ARCHIVUM MATHEMATICUM (BRNO)
Tomus 53 (2017), 325–334

ALMOST C-SPINORIAL GEOMETRY

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Abstract. Almost c-spinorial geometry arises as an interesting example of the metrisability problem for parabolic geometries. It is a complex analogue of real spinorial geometry. In this paper, we first define the type of parabolic geometry in question, then we discuss its underlying geometry and its homogeneous model. We compute irreducible components of the harmonic curvature and discuss the conditions for regularity. In the second part of the paper, we describe the linearisation of the metrisability problem for Hermitian and skew-Hermitian metrics, state the corresponding first BGG equations and present explicit formulae for their solutions on the homogeneous model.

1. Introduction

Let G be a real simple Lie group and P a parabolic subgroup with Lie algebras $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Let $\mathfrak{g} = \bigoplus_{j=-k}^{k} \mathfrak{g}_j$ be a $|k|$-gradation of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ and let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G,P) (see [6]). Now, consider the natural bundle $H := \mathcal{G} \times_P \mathfrak{g}_{-1}$. A parabolic geometry is called metrisable if there exists a (pseudo-) Riemannian metric g on the distribution H with the property that there exists a Weyl structure on M such that g is covariantly constant in the directions of H with respect to the corresponding Weyl covariant derivative. According to a recent classification of metrisable parabolic geometries with irreducible $\mathfrak{g}_{-1}$-part (see [4], [12]), there are three interesting geometries in the following sense. By definition, a $\mathfrak{g}_0$-representation V is called absolutely irreducible, if $V \otimes \mathbb{C}$ is $\mathfrak{g}_0$-irreducible. If $\mathfrak{g}$ is a complex Lie algebra which is considered as the real Lie algebra, and a $\mathfrak{g}_0$-module $B \subset \odot^2 \mathfrak{g}_{-1}$ is absolutely irreducible, then there are only three types of parabolic geometries which are metrisable by a procedure describe below. A version of one of these geometries was recently studied in [3], which is called c-projective geometry. It is a natural analogue of real projective geometry.

In this paper we discuss the equivalence problem between regular normal parabolic geometries of type (G,P) and infinitesimal flag structures, where the pair (G,P) defines a c-spinorial geometry. To solve this problem we study the first and

2010 *Mathematics Subject Classification*: primary 53A40; secondary 53B35.
Key words and phrases: spinorial geometry, metrisability problem, equivalence problem, first BGG operator.
Received April 18, 2017, revised December 2017. Editor M. Čadek.
DOI: 10.5817/AM2017-5-325

second cohomology groups $H^i(\mathfrak{g}_-, \mathfrak{g})$ with coefficients in $\mathfrak{g}$, where $i = 1, 2$. We restrict our attention to the case $\operatorname{rank}(\mathfrak{g}) \geq 3$.

Since we want to study the above-mentioned metrisability problem, we briefly explain how the metrisability procedure works (for further details see [4], [12]). In order to find a metric on a distribution $\mathcal{G} \times_P \mathfrak{g}_{-1} = H \subset TM$, we need to consider nondegenerate sections of $\odot^2 H^*$. For this it turns out to be useful to employ a dual picture by looking for nondegenerate inverse metrics in $\Gamma(\odot^2 H)$. For the almost c-spinorial geometry the space $\odot^2 H$ decomposes into two $\mathfrak{g}_0$-irreducible bundles $\mathcal{G} \times_P \hat{V} \oplus \mathcal{G} \times_P \hat{W}$. To ensure that elements of $\Gamma(\odot^2 H)$ are covariantly constant in the directions of H (with respect to a suitable Weyl covariant derivative), we use a suitable invariant differential operator of first order (see [14]). Such an operator arises as a composition of a covariant derivative and a projection onto an irreducible subspace. A typical example of this operator is the first BGG operator $\mathcal{D} : \Gamma(\odot^2 H) \to \Gamma(\odot^2 H \circledcirc (\mathcal{G} \times_P \mathfrak{g}_1))$, where $\circledcirc$ is the Cartan product (for BGG operators see [?],[1]). It may happen that this BGG operator is not of the first order, but if we tensor the bundles by a suitable line bundle L, then the resulting operator $\mathcal{D}' \colon \Gamma(\odot^2 H \otimes L) \to \Gamma(\odot^2 H \otimes L \circledcirc (\mathcal{G} \times_P \mathfrak{g}_1))$ is of first order. Therefore $\pi \circ \nabla|_H \sigma = 0$ holds for every Weyl covariant derivative $\nabla|_H$ acting in the directions of H, where $\pi \colon \Gamma(\odot^2 H \otimes L \otimes (\mathcal{G} \times_P \mathfrak{g}_1)) \to \Gamma(\odot^2 H \otimes L \circledcirc (\mathcal{G} \times_P \mathfrak{g}_1))$ is the natural projection and σ is an element of the kernel of $\mathcal{D}'$. If the algebraic linearisation condition (see [4], [12]) is satisfied, one can choose a Weyl covariant derivative $\hat{\nabla}$ in such a way that $\hat{\nabla}|_H \sigma = 0$. The only complication is that we are no longer dealing with inverse metrics (sections of $\odot^2 H$), but with sections of $\odot^2 H \otimes L$.

Let us briefly recall a version of the algebraic linearisation condition. Suppose that $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module and let B be a $\mathfrak{g}_0$-irreducible subspace of $\odot^2 \mathfrak{g}_1$ such that it contains nondegenerate elements. We say that B satisfies the algebraic linearisation condition if $B \otimes \mathfrak{g}_1 \simeq B \circledcirc \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, where $\circledcirc$ is the Cartan product. For detailed description, see [4], [12].

It turns out that in the case of almost c-spinorial geometry there exist Hermitian metrics as well as skew-Hermitian metrics on H. We describe explicitly solutions of metrisability equations in the homogeneous case.

The structure of the paper is as follows. In Section 2 we describe almost c-spinorial geometry and its equivalence problem. We turn to metrisability in Section 3, where we consider Hermitian metrics in subsection 3.1 and skew-Hermitian metrics in subsection 3.2. In each subsection we first solve the metrisability problem, then we describe the solutions for the flat model.

Acknowledgement. The author would like to thank Vladimír Souček for helpful discussions. This research was supported by the institutional grant of the Charles University SVV 260456 and by the grant GACR 17-01171S.

2. Almost c-spinorial geometry

2.1. **Definition.** Let $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{C})$. We equip the algebra $\mathfrak{g}$ with a gradation $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ which corresponds to the last node crossed in the Dynkin diagram, or equivalently to block matrices of the form (see [15])

$$\begin{pmatrix} A & a & B \\ b & 0 & -a' \\ C & -b' & -A' \end{pmatrix} \colon A, B, C \in \mathfrak{gl}(n, \mathbb{C}), a \in \mathbb{C}^n, b \in (\mathbb{C}^n)^*, B = -B', C = -C',$$

where $\mathfrak{g}_0$ is the block diagonal subalgebra, $\mathfrak{g}_1$ is generated by a, $\mathfrak{g}_{-1}$ by b, $\mathfrak{g}_2$ by B, $\mathfrak{g}_{-2}$ by C, and the prime operation means transposition with respect to the anti-diagonal. Let us denote by $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ its parabolic subalgebra and let $\mathfrak{g}_-$ be the sum $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Moreover, we define the filtration $\mathfrak{g}^i = \oplus_{j \geq i} \mathfrak{g}_j,$.

Let G be the special orthogonal group $SO(2n+1, \mathbb{C})$, and let $P \subset G$ be the subgroup with the Lie algebra $\mathfrak{p}$, and let G_0 be the subgroup of P with the Lie algebra $\mathfrak{g}_0$.

From now on we shall view $G_0 \subset P \subset G$ as real Lie groups embedded in the standard way to $GL(4n+2, \mathbb{R})$.

Definition 1. Let G and P be as above. A parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) is called almost c-spinorial geometry.

We fix the following notation. A complex Lie algebra will be denoted by $\mathfrak{g}^{\mathbb{C}}$ and its underlying real Lie algebra will be denoted by $\mathfrak{g}$. Let us mention the relation $\mathfrak{g} \otimes \mathbb{C} \simeq \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$.

2.2. **The equivalence problem.** In this section we discuss the equivalence problem between infinitesimal flag structure and normal Cartan connection for the case of almost c-spinorial geometry. We shall consider only $\operatorname{rank}(\mathfrak{g}) \geq 3$. The case $\operatorname{rank}(\mathfrak{g}) = 2$ is quite special and will be discussed elsewhere.

Theorem 1. *Let* $\operatorname{rank}(\mathfrak{g}) \geq 3$. *Then normal regular parabolic geometries* $(\mathcal{G} \to M, \omega)$ *of type* (G, P) *are in bijective correspondence up to isomorphism with triples* (M, H, J), *where* M *is a manifold of (real) dimension* $n(n+1)$ *with a filtration on* TM *given by a generic distribution* H *of (real) dimension* $2n$, *and* J *is an almost complex structure on* H. *The homogeneous model* $(G \to G/P)$ *corresponds to isotropic complex Grassmannian* $\mathrm{IGr}(n, 2n+1)$ *equipped with the tautological bundle.*

Proof. According to Observation 3.1.7 in [6], regular infinitesimal flag structure is equivalent to:

(i) a filtration $\{T^i M\}$ such that M is a filtered manifold with locally trivial bundle of symbol algebras with the fiber $\mathfrak{g}_-$

(ii) the reduction $\mathrm{Ad} \colon G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$ of structure group of the natural frame bundle of $\mathrm{gr}(TM)$

As H is a generic distribution, (i) holds. By an easy computation one gets that the homomorphism Ad is injective. Since $\mathfrak{g}_{-2}$ is bracket-generated by $\mathfrak{g}_{-1}$, $\mathrm{Aut}(\mathfrak{g}_{-2})$ is fully determined by $\mathrm{Aut}(\mathfrak{g}_{-1})$. As $\mathrm{Aut}(\mathfrak{g}_{-1})$ are real automorphisms of the real $2n$-dimensional vector space, the reduction is equivalent to a choice of an almost complex structure on H. Therefore, the claim of the theorem follows by the general theorem (Theorem 3.1.14 in [6]) on the equivalence of categories between normal regular parabolic geometries of type (G, P) and underlying regular infinitesimal flag structure on M.

Let Q be the quadratic form given by $z_{n+1}^2 + \sum_{i=1}^{n} z_i z_{2n+2-i}$. It is easy to see that the set of vectors $\{e_i\}_{i=1}^{n}$ form isotropic subspace, where $\{e_i\}_{i=1}^{2n+1}$ is standard basis of $\mathbb{C}^{2n+1}$. Under the action of G, the orbit of the subspace is the Grassmannian and isotropy subgroup is P. $\square$

In order to compute the regularity condition of a parabolic geometry we shall consider the second cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$. Corollary 3.1.8 from [6] gives us an equivalent characterization of a regular parabolic geometry in terms of the filtration of $\mathfrak{g}$. Therefore, if κ is the curvature function then the regularity is equivalent to

$$\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1} \quad \text{for all} \quad i, j < 0\,, \tag{1}$$

where $\mathfrak{g}^i = \oplus_{l \geq i} \mathfrak{g}_l$. The only nontrivial condition (1) for $|2|$-graded algebra is $\kappa(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \subset \mathfrak{g}^{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. In other words the $\wedge^2(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-2}$ component of the curvature function is zero. To proceed further we compute the second cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$, where the harmonic curvature κ_H takes values.

To describe the result, we need a suitable notation. A real irreducible representation V of $\mathfrak{g}$ can be described as a real part of an irreducible module for $\mathfrak{g} \otimes \mathbb{C}$. Because $\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$, they can have the following two forms:

(i) either the module $V \otimes \mathbb{C}$ is the (outer) product $V_\lambda \boxtimes V_\lambda$, which will be denoted by $V = [V_\lambda \boxtimes V_\lambda]_{\mathbb{R}}$, where λ is an integral dominant weight for $\mathfrak{g}^{\mathbb{C}}$,

(ii) or the module $V \otimes \mathbb{C}$ is $V_\lambda \boxtimes V_{\lambda'} \oplus V_{\lambda'} \boxtimes V_\lambda$, which will be denoted by $V = [V_\lambda \boxtimes V_{\lambda'} \oplus V_{\lambda'} \boxtimes V_\lambda]_{\mathbb{R}}$, where (λ, λ') is a pair of dominant integral weights for $\mathfrak{g}^{\mathbb{C}}$.

The highest (fundamental) weights of fundamental representations of $\mathfrak{g}^{\mathbb{C}}$ will be denoted by $\omega_i, i = 1, \dots, \mathrm{rank}(\mathfrak{g}^{\mathbb{C}})$. Recall that a $\mathfrak{g}^{\mathbb{C}}$-dominant algebraically integral weight is a linear combination of the fundamental weights with nonnegative integral coefficients.

Since $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, we can decompose it into its semisimple part and its centre, $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0)$. In our case, $\mathfrak{g}_0^{ss}$ is $\mathfrak{sl}(n, \mathbb{C})$ considered as the underlying real Lie algebra and hence the semisimple part is simple. Recall that the compatible Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}_0$ which splits into $\mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{g}_0)$, where $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}_0^{ss}$. Hence, complex irreducible representations of $\mathfrak{g}_0$ are in bijective correspondence with a subset of $\mathfrak{h}^*$, but the dominance and integrality conditions refer only to the restrictions to $\mathfrak{h}'$. In the case of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2n+1, \mathbb{C})$, complex irreducible representations of $\mathfrak{g}_0^{\mathbb{C}}$ are in bijective correspondence with $\{\sum_{i=1}^{n} \lambda_i \omega_i : \lambda_i \in \mathbb{N} \cup \{0\} \text{ for } 1 \leq i \leq n-1, \lambda_n \in \mathbb{R}\}$. For detailed description see sections 3.2.12 and 3.2.13 in [6]. It will always be clear from the context whether we are considering an irreducible module for the Lie algebra $\mathfrak{g}$ or $\mathfrak{g}_0$.

Let us clarify the notation on a few distinguished examples. Suppose that $(\mathcal{G} \to M, \omega)$ is c-spinorial geometry. The generic distribution H is generated by the $\mathfrak{g}_0$-representation $\mathfrak{g}_{-1} = [V_{\omega_{n-1} - 2\omega_n} \boxtimes V_0 \oplus V_0 \boxtimes V_{\omega_{n-1} - 2\omega_n}]_{\mathbb{R}}$, where V_0 is the trivial representation $\mathbb{C}$. The tangent bundle TM is isomorphic with the natural bundle generated by the representations $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, where $\mathfrak{g}_{-2} = [V_{\omega_{n-2} - 2\omega_n} \boxtimes$

$V_0 \oplus V_0 \boxtimes V_{\omega_{n-2}-2\omega_n}]_{\mathbb{R}}$. Taking dual representations one gets that T^*M is given by the representation $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = [V_{\omega_1} \boxtimes V_0 \oplus V_0 \boxtimes V_{\omega_1}]_{\mathbb{R}} \oplus [V_{\omega_2} \boxtimes V_0 \oplus V_0 \boxtimes V_{\omega_2}]_{\mathbb{R}}$.

Lemma 2. *The second cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$ has three irreducible components with respect to the action of $\mathfrak{g}_0$.*

(i) *Their highest weight description for rank $\mathfrak{g} \geq 5$ is:*

(1) $V_1 = [V_{\omega_{n-2}-2\omega_n} \boxtimes V_{\omega_1+\omega_2} \oplus V_{\omega_1+\omega_2} \boxtimes V_{\omega_{n-2}-2\omega_n}]_{\mathbb{R}}$
(2) $V_2 = [V_{\omega_1+\omega_{n-2}-2\omega_n} \boxtimes V_{\omega_1} \oplus V_{\omega_1} \boxtimes V_{\omega_1+\omega_{n-2}-2\omega_n}]_{\mathbb{R}}$
(3) $V_3 = [V_{\omega_1+\omega_2+\omega_{n-2}-2\omega_n} \boxtimes V_0 \oplus V_0 \boxtimes V_{\omega_1+\omega_2+\omega_{n-2}-2\omega_n}]_{\mathbb{R}}$

Moreover, V_1 has homogeneity 1, V_2 has homogeneity 0, V_3 has homogeneity 1.

(ii) *Their highest weight description for rank of $\mathfrak{g}$ equal to 4 is:*

(1) $V_1 = [V_{\omega_2-2\omega_4} \boxtimes V_{\omega_1+\omega_2} \oplus V_{\omega_1+\omega_2} \boxtimes V_{\omega_2-2\omega_4}]_{\mathbb{R}}$
(2) $V_2 = [V_{\omega_1+\omega_2-2\omega_4} \boxtimes V_{\omega_1} \oplus V_{\omega_1} \boxtimes V_{\omega_1+\omega_2-2\omega_4}]_{\mathbb{R}}$
(3) $V_3 = [V_{\omega_1+2\omega_2-2\omega_4} \boxtimes V_0 \oplus V_0 \boxtimes V_{\omega_1+2\omega_2-2\omega_4}]_{\mathbb{R}}$

Moreover, V_1 has homogeneity 1, V_2 has homogeneity 0, V_3 has homogeneity 3.

(iii) *Their highest weight description for rank of $\mathfrak{g}$ equal to 3 is:*

(1) $V_1 = [V_{\omega_1-2\omega_3} \boxtimes V_{\omega_1+\omega_2} \oplus V_{\omega_1+\omega_2} \boxtimes V_{\omega_1-2\omega_3}]_{\mathbb{R}}$
(2) $V_2 = [V_{2\omega_1-2\omega_3} \boxtimes V_{\omega_1} \oplus V_{\omega_1} \boxtimes V_{2\omega_1-2\omega_3}]_{\mathbb{R}}$
(3) $V_3 = [V_{2\omega_1+2\omega_2-2\omega_3} \boxtimes V_0 \oplus V_0 \boxtimes V_{2\omega_1+2\omega_2-2\omega_3}]_{\mathbb{R}}$

Moreover, V_1 has homogeneity 1, V_2 has homogeneity 0, V_3 has homogeneity 3.

Proof. We give the proof for the general case $n \geq 5$, the other two cases are similar. For the low rank cases, it is also possible to use the online service provided by J. Šilhan [13]. Recall the decompositions $\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$, $\mathfrak{g}_- \otimes \mathbb{C} \cong \mathfrak{g}_-^{\mathbb{C}} \oplus \mathfrak{g}_-^{\mathbb{C}}$ and similarly for $\mathfrak{g}_0$. The adjoint representation of $\mathfrak{g} \otimes \mathbb{C}$ on itself acts as $\mathfrak{g}^{\mathbb{C}} \boxtimes \mathbb{C} \oplus \mathbb{C} \boxtimes \mathfrak{g}^{\mathbb{C}}$. Due to this we have

$$(2)\quad H^2(\mathfrak{g}_-, \mathfrak{g}) \otimes \mathbb{C} \cong \bigoplus_{i+j=2} \left(H^i(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \boxtimes H^j(\mathfrak{g}_-^{\mathbb{C}}, \mathbb{C}) \oplus H^i(\mathfrak{g}_-^{\mathbb{C}}, \mathbb{C}) \boxtimes H^j(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})\right).$$

Hence it is sufficient to compute the (complex) cohomology groups with values in the trivial, resp. adjoint representations up to the order two using the Kostant theorem. We get

$$H^0(\mathfrak{g}_-^{\mathbb{C}}, \mathbb{C}) \cong V_0 \cong \mathbb{C},\quad H^1(\mathfrak{g}_-^{\mathbb{C}}, \mathbb{C}) \cong V_{\omega_1} \cong \mathfrak{g}_1^{\mathbb{C}},\quad H^2(\mathfrak{g}_-^{\mathbb{C}}, \mathbb{C}) \cong V_{\omega_1+\omega_2} \cong \mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}},$$

and (symbol $\circledcirc$ denotes the Cartan product)

$$H^0(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \cong V_{\omega_{n-2}-2\omega_n} \cong \mathfrak{g}_{-2}^{\mathbb{C}},\quad H^1(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \cong V_{\omega_1+\omega_{n-2}-2\omega_n} \cong \mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}},$$
$$H^2(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \cong V_{\omega_1+\omega_2+\omega_{n-2}-2\omega_n} \cong \mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}}.$$

We get the corresponding homogenities from relations

$$V_1 \cong [(\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}}) \boxtimes \mathfrak{g}_{-2}^{\mathbb{C}} \oplus \mathfrak{g}_{-2}^{\mathbb{C}} \boxtimes (\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}})]_{\mathbb{R}},$$

$$V_2 \cong [\mathfrak{g}_1^{\mathbb{C}} \boxtimes (\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}}) \oplus (\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}}) \boxtimes \mathfrak{g}_1^{\mathbb{C}}]_{\mathbb{R}},$$

$$V_3 \cong [\mathbb{C} \boxtimes (\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}}) \oplus (\mathfrak{g}_1^{\mathbb{C}} \circledcirc \mathfrak{g}_2^{\mathbb{C}} \circledcirc \mathfrak{g}_{-2}^{\mathbb{C}}) \boxtimes \mathbb{C}]_{\mathbb{R}}.$$

□

As a corollary, we get from Cor. 3.1.8 [6].

Corollary. *The c-spinorial geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) is regular if and only if the second piece V_2 in $H^2(\mathfrak{g}_-, \mathfrak{g})$ is trivial.*

3. Metrisability

In this section we shall discuss the metrisability problem for an almost c-spinorial geometry (M, H, J) given by the distribution $H \subset TM$ and the complex structure J on H. We shall construct certain invariant subspaces $B \subset \odot^2 H$ and we shall apply the linearisation method translating metrisability problem to the problem of finding all nondegenerate solutions of certain invariant overdetermined first order system of linear PDE's.

Let V be the $\mathfrak{g}$-irreducible representation given by $[V_{\omega_1} \boxtimes V_{\omega_1}]_{\mathbb{R}}$. The representation V give rise to the first BGG operator $\mathcal{D}^V : \Gamma(\mathcal{H}_0(\mathfrak{g}_+, V)) \to \Gamma(\mathcal{H}_1(\mathfrak{g}_+, V))$ of first order, where $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathcal{H}_i(\mathfrak{g}_+, V)$ is the natural bundle given by the homology $H_i(\mathfrak{g}_+, V)$. Similarly, for the representation $W = [W_{2\omega_1} \boxtimes W_0 \oplus W_0 \boxtimes W_{2\omega_1}]_{\mathbb{R}}$, there is the first BGG operator $\mathcal{D}^W$ of first order.

3.1. **Hermitian metrics.** We shall use abstract indices as in [3]. Sections of the bundle H or elements of $\mathfrak{g}_{-1}$ will be denoted by h^α, sections of the dual H^* or elements of $\mathfrak{g}_1$ by h_α. The almost complex structure J on H is denoted by $J_\alpha{}^\beta$ and the same notation is used in the case of the almost complex structure on $\mathfrak{g}_{-1}$. An analogue of the Einstein summation convention is used as, e.g., in the action of J on a section h of H^* in $-J_\alpha{}^\beta h_\beta$. A Hermitian metric on (H, J) is a J-invariant (pseudo-)Riemannian metric metric $g_{\alpha\beta} \in \Gamma(\odot^2 H^*)$, i.e. it satisfies the relation

$$J_\alpha^\gamma J_\beta^\delta g_{\gamma\delta} = g_{\alpha\beta} \,.$$

The complexification $H \otimes \mathbb{C}$ decomposes as $H^{\mathbb{C}} \oplus \overline{H}^{\mathbb{C}}$. We use Greek indices to denote real sections of H resp. $H \otimes \mathbb{C}$, and we use Latin indices (with or without the bar) for sections of $H^{\mathbb{C}}$ or $\overline{H}^{\mathbb{C}}$.

In local coordinates, indices for sections h^α of H are in $\{1, \dots, 2n\}$ and indices for sections h^a, resp. $\bar{h}^{\bar{a}}$, are in $\{1, \dots, n\}$.

Let us note that the representation V corresponds to a Hermitian metrics.

Theorem 3. *Let $(\mathcal{G} \to M, \omega)$ be an almost c-spinorial geometry. There exists a bijection between nondegenerate Hermitian metrics on $H \subset TM$, which are covariantly constant in the directions of H with respect to a suitable Weyl connection of the geometry, and nondegenerate sections in the kernel of $\mathcal{D}^V$.*

Proof. The general scheme of proof described in the introduction can be used (for details see [12]) in this particular case. A simplification is that the first BGG operator corresponding to the choice of V is a first order operator, so that it is not necessary to twist the corresponding associated bundle by a line bundle. Let us show how the BGG operator arises. Consider the $\mathfrak{g}$-representation $V = [V_{\omega_1} \boxtimes V_{\omega_1}]_{\mathbb{R}}$ which defines the BGG sequence. The first BGG operator $\mathcal{D}^V$ acts on a sections of the natural bundle given by the zeroth homology group $H_0(\mathfrak{g}_+, V)$ and it has values in a sections of the natural bundle given by the first homology group $H_1(\mathfrak{g}_+, V)$. By an easy computation one gets $H_0(\mathfrak{g}_+, V) = [V_{\omega_{n-1}-2\omega_n} \boxtimes V_{\omega_{n-1}-2\omega_n}]_{\mathbb{R}}$ and $H_1(\mathfrak{g}_+, V) = [V_{\omega_1+\omega_{n-1}-2\omega_n} \boxtimes V_{\omega_{n-1}-2\omega_n} \oplus V_{\omega_{n-1}-2\omega_n} \boxtimes V_{\omega_1+\omega_{n-1}-2\omega_n}]_{\mathbb{R}}$ as $\mathfrak{g}_0$-representations. This shows that the first BGG operator acts on the correct bundle (inverse Hermitian metrics). Since $H_1(\mathfrak{g}_+, V)$ is the Cartan product of $H_0(\mathfrak{g}_+, V)$ and $\mathfrak{g}_1$, the operator is of first order. $\square$

Let $(p\colon G \to G/P, \omega)$ be the homogeneous model of a parabolic geometry and let $\pi\colon G \times_P \mathbb{V} \to G/P$ be an associated bundle. The map φ given by $X \mapsto exp(X)P$ is a diffeomorphism from $\mathfrak{g}_-$ onto dense open subset $U \subset G/P$. Recall that, in particular, smooth sections $\Gamma(U, G \times_P \mathbb{V})$ are in bijective correspondence with P-equivariant smooth functions $C^\infty(p^{-1}(U), \mathbb{V})$. For any $\sigma \in \Gamma(U, G \times_P \mathbb{V})$ one can construct the local trivialisation induced by σ. Indeed, there is a map $U \times \mathbb{V} \to \pi^{-1}(U)$ which maps (x, v) to the P-orbit $P \cdot (\sigma(x), v)$. On the other hand, the inverse map is given by $[\![u, v]\!] \mapsto (x, b \cdot v)$, where $u = \sigma(x) \cdot b$ and $[\![u, v]\!]$ stands for the equivalence class of the element (u, v). In such a trivialisation every $s \in \Gamma(U, G \times_P \mathbb{V})$ has the form $x \mapsto (x, f(\sigma(x)))$, where f is the P-equivariant function which corresponds to s. Therefore, after a choice of a section σ, any section $s \in \Gamma(U, G \times_P \mathbb{V})$ can be viewed as the unique function $f : U \to \mathbb{V}$, and vice versa. Moreover, by precomposition with the diffeomorphism φ one gets functions defined on $\mathfrak{g}_-$. In particular, a solution to a BGG equation can be characterized by such a function. In the sequel the section σ is assumed to be a normal section. For closer discussion see [5] and Example 5.1.12 in [6], which concern the very flat Weyl structure.

Now we describe $\mathrm{Ker}(\mathcal{D}^V)$. This operator is called metrisability operator. The tractor bundle which defines the metrisability operator is called the metric tractor bundle. The standard tractor bundle is the natural bundle which arises from the defining representation of $\mathfrak{g}$ and it is denoted by $\mathcal{T}$. Clearly, $\mathfrak{g}$-defining representation induces filtration $\mathcal{T} = \mathcal{T}^0 \supset \mathcal{T}^1 \supset \mathcal{T}^2$ such that $\mathcal{T}^0/\mathcal{T}^1 \simeq H^* = \mathcal{G} \times_P \mathfrak{g}_1$, $\mathcal{T}^1/\mathcal{T}^2 \simeq \mathbb{R}$ the trivial representation, and $\mathcal{T}^2 \simeq H = \mathcal{G} \times_P \mathfrak{g}_{-1}$.

The metric tractor bundle of $\mathcal{D}^V$ is the tractor bundle $\mathcal{G} \times_P V \subset \odot^2\mathcal{T}$, where V is the module $V = [V_{\omega_1} \boxtimes V_{\omega_1}]_{\mathbb{R}}$. Its elements can be decomposed as $\begin{pmatrix} \tau_{\beta\gamma} \\ \xi_\beta \\ \rho \mid {\psi_\beta}^\gamma \mid {\hat{\psi}^\gamma}_\beta \\ \sigma^\beta \\ \nu^{\beta\gamma} \end{pmatrix}$,

where $\tau_{\beta\gamma} \in \odot^2\mathfrak{g}_1$, $\nu^{\beta\gamma} \in \odot^2\mathfrak{g}_{-1}$ and the following equations hold: $J^\alpha_\beta J^\delta_\gamma \tau_{\alpha\delta} = \tau_{\beta\gamma}$, $J^\beta_\alpha J^\gamma_\delta \nu^{\alpha\delta} = \nu^{\beta\gamma}$, $J^\alpha_\beta J^\gamma_\delta {\psi_\alpha}^\delta = {\psi_\beta}^\gamma$, $J^\alpha_\beta J^\gamma_\delta {\hat{\psi}^\delta}_\alpha = {\hat{\psi}^\gamma}_\beta$.

Theorem 4. *In the situation as above, the metrisability operator can be equivalently characterized as:*

$$\nabla_\alpha \eta^{\beta\gamma} - \delta_\alpha^{(\beta}\mu^{\gamma)} = 0 \quad \Leftrightarrow \quad \nabla_a \eta^{b\bar{c}} - \delta^b_a \mu^{\bar{c}} = 0, \nabla_{\bar{a}} \eta^{b\bar{c}} - \delta^{\bar{c}}_{\bar{a}} \mu^b = 0$$

for some μ, and $\eta \in \Gamma(\odot^2 H)$ such that $J^\beta_\alpha J^\gamma_\delta \eta^{\alpha\delta} = \eta^{\beta\gamma}$, where ∇ is a Weyl covariant derivative. Moreover, in the homogeneous case, solutions have the form

$$\begin{aligned}\eta^{\alpha\beta}(x,y) = \nu^{\alpha\beta} &+ y^{\alpha\gamma}\psi_\gamma{}^\beta + \hat{\psi}^\alpha{}_\gamma y^{\gamma\beta} - 2x^{(\alpha}\sigma^{\beta)} - \frac{1}{2}(x^\alpha x^\gamma \psi_\gamma{}^\beta + \hat{\psi}^\alpha{}_\gamma x^\gamma x^\beta)\\ &+ y^{\alpha\gamma}y^{\beta\delta}\tau_{\gamma\delta} + \rho x^\alpha x^\beta - y^{\alpha\gamma}\xi_\gamma x^\beta - x^\alpha y^{\beta\gamma}\xi_\gamma\\ &+ \frac{1}{2}(x^\alpha y^{\beta\gamma}x^\delta\tau_{\delta\gamma} + y^{\alpha\gamma}x^\delta\tau_{\gamma\delta}x^\beta)\\ &- \frac{1}{2}x^\alpha x^\beta x^\gamma \xi_\gamma + \frac{1}{4}x^\alpha x^\beta x^\gamma x^\delta \tau_{\delta\gamma}\,,\end{aligned}$$

where $x \in \mathfrak{g}_{-1}$, $y \in \mathfrak{g}_{-2}$ and ν, ψ, $\hat{\psi}$, σ, τ, ρ, $\xi \in V$ as stated above the theorem.

Remark. In particular we can take ∇ to be the flat Weyl connection induced by the normal coordinates and then ∇ is ordinary partial differentiation.

Proof. The metrisability operator is characterized by the action of the tractor covariant derivative and the projection on the homology part. Solutions can be computed algorithmically using [5]. For $X \in \mathfrak{g}_-$ and for $v \in V$ a normal solution has the form $\Pi(exp(-X)\cdot v)$ which is a finite sum by the nilpotence of the operator X, where $\Pi\colon \Gamma(G\times_P V) \to \Gamma(G\times_P H_0)$. Let us recall that in the case of a homogeneous model all solutions to a first BGG equation are normal. $\square$

Remark. The second part of the Theorem 4 can be generalized. Following a choice of normal coordinates and a normal section as in [5] for a general parabolic geometry $(\mathcal{G} \to M, \omega)$ then the following holds. If a normal solution to a first BGG equation exists then it has the form as in Theorem 4.

3.2. **Skew-Hermitian metrics.** There are also covariantly constant metrics (in the above sense) which are not Hermitian metrics. As in the previous subsection we first consider the kernel and then the homogeneous case.

The representation $W = [W_{2\omega_1}\boxtimes W_0 \oplus W_0 \boxtimes W_{2\omega_1}]_{\mathbb{R}}$ corresponds to skew-Hermitian metrics which can be characterized by the equation:

$$J^\gamma_\alpha J^\delta_\beta g_{\gamma\delta} = -g_{\alpha\beta}\,.$$

Theorem 5. *Let $(\mathcal{G} \to M, \omega)$ be an almost c-spinorial geometry. There exists a bijection between nondegenerate skew-Hermitian metrics on $H \subset TM$, which are covariantly constant in the directions of H with respect to a suitable Weyl connection of the geometry, and nondegenerate sections in kernel of $\mathcal{D}^W$.*

The proof of this theorem is analogous as in Theorem 3.

Now we proceed similarly as in the case of Hermitian metrics. If we consider the representation W, its tractor bundle can be realized in $\odot^2\mathcal{T}$. An element of W can be decomposed as $\begin{pmatrix} \tau_{\beta\gamma} \\ \xi_\beta \\ \rho \mid \psi_\beta{}^\gamma \mid \hat{\psi}^\gamma{}_\beta \\ \sigma^\beta \\ \nu^{\beta\gamma}\end{pmatrix}$, where $\tau_{\beta\gamma} \in \odot^2\mathfrak{g}_1$, $\nu^{\beta\gamma} \in \odot^2\mathfrak{g}_{-1}$ and the following equations hold: $J^{\alpha}_{\beta} J^{\delta}_{\gamma} \tau_{\alpha\delta} = -\tau_{\beta\gamma}$, $J^{\beta}_{\alpha} J^{\gamma}_{\delta} \nu^{\alpha\delta} = -\nu^{\beta\gamma}$, $J^{\alpha}_{\beta} J^{\gamma}_{\delta} \psi_{\alpha}{}^{\delta} = -\psi_{\beta}{}^{\gamma}$, $J^{\alpha}_{\beta} J^{\gamma}_{\delta} \hat{\psi}^{\delta}{}_{\alpha} = -\hat{\psi}^{\gamma}{}_{\beta}$.

Theorem 6. *In the situation as above, the metrisability operator can equivalently be characterized as:*

$$\nabla_\alpha \eta^{\beta\gamma} - \mu^{(\beta}\delta_\alpha^{\gamma)} = 0 \Leftrightarrow$$
$$\nabla_a \eta^{bc} - \mu^{(b}\delta_a^{c)} = 0, \qquad \nabla_{\bar{a}} \eta^{\bar{b}\bar{c}} - \mu^{(\bar{b}}\delta_{\bar{a}}^{\bar{c})} = 0$$

for some μ, and $\eta \in \Gamma(\odot^2 H)$ such that $J^{\beta}_{\alpha} J^{\gamma}_{\delta} \eta^{\alpha\delta} = -\eta^{\beta\gamma}$. Moreover, in the homogeneous case, solutions have the form

$$\begin{aligned}\eta^{\alpha\beta}(x,y) = \nu^{\alpha\beta} - y^{\alpha\gamma}\psi_{\gamma}{}^{\beta} - \hat{\psi}^{\alpha}{}_{\gamma} y^{\beta\gamma} + x^{(\alpha}\sigma^{\beta)} - \frac{1}{2}(x'^{\alpha} x^{\gamma}\psi_{\gamma}{}^{\beta} + \hat{\psi}^{\alpha}{}_{\gamma} x^{\gamma} x'^{\beta}) \\ + \rho x'^{\alpha} x'^{\beta} + y^{\alpha\gamma} y^{\beta\delta}\tau_{\gamma\delta} - \frac{1}{2}(y^{\alpha\gamma}\xi_{\gamma} x'^{\beta} + x'^{\alpha}\xi_{\gamma} y^{\beta\gamma}) \\ + \frac{1}{2}(x'^{\alpha} x^{\gamma}\tau_{\gamma\delta} y^{\beta\delta} + y^{\beta\gamma}\tau_{\gamma\delta} x^{\delta} x'^{\beta}) \\ - \frac{1}{12} x'^{\alpha} x'^{\beta} x^{\gamma}\xi_{\gamma} + \frac{1}{4} x'^{\alpha} x'^{\beta} x^{\gamma} x^{\delta}\tau_{\gamma\delta}\,,\end{aligned}$$

where $x \in \mathfrak{g}_{-1}$, $y \in \mathfrak{g}_{-2}$ and ν, ψ, $\hat{\psi}$, σ, τ, ρ, $\xi \in W$ similarly as in Theorem 4.

The proof is a straightforward computation.

Remark. Similarly to the remark below Theorem 4 we can choose ∇ in Theorem 6 to be the flat Weyl connection and then ∇ is ordinary partial differentiation.

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