{2.25}$$

From (2.9) we see that

$$\begin{aligned} y(n_2+1) &= f\big(n_2, y(n_2) + \tilde{x}(n_2)\big) - f\big(n_2, \tilde{x}(n_2)\big) \\ &\qquad + g\big(n_2, y(n_2-k) + \tilde{x}(n_2-k)\big) - g\big(n_2, \tilde{x}(n_2-k)\big)\,. \end{aligned} \tag{2.26}$$

Since $y(n_2) > 0$ and $f(n,x)$ is nondecreasing in x, $f(n_2, y(n_2)+\tilde{x}(n_2)) \ge f(n_2, \tilde{x}(n_2))$. Hence, it follows from (2.26) that

$$y(n_2+1) \ge g\big(n_2, y(n_2-k) + \tilde{x}(n_2-k)\big) - g\big(n_2, \tilde{x}(n_2-k)\big)\,,$$

that is, (2.25) holds when $n = n_2+1$. Next, assume that (2.25) holds when $n = m$ with $n_2 < m < n_3$,

$$y(m) \ge \sum_{j=n_2}^{m-1} \Big(\prod_{i=j+1}^{m-1} a(i) \Big) \big[g\big(j, y(j-k) + \tilde{x}(j-k)\big) - g\big(j, \tilde{x}(j-k)\big) \big]\,. \tag{2.27}$$

From (2.9) we see that

$$\begin{aligned} y(m+1) - a(m)y(m) &= f\big(m, y(m) + \tilde{x}(m)\big) - f\big(m, \tilde{x}(m)\big) - a(m)y(m) \\ &\qquad + g\big(m, y(m-k) + \tilde{x}(m-k)\big) - g\big(m, \tilde{x}(m-k)\big)\,. \end{aligned} \tag{2.28}$$

Since $f(n,x) - a(n)x$ is nonincreasing in x and $y(m) \le 0$,

$$f\big(m, y(m) + \tilde{x}(m)\big) - a(m)\big(y(m) + \tilde{x}(m)\big) \ge f\big(m, \tilde{x}(m)\big) - a(m)\tilde{x}(m)$$

which yields

$$f\big(m, y(m) + \tilde{x}(m)\big) - f\big(m, \tilde{x}(m)\big) - a(m)y(m) \ge 0\,.$$

Hence, it follows from (2.28) that

$$y(m+1) \ge a(m)y(m) + g\big(m, y(m-k) + \tilde{x}(m-k)\big) - g\big(m, \tilde{x}(m-k)\big)\,.$$

Then by noting (2.27) we see that

$$\begin{aligned} y(m+1) &\geq a(m)\sum_{j=n_2}^{m-1}\Big(\prod_{i=j+1}^{m-1}a(i)\Big)\big[g\big(j,y(j-k)+\tilde{x}(j-k)\big)-g\big(j,\tilde{x}(j-k)\big)\big] \\ &\quad + g\big(m,y(m-k)+\tilde{x}(m-k)\big)-g\big(m,\tilde{x}(m-k)\big) \\ &= \sum_{j=n_2}^{m}\Big(\prod_{i=j+1}^{m}a(i)\Big)\big[g\big(j,y(j-k)+\tilde{x}(j-k)\big)-g\big(j,\tilde{x}(j-k)\big)\big]\,, \end{aligned}$$

that is, (2.25) holds when $n=m+1$. Therefore, by mathematical induction, (2.25) holds when $n_2<n\leq n_3$.

As before, since $g(n,x)$ is periodic in n with period p and (2.5) holds, we see that

$$|g(n,x)-g(n,y)|\leq L(n)|x-y|\,,\quad n=0,1,\ldots\,.$$

It follows from (2.25) that when $n_2<n\leq n_3$, as $y(n)\leq 0$,

$$\begin{aligned} |y(n)| = -y(n) &\leq \Big|\sum_{j=n_2}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)\big[g\big(j,y(j-k)+\tilde{x}(j-k)\big)-g\big(j,\tilde{x}(j-k)\big)\big]\Big| \\ &\leq \sum_{j=n_2}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)\big|g\big(j,y(j-k)+\tilde{x}(j-k)\big)-g\big(j,\tilde{x}(j-k)\big)\big| \\ &\leq \sum_{j=n_2}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)L(j)\,\big|y(j-k)\big|\,. \end{aligned} \tag{2.29}$$

To show (2.24) holds, we consider the two cases $n_3\leq n_2+k+1$ and $n_3>n_2+k+1$, respectively. When $n_3\leq n_2+k+1$, then for any $n_2<n\leq n_3$, $n-k-1\leq n_2$ and so (2.29) yields

$$\begin{aligned} -y(n) &\leq \sum_{j=n_2}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)L(j)\max_{n_2-k\leq s\leq n_2}\big|y(s)\big| \\ &\leq \sum_{j=n-k-1}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)L(j)\max_{n_2-k\leq s\leq n_2}\big|y(s)\big|\,. \end{aligned} \tag{2.30}$$

By noting (2.19),

$$-y(n)\leq c\max_{n_2-k\leq s\leq n_2}\big|y(s)\big|\,,$$

thus (2.24) holds. Next, consider the case that $n_3>n_2+k+1$. When $n_2<n\leq n_2+k+1$, as we have shown above, (2.24) holds. Hence, we only need to show that (2.24) holds also when $n_2+k+1<n\leq n_3$. To this end, we first show that

$$y(n)\geq a(n-1)y(n-1)\,,\quad n_2+k+1<n\leq n_3\,. \tag{2.31}$$

By noting that when $n_2+k+1<n\le n_3$, $y(n-1)\le 0$ and $f(n,x)-a(n)x$ is nonincreasing in x, we see that

$$\begin{aligned}f\bigl(n-1,y(n-1)+\tilde{x}(n-1)\bigr)-a(n-1)\bigl(y(n-1)+\tilde{x}(n-1)\bigr)\\ \ge f\bigl(n-1,\tilde{x}(n-1)\bigr)-a(n-1)\tilde{x}(n-1)\end{aligned}$$

which yields

$$f\bigl(n-1,y(n-1)+\tilde{x}(n-1)\bigr)-f\bigl(n-1,\tilde{x}(n-1)\bigr)\ge a(n-1)y(n-1)\,.$$

In addition, by noting that when $n_2+k+1<n\le n_3$, $y(n-1-k)\le 0$ and $g(\cdot,x)$ is nonincreasing in x, we see that

$$g\bigl(n-1,y(n-1-k)+\tilde{x}(n-1-k)\bigr)-g\bigl(n-1,\tilde{x}(n-1-k)\bigr)\ge 0\,.$$

Then it follows from (2.9) that

$$\begin{aligned}y(n)&=f\bigl(n-1,y(n-1)+\tilde{x}(n-1)\bigr)-f\bigl(n-1,\tilde{x}(n-1)\bigr)\\&\quad+g\bigl(n-1,y(n-1-k)+\tilde{x}(n-1-k)\bigr)-g\bigl(n-1,\tilde{x}(n-1-k)\bigr)\\&\ge a(n-1)y(n-1)\end{aligned}$$

and so (2.31) holds. Then by noting $a(n)\in[0,1]$, as $y(n)\le 0$, it follows that $y(n_3)\ge y(n_3-1)\ge\cdots\ge y(n_2+1+k)$, which implies that (2.24) holds when $n_2+k+1<n\le n_3$. Hence, for any $n_2<n\le n_3$, (2.24) holds.

If $n_2-k>n_1$, we see that when $n_2-k\le n\le n_2$, (2.21) holds and so

$$\max_{n_2-k\le s\le n_2}|y(s)|\le c\max_{n_1-k\le s\le n_1}|y(s)|\le\max_{n_1-k\le s\le n_1}|y(s)|.$$

If $n_2-k<n_1$, we see that (2.21) holds when $n_1<n\le n_2$; while when $n_2-k\le n\le n_1$, by noting $n_1-k<n_2-k$, we see that $|y(n)|\le\max_{n_1-k\le s\le n_1}|y(s)|$. Hence, from the above discussion, we see that $\max_{n_2-k\le s\le n_2}|y(s)|\le\max_{n_1-k\le s\le n_1}|y(s)|$ and so it follows from (2.24) that when $n_2<n\le n_3$,

$$y(n)\ge -c\max_{n_1-k\le s\le n_1}\bigl|y(s)\bigr|\,. \tag{2.32}$$

By combining (2.21) and (2.32), we see that

$$\bigl|y(n)\bigr|\le c\max_{n_1-k\le s\le n_1}\bigl|y(s)\bigr|\quad\text{for}\quad n_1<n\le n_3\,.$$

Then by the method of steps, we may show that

$$\bigl|y(n)\bigr|\le c\max_{n_1-k\le s\le n_1}\bigl|y(s)\bigr|\quad\text{for}\quad n>n_1\,.$$

Now, by choosing an $n_{r_2}\in\{n_r\}$ with $n_{r_2}>n_1+k$ and then by using a similar argument, we may show that

$$\bigl|y(n)\bigr|\le c^2\max_{n_1-k\le s\le n_1}\bigl|y(s)\bigr|\quad\text{for}\quad n>n_{r_2}\,.$$

Finally, by induction, we may show that for any positive integer $m>1$, (2.20) holds. Since $0<c<1$, we see that $y(n)\to 0$ as $n\to\infty$ and then it follows that (2.8) holds.

Next, assume that (2.7) holds. By the periodic property of $\{a(n)\}$ and $\{L(n)\}$, there is a positive constant $d<1$ such that

$$\sum_{j=n}^{n+k+p-1}\Big(\prod_{i=j+1}^{n+k+p-1}a(i)\Big)L(j)\le d\,,\quad n=0,1,\dots\,. \tag{2.33}$$

We claim that when $n_1<n\le n_2$,

$$y(n)\le d\max_{n_1-k\le s\le n_1+p-1}\big|y(s)\big|\,. \tag{2.34}$$

To this end, consider the two cases $n_2\le n_1+k+p$ and $n_2>n_1+k+p$, respectively. When $n_2\le n_1+k+p$, then for any $n_1<n\le n_2$, $n-k-p\le n_1$ and so (2.18) yields

$$\begin{aligned} y(n)&\le \sum_{j=n_1}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)L(j)\max_{n_1-k\le s\le n_1+p-1}\big|y(s)\big|\\ &\le \sum_{j=n-k-p}^{n-1}\Big(\prod_{i=j+1}^{n-1}a(i)\Big)L(j)\max_{n_1-k\le s\le n_1+p-1}\big|y(s)\big|\,. \end{aligned}$$

Then by noting (2.33), we see that (2.34) holds. Next, consider the case that $n_2>n_1+k+p$. When $n_1<n\le n_1+k+p$, as we have shown above, (2.34) holds. Hence, we only need to show that (2.34) holds also when $n_1+k+p<n\le n_2$. First, by the same argument used for the case (2.6) above, we may have

$$y(n)\le a(n-1)y(n-1)\,,\quad n_1+k+1<n\le n_2\,. \tag{2.35}$$

Hence,

$$\begin{aligned} y(n_1+k+p+1)&\le a(n_1+k+p)y(n_1+k+p)\\ &\le a(n_1+k+p)a(n_1+k+p-1)y(n_1+k+p-1)\\ &\le\cdots\le\Big(\prod_{i=n_1+k+1}^{n_1+k+p}a(i)\Big)y(n_1+k+1)<y(n_1+k+1) \end{aligned}$$

and similarly,

$$y(n_1+k+p+2)<y(n_1+k+2),\dots,y(n_2)<y(n_2-p)\,.$$

Since $y(n)$ satisfies (2.34) when $n_1<n\le n_1+k+p$, from the above inequalities we see that $y(n)$ satisfies (2.34) also when $n_1+k+p<n\le n_2$. Hence, (2.34) holds when $n_1<n\le n_2$. Then by an argument similar to that used for the case when (2.6) holds, we may show that there is a subsequence $\{n_{t_l}\}$ of $\{n_t\}$ with $n_{t_{l+1}}\ge n_{t_l}+k,\ \ l=1,2,\dots$ such that

$$\big|y(n)\big|\le d^l\max_{n_1-k\le s\le n_1+p-1}\big|y(s)\big|\quad\text{for}\quad n>n_{t_l}\,.$$

Since $0<d<1$, we see that $y(n)\to0$ as $n\to\infty$ and then it follows that (2.8) holds. The proof is complete. □

By combining Theorems 1 and 2, we have the following result immediately.

Theorem 3. *Assume that $f(n,x)$ is nondecreasing in x and there is a nonnegative sequence $\{a(n)\}$ with period p such that* (2.1) *holds and $f(n,x)-a(n)x$ is nonincreasing in x. Suppose also that $g(n,x)$ is nonincreasing in x and there are a positive constant B and a nonnegative periodic sequence $\{L(n)\}$ with period p such that* (2.2), (2.3), (2.5) *and either* (2.6) *or* (2.7) *hold. Then Eq.* (1.1) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (1.1).

When $f(n,x)=a(n)x$ where $\{a(n)\}$ is a nonnegative periodic sequence with period p, Eq. (1.1) reduces to Eq. (1.2). Clearly, (2.2) holds for any positive B, and (2.3) holds when B is large. Hence, the following conclusion is a direct consequence of Theorem 3.

Corollary 1. *Assume that $\prod_{i=0}^{p-1} a(i) < 1$, and that $g(n,x)$ is nonincreasing in x and there is a nonnegative periodic sequence $\{L(n)\}$ with period p such that* (2.5) *and either* (2.6) *or* (2.7) *hold. Then Eq.* (1.2) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (1.2).

When g is free of x, that is, $g(n,x)=b(n)$ where $\{b(n)\}$ is a nonnegative periodic sequence with period p, Eq. (1.1) reduces to the first order equation

$$x(n+1) = f\big(n, x(n)\big) + b(n)\,. \tag{2.36}$$

Clearly, (2.5) and (2.7) hold with $L(n)\equiv 0$. (2.2) and (2.3) become

$$\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1} a(i)\Big)\big[f(j,B)-a(j)B+b(j)\big] \ge 0\,, \quad n=0,1,\ldots,p-1 \tag{2.37}$$

and

$$\frac{1}{1-\hat{a}}\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1} a(i)\Big)b(j) \le B\,, \quad n=0,1,\ldots,p-1 \tag{2.38}$$

respectively. Hence, the following conclusion is a direct consequence of Theorem 3.

Corollary 2. *Assume that $f(n,x)$ is nondecreasing in x and there is a nonnegative sequence $\{a(n)\}$ with period p such that* (2.1) *holds and $f(n,x)-a(n)x$ is nonincreasing in x, and that there is a positive constant B such that* (2.37) *and* (2.38) *hold. Then Eq.* (2.36) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (2.36).

In particular, when $f(n,x)=a(n)x$ and $g(n,x)=b(n)$ where $\{a(n)\}$ and $\{b(n)\}$ are nonnegative periodic sequences with period p, Eq. (1.1) reduces to the following first order linear equation

$$x(n+1) = a(n)x(n) + b(n)\,. \tag{2.39}$$

From the above we have the following conclusion immediately.

Corollary 3. *Assume that $\prod_{i=0}^{p-1} a(i) < 1$. Then Eq.* (2.39) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (2.39).

Remark 1. Recently, bounded and periodic solutions of the linear first order difference equation have been studied extensively in [21]. Several interesting results are obtained, one of which is the following: if $\{a(n)\}$ and $\{b(n)\}$ are two periodic sequences with period p and $\prod_{i=0}^{p-1} a(i)$ is different from zero and one, then Eq. (2.39) has a unique p-periodic solution $\{\tilde{x}(n)\}$. Furthermore, if $|\prod_{i=0}^{p-1} a(i)| < 1$, then every solution of Eq. (2.39) converges geometrically to $\{\tilde{x}(n)\}$ as $n \to \infty$, and it is getting away geometrically from $\{\tilde{x}(n)\}$ as $n \to -\infty$. Comparing this result with Corollary 3, we see that $\{a(n)\}$ and $\{b(n)\}$ are not required to be nonnegative and the conclusion is stronger in this result, but $a(n) \neq 0,\ \ n = 0, 1, 2, \dots$ is not required in Corollary 3.

3. Applications

In this section, we apply the results obtained above to some equations derived from applications. First consider the following equation mentioned in Section 1

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}}\,, \quad n = 0, 1, \dots \tag{3.1}$$

where $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$, $\{q(n)\}$ and $\{r(n)\}$ are nonnegative periodic sequences with period p, and k is a nonnegative integer. Eq. (3.1) is in the form of Eq. (1.1) with

$$f(n,x) = \frac{a(n)x^2}{b(n)+x} \quad \text{and} \quad g(n,x) = c(n)\frac{e^{r(n)-q(n)x}}{1+e^{r(n)-q(n)x}}\,.$$

Clearly, f is nondecreasing in x and $f(n,x) \le a(n)x$. Noting

$$\frac{d}{dx}(f(n,x) - a(n)x)' = -\frac{a(n)b^2(n)}{(b(n)+x)^2} \le 0$$

we see that $f(n,x) - a(n)x$ is nonincreasing in x. Observe that

$$\frac{d}{dx}\big(g(n,x)\big) = -c(n)q(n)\frac{e^{r(n)-q(n)x}}{(1+e^{r(n)-q(n)x})^2}$$

and

$$\frac{d^2}{dx^2}\big(g(n,x)\big) = c(n)q^2(n)\frac{e^{r(n)-q(n)x}(1-e^{r(n)-q(n)x})}{(1+e^{r(n)-q(n)x})^3}\,.$$

Clearly, $g(n,x)$ is nonincreasing in x, and for each n, $|\frac{dg(n,x)}{dx}|$ takes maximum when $x = \frac{r(n)}{q(n)}$ and $|\frac{dg(n,x)}{dx}|_{x=\frac{r(n)}{q(n)}} = \frac{c(n)q(n)}{4}$. Hence, $g(n,x)$ is L-Lipschitz in x with $L(n) = \frac{c(n)q(n)}{4}$ for each n. Then by Theorems 1 and 2, we have the following conclusion immediately.

Theorem 4. *Assume that* $\prod_{i=0}^{p-1} a(i) < 1$*, and that there is a positive number* B *such that*

$$\sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i)\Big)\Big[c(j)\frac{e^{r(j)-q(j)B}}{1+e^{r(j)-q(j)B}} - \frac{a(j)b(j)B}{b(j)+B}\Big] \geq 0\,, \qquad n = 0,1,\ldots,p-1 \tag{3.2}$$

and

$$\frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \Big(\prod_{i=j+1}^{n+p-1} a(i)\Big)c(j)\frac{e^{r(j)}}{1+e^{r(j)}} \leq B\,, \quad n = 0,1,\ldots,p-1 \tag{3.3}$$

where $\hat{a} = \prod_{i=0}^{p-1} a(i)$*. Then Eq.* (3.1) *has a nonnegative periodic solution* $\{\tilde{x}(n)\}$ *with period* p*. Furthermore, if either*

$$a(n) \leq 1 \quad \text{and} \quad \sum_{j=n}^{n+k} \Big(\prod_{i=j+1}^{n+k} a(i)\Big)\frac{c(j)q(j)}{4} < 1\,, \quad n = 0,1,\ldots,p-1 \tag{3.4}$$

or

$$\sum_{j=n}^{n+k+p-1} \Big(\prod_{i=j+1}^{n+k+p-1} a(i)\Big)\frac{c(j)q(j)}{4} < 1\,, \quad n = 0,1,\ldots,p-1\,, \tag{3.5}$$

then Eq. (3.1) *has a unique nonnegative periodic solution* $\{\tilde{x}(n)\}$ *with period* p *and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Eq.* (3.1).

Clearly, if

$$c(n)\frac{e^{r(n)-q(n)}}{1+e^{r(n)-q(n)}} \geq \frac{a(n)b(n)}{b(n)+1}\,, \quad n = 0,1,\ldots,p-1 \tag{3.6}$$

and

$$c(n)\frac{e^{r(n)}}{1+e^{r(n)}} \leq \frac{1-\hat{a}}{\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1} a(i)\Big)}\,, \quad n = 0,1,\ldots,p-1 \tag{3.7}$$

then (3.2) and (3.3) hold with $B = 1$. Hence, the following corollary is a direct consequence of the above theorem.

Corollary 4. *Assume that* $\prod_{i=0}^{p-1} a(i) < 1$*, and that* (3.6) *and* (3.7) *hold. Then Eq.* (3.1) *has a nonnegative periodic solution* $\{\tilde{x}(n)\}$*. Furthermore, if either* (3.4) *or* (3.5) *holds also, then* $\{\tilde{x}(n)\}$ *is the only periodic solution of Eq.* (3.1) *and it is a global attractor of all nonnegative solutions of Eq.* (3.1).

Next, we consider some special cases of Eq. (3.1). If $a(n)b(n) \equiv 0$, then (3.2) holds for any B and (3.3) holds when B is large. Hence the following corollary is a direct consequence of Theorem 4.

Corollary 5. *Assume that $a(n)b(n) \equiv 0$ and $\prod_{i=0}^{p-1} a(i) < 1$. Then Eq.* (3.1) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$. Furthermore, if either* (3.4) *or* (3.5) *holds, then $\{\tilde{x}(n)\}$ is the only periodic solution and it is a global attractor of all nonnegative solutions of Eq.* (3.1)*.*

In particular, when $a(n) \equiv 0$, Eq. (3.1) reduces to

$$x(n+1) = c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}}\,, \quad n = 0, 1, \ldots \tag{3.8}$$

and the following result is a direct consequence of Corollary 5.

Corollary 6. *Eq.* (3.8) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ which is also a global attractor of all nonnegative solutions of Eq.* (3.8)*.*

When $b(n) \equiv 0$, Eq. (1.1) reduces to

$$x(n+1) = a(n)x(n) + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}}\,, \quad n = 0, 1, \ldots . \tag{3.9}$$

The following result comes from Corollary 5 immediately.

Corollary 7. *Assume that $\prod_{i=0}^{p-1} a(i) < 1$. Then Eq.* (3.9) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$. If either* (3.4) *or* (3.5) *holds, then $\{\tilde{x}(n)\}$ is the only periodic solution and it is a global attractor of all nonnegative solutions of Eq.* (3.9)*.*

In addition, when $q(n) \equiv 0$, Eq. (3.1) reduces to

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)}}{1+e^{r(n)}}\,, \quad n = 0, 1, \ldots . \tag{3.10}$$

In this case, (3.5) holds automatically. Hence, the following conclusion is a direct consequence of Theorem 4 and Corollary 4.

Corollary 8. *Assume that $\prod_{i=0}^{p-1} a(i) < 1$. Suppose also either there is a positive number B such that* (3.2) *and* (3.3) *hold with $q(n) \equiv 0$, or in particular* (3.6) *and* (3.7) *hold with $q(n) \equiv 0$. Then Eq.* (3.10) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (3.10)*.*

Next, consider the difference equation

$$x(n+1) = \frac{\alpha(n)x(n)}{\gamma(n)x(n)+\delta(n)} + \frac{\beta(n)}{\mu(n)x(n-k)+\eta(n)}\,, \quad n = 0, 1, \ldots \tag{3.11}$$

where $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$, $\{\delta(n)\}$, $\{\mu(n)\}$ and $\{\eta(n)\}$ are nonnegative sequences. Eq. (3.11) is in the form of Eq. (1.1) with

$$f(n,x) = \frac{\alpha(n)x}{\gamma(n)x+\delta(n)} \quad \text{and} \quad g(n,x) = \frac{\beta(n)}{\mu(n)x+\eta(n)}\,.$$

Clearly f is nondecreasing in x. Noting

$$\frac{d}{dx}\Big(f(n,x) - \frac{\alpha(n)}{\delta(n)}x\Big) = -\frac{\alpha(n)\gamma(n)x(\gamma(n)x+2\delta(n))}{\delta(n)(\gamma(n)x+\delta(n))^2} \le 0$$

we see that $f(n,x) - \frac{\alpha(n)}{\delta(n)}x$ is nonincreasing in x. Since

$$\frac{d}{dx}\big(g(n,x)\big) = -\frac{\beta(n)\mu(n)}{(\mu(n)x+\eta(n))^2}\,,$$

we see that for each n,

$$\max_{x\ge 0}\left|\frac{dg(n,x)}{dx}\right| = \frac{\beta(n)\mu(n)}{\eta^2(n)}\,.$$

Hence, by Theorems 1 and 2, we have the following result immediately.

Theorem 5. *Assume that*

$$\delta(n) \neq 0 \quad \textit{and} \quad \prod_{i=0}^{p-1}\frac{\alpha(i)}{\delta(i)} < 1\,, \tag{3.12}$$

and that there is a positive number B such that

$$\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\Big[\frac{\beta(j)}{\mu(j)B+\eta(j)} - \frac{\alpha(j)\gamma(j)B^2}{\delta(j)(\gamma(j)B+\delta(j))}\Big] \ge 0\,, \quad n = 0,1,\dots,p-1 \tag{3.13}$$

and

$$\frac{1}{1-\hat{a}}\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)}{\eta(j)} \le B\,, \quad n = 0,1,\dots,p-1 \tag{3.14}$$

where $\hat{a} = \prod_{i=0}^{p-1}\frac{\alpha(i)}{\delta(i)}$. Then Eq. (3.11) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either*

$$\frac{\alpha(n)}{\delta(n)} \le 1 \quad \textit{and} \quad \sum_{j=n}^{n+k}\Big(\prod_{i=j+1}^{n+k}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)\mu(j)}{\eta^2(j)} < 1\,, \quad n = 0,1,\dots,p-1\,, \tag{3.15}$$

or

$$\sum_{j=n}^{n+k+p-1}\Big(\prod_{i=j+1}^{n+k+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)\mu(j)}{\eta^2(j)} < 1\,, \quad n = 0,1,\dots,p-1\,, \tag{3.16}$$

then Eq. (3.11) *has a unique periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative periodic solutions of Eq.* (3.11).

When $\mu(n) \equiv \gamma(n)$, $\eta(n) \equiv \delta(n)$ and $k = 0$, Eq. (3.11) reduces to the following Riccati equation

$$x(n+1) = \frac{\alpha(n)x(n)+\beta(n)}{\gamma(n)x(n)+\delta(n)}\,, \quad n = 0,1,\dots\,. \tag{3.17}$$

(3.13), (3.14), (3.15) and (3.16) reduce to

$$\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)\delta(j)-\alpha(j)\gamma(j)B^2}{\delta(j)(\gamma(j)B+\delta(j))}\geq 0\,,\quad n=0,1,\ldots,p-1\,, \tag{3.18}$$

$$\frac{1}{1-\hat{a}}\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)}{\delta(j)}\leq B\,,\quad n=0,1,\ldots,p-1\,, \tag{3.19}$$

$$\frac{\alpha(n)}{\delta(n)}\leq 1\quad\text{and}\quad\frac{\beta(n)\gamma(n)}{\delta^2(n)}<1\,,\quad n=0,1,\ldots,p-1\,. \tag{3.20}$$

and

$$\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)\frac{\beta(j)\gamma(j)}{\delta^2(j)}<1\,,\quad n=0,1,\ldots,p-1\,, \tag{3.21}$$

respectively. Hence, the following is a direct consequence of the above theorem.

Corollary 9. *Assume that* (3.12) *holds and there is a positive number B such that* (3.18) *and* (3.19) *hold. Then Eq.* (3.17) *has a periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either* (3.20) *or* (3.21) *holds, then Eq.* (3.17) *has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (3.17).

Clearly, if

$$\beta(n)\delta(n)\geq\alpha(n)\gamma(n)\,,\quad n=0,1,\ldots \tag{3.22}$$

and

$$\frac{\beta(n)}{\delta(n)}\leq\frac{1-\hat{a}}{\sum_{j=n}^{n+p-1}\Big(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\Big)}\,,\quad n=0,1,\ldots \tag{3.23}$$

then (3.18) and (3.19) hold with $B=1$. Hence, the following conclusion comes from Corollary 9 directly.

Corollary 10. *Assume that* (3.12), (3.22) *and* (3.23) *hold. Then Eq.* (3.17) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either* (3.20) *or* (3.21) *holds, then $\{\tilde{x}(n)\}$ is the only periodic solution and it is a global attractor of all nonnegative solutions of Eq.* (3.17).

If $\alpha(n)\gamma(n)\equiv 0$, then (3.18) holds for any B, and (3.19) holds when B is large. Hence, the following corollary is a direct consequence of Theorem 5.

Corollary 11. *Assume that $\alpha(n)\gamma(n)\equiv 0$ and* (3.12) *holds. Then Eq.* (3.17) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$. Furthermore, if either* (3.20) *or* (3.21) *holds, then $\{\tilde{x}(n)\}$ is the only periodic solution and it is a global attractor of all nonnegative solutions of Eq.* (3.17).

In particular, when $\alpha(n) \equiv 0$, Eq.(3.17) reduces to

$$x(n+1) = \frac{\beta(n)}{\gamma(n)x(n) + \delta(n)}, \quad n = 0, 1, \dots . \tag{3.24}$$

Clearly, (3.12) holds when $\delta(n) \neq 0$. Hence, from Corollary 8 we have the following conclusion immediately.

Corollary 12. *Assume that $\delta(n) \neq 0$. Then, Eq.* (3.24) *has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either* (3.20) *or* (3.21) *holds, then $\{\tilde{x}(n)\}$ is the only periodic solution of Eq.* (3.24) *and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Eq.* (3.24).

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