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SCALAR PERTURBATIONS IN F(R) COSMOLOGIES IN THE LATE UNIVERSE

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Abstract. Standard approach in cosmology is hydrodynamical approach, when galaxies are smoothed distributions of matter. Then we model the Universe as a fluid. But we know, that the Universe has a discrete structure on scales 150 - 370 MPc. Therefore we must use the generalized mechanical approach, when is the mass concentrated in points. Methods of computations are then different. We focus on $f(R)$-theories of gravity and we work in the cell of uniformity in the late Universe. We do the scalar perturbations and we use 3 approximations. First we neglect the time derivatives and we do the astrophysical approach and we find the potentials Φ and Ψ in this case. Then we do the large scalaron mass approximation and we again obtain the potentials. Final step is the quasi-static approximation, when we use the equations from astrophysical approach and the result are the potentials Φ and Ψ. The resulting potentials are combination of Yukawa terms, which are characteristic for $f(R)$-theories, and standard potential.

1. Introduction

Modern observational phenomena, such as dark energy and dark matter, are the great challenge for present cosmology, astrophysics and theoretical physics. Within the scope of standard models, a satisfactory explanation to these problems has not been offered yet. There are two major ways, how to deal with accelerated expansion in our Universe, [14]. The first one is the standard lore based on Einstein theory of general relativity with a supplement of energy momentum tensor by an exotic component, dubbed dark energy, and the scenarios based upon large scale modifications of gravity.

A promising way, how to build a model with accelerated expansion epoch, are so called $f(R)$-theories of gravity. We obtain the Einstein equations by variation of Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2} \int R\sqrt{-g}\; d^4x \,, \tag{1}$$

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with respect to metric tensor, where R is the Ricci scalar, g is the determinant of metric tensor and $\kappa^2 = 8\pi G$. We take a function $f(R)$ instead of Ricci scalar R and we calculate the variation with respect to metric tensor $g_{\mu\nu}$. Then the resulted field equations are more complicated than in the case of general relativity:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \Box F g_{\mu\nu} - \nabla_\mu \nabla_\nu F = \kappa T_{\mu\nu}\,, \tag{2}$$

where $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M(\psi, g_{\mu\nu})}{\delta g^{\mu\nu}}$, S_M is the matter part of the action and $F = \frac{df}{dR}$.

We can also calculate the variation with respect to metric and independent connection. Such theories are called Palatini theories. Interesting feature is that we obtain the Levi-Civita connection like dynamical consequence of the theory and we do not need to consider it like an a priori assumption. However, we will concentrate on metric $f(R)$-theories.

We can take the trace of equation (2) and we obtain the following:

$$F(R)R - 2f(R) + 3\Box F = \kappa^2 T\,, \tag{3}$$

where $\kappa^2 = 8\pi G$ and $\Box F = 1/\sqrt{-g}\ \partial_\mu(\sqrt{-g} g^{\mu\nu}\partial_\nu F)$. We could see from this equation that there is an extra degree of freedom, so called scalaron. This equation plays a role of wave equation for this field F.

We will consider a special class of $f(R)$-functions, which have de Sitter point. This is a solution of

$$F(R)R - 2f(R) = 0\,. \tag{4}$$

We obtain this equation when $R =$ const. and $T = 0$. This is an algebraic equation for the de Sitter points. For example, for the case of Starobinsky model $f(R) = R^2$, there is infinitely many de Sitter points. We search for models with at least 2 de Sitter points, one is for hypothetical cosmological inflation and the second one is for accelerated expansion in the late stage of the evolution of the Universe. The, so called, Hu-Sawicky model has 2 de Sitter points, [9],

$$f(R) = R - \mu^2 \frac{c_1(\frac{R}{\mu^2})^k + c_3}{c_2(\frac{R}{\mu^2})^k + 1} \tag{5}$$

and MJWQ (Miranda-Joras-Waga-Quartin), [12],

$$f(R) = R - a\Big[\tanh\Big(b\frac{(R-R_0)}{2}\Big) + \tanh\Big(\frac{bR_0}{2}\Big)\Big] \tag{6}$$

also 2 de Sitter points. Both these 3 functions fulfill the local condition on the function $f(R)$, which is $F(R) > 0$, because gravity is attractive and $F'(R) > 0$, because scalaron is not a tachyon.

Because $f(R)$ is an analytic function of R, we could make the Taylor expansion around the de Sitter point and we obtain:

$$f(R) = f(R_{dS}) + f'(R_{dS})(R - R_{dS}) + o(R - R_{dS})\,. \tag{7}$$

We could rewrite it by equation (4), so we get

$$f(R) = -f(R_{dS}) + \frac{2f(R_{dS})}{R_{dS}} R + o(R - R_{dS}) \,. \tag{8}$$

Now, in order to have linear gravity in the late stage of Universe evolution, we can choose $2\frac{f(R_{dS})}{R_{dS}} = 1$ and we obtain

$$f(R) = R - 2\Lambda + o(R - R_{dS}) \,, \tag{9}$$

with $\Lambda = \frac{R_{dS}}{4}$. It is clear that these models go asymptotically to the de Sitter space, when $R \to R_{dS} \neq 0$. This is exactly the case for the late FLRW Universe, when the matter content becomes negligible with respect to cosmological constant.

We will now take the FLRW metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{10}$$

and we plug it to the generalized field equations (2). We use for the background tensor energy-momentum tensor

$$\bar{T}^{\mu}_{\nu} = (-\bar{\rho}, \bar{P}, \bar{P}, \bar{P}) \,. \tag{11}$$

As in the case of General Relativity, we obtain two equations:

$$\begin{aligned} 3FH^2 &= \frac{(FR - f)}{2} - 3H\dot{F} + \kappa^2 \bar{\rho} \,, \\ -2F\dot{H} &= \ddot{F} - H\dot{F} + \kappa^2(\bar{\rho} + \bar{P}) \,, \end{aligned} \tag{12}$$

where $H = \frac{\dot{a}}{a}$, the dot is a derivative with respect to synchronous time and $R = 6(2H^2 + \dot{H})$. When we add also the continuity equation

$$\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P}) = 0 \,, \tag{13}$$

we obtain a system of 3 equations, which are not independent. We can construct the second Friedmann equation from the first Friedmann equation and continuity equation. The procedure is similar to the case of General Relativity.

The solution to the (13), for non-relativistic matter, with $\bar{P} = 0$ is

$$\bar{\rho} = \frac{\bar{\rho}_c}{a^3} \,. \tag{14}$$

2. Mechanical approach

Above equations describe the homogeneous background. We consider the Universe in the late stage of its evolution when galaxies and cluster galaxies are already formed. We could describe it by hydrodynamical approach, but inside the cell of uniformity 150 - 370 Mpc it is highly inhomogeneous and the hydrodynamical approach is already not adequate. We must use the mechanical approach, [5] and [7].

In the framework of mechanical approach galaxies and clusters of galaxies (composed from baryonic and dark matter) can be considered like separate compact objects. They could be described by the following rest mass density, when we are far from their location:

$$\rho = \frac{1}{a^3} \sum_i m_i \delta(\vec{r} - \vec{r}_i) \equiv \frac{\rho_c}{a^3}\,, \tag{15}$$

where $\vec{r}_i$ is the radius vector of the considered mass in the comoving coordinates. This is the generalization of astrophysical approach to cosmological background. Usually are the peculiar velocities of the objects small and the produced fields are also small. The galaxies perturb the FLRW background and they produce the following metric in conformal-Newtonian gauge:

$$ds^2 = (-1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)(dx^2 + dy^2 + dz^2)\,, \tag{16}$$

where Φ, $\Psi << 1$.

It is important to note that smallness of peculiar velocities and the smallness of non-relativistic gravitational potentials Φ and Ψ are two independent conditions. We could have very light relativistic bodies and the produced potentials are still small. So we can work in two steps. First we neglect peculiar velocities and we define gravitational potentials Φ and Ψ. And then we use this potential for obtaining the dynamical behavior of the galaxies. This procedure is important because it enables us to take into consideration the gravitational attraction as well as the cosmological expansion of the Universe. This work is devoted to the first step of this program.

We now write the perturbed field equations for the case of scalar perturbations:

$$\begin{aligned} -\frac{\Delta\Psi}{a^2} + 3H\left(H\Phi + \dot{\Psi}\right) = {} & -\frac{1}{2F}\Big[\Big(3H^2 + 3\dot{H} + \frac{\Delta}{a^2}\Big)\delta F \\ & - 3H\dot{\delta F} + 3H\dot{F}\Phi + 3\dot{F}\left(H\Phi + \dot{\Psi}\right) + \kappa^2\delta\rho\Big]\,, \end{aligned} \tag{17}$$

$$H\Phi + \dot{\Psi} = \frac{1}{2F}\left(\dot{\delta F} - H\delta F - \dot{F}\Phi\right)\,, \tag{18}$$

$$-F(\Phi - \Psi) = \delta F\,, \tag{19}$$

$$\begin{aligned} & 3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Phi}) + 6H(H\Phi + \dot{\Psi}) + 3\dot{H}\Phi + \frac{\nabla\Phi}{a^2} \\ & \quad = \frac{1}{2F}[3\ddot{\delta F} + 3H\dot{\delta F} - -6H^2\delta F - \frac{\Delta\delta F}{a^2} - 3\dot{F}\dot{\Phi} \\ & \quad - 3\dot{F}(H\Phi + \dot{\Psi}) - (3H\dot{F} + 6\ddot{F})\Phi + \kappa^2\delta\rho]\,, \end{aligned} \tag{20}$$

$$\begin{aligned} \ddot{\delta F} + 3H\dot{\delta F} - \frac{\Delta\delta F}{a^2} - \frac{1}{3}R\delta F = {} & \frac{1}{3}\kappa^2(\delta\rho - 3\delta P) \\ & + \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) + 2\ddot{F}\Phi + 3H\dot{F}\Phi - \frac{1}{3}F\delta R\,, \end{aligned} \tag{21}$$

$$\delta R = -2\Big[3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) + 12H(H\Phi + \dot{\Psi}) + \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2}\Big]\,,$$

$$\delta F = F'\delta R\,. \tag{22}$$

F and F' are unperturbed quantities and we consider Laplacian Δ in comoving coordinates. As a source term we consider dust-like matter and therefore $\delta p = 0$ and

$$\delta\rho = \rho - \bar{\rho} = \frac{(\rho_c - \bar{\rho}_c)}{a^3}\,, \tag{23}$$

where ρ_c and ρ are defined in (14) and (15). And it is also clear that this system of equations is reduced to (2.18)–(2.20) in [5] for $F(R) = 1$.

3. Astrophysical approach

We will consider the equations (17)–(22) in, so called, astrophysical approach. We will neglect all time dependences in these equations and we put time derivatives to zero. The background model is also matter-free, $\bar{\rho} = 0$. There are 2 types of limits: de Sitter space with $R = 12H^2 = \text{const.} \neq 0$ and Minkowski space with $R = 0$ and $H = 0$. However the system of equations (17)–(22) was obtained for FLRW metric, where we had explicitly time dependent a. Therefore if we want to get time independent astrophysical equations from (17)–(22), we put $H = 0$. The background solution is then the Minkowski spacetime. And the background is perturbed by dust-like matter with $\bar{\rho} = 0$, which means $\delta\rho = \rho$.

In the case of Minkowski background and dropping the time derivatives, equations (17)–(22) in the astrophysical approach are reduced to the following system:

$$-\frac{\Delta}{a^2}\Psi = -\frac{1}{2F}\Big(\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\Big)\,, \tag{24}$$

$$-F(\Phi - \Psi) = \delta F\,, \tag{25}$$

$$\frac{\Delta}{a^2}\Phi = \frac{1}{2F}\Big(-\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\Big)\,, \tag{26}$$

$$-\frac{\Delta}{a^2}\delta F = \frac{1}{3}\kappa^2\delta\rho - \frac{1}{3}F\delta R\,, \tag{27}$$

$$\delta F = F'\delta R\,, \qquad \delta R = -2\Big(\frac{\Delta}{a^2}\Phi - 2\frac{\Delta}{a^2}\Psi\Big)\,, \tag{28}$$

From (24) and (26) we obtain respectively

$$\Psi = \frac{1}{2F}\delta F + \frac{\varphi}{a} = \frac{F'}{2F}\delta R + \frac{\varphi}{a}, \quad \Phi = -\frac{1}{2F}\delta F + \frac{\varphi}{a} = -\frac{F'}{2F}\delta R + \frac{\varphi}{a}\,, \tag{29}$$

where the function φ satisfies the equation

$$\Delta\varphi = \frac{1}{2F}\kappa^2 a^3\delta\rho = \frac{1}{2F}\kappa^2\delta\rho_c = 4\pi G_N\delta\rho_c\,, \qquad G_N = \frac{\kappa^2}{8\pi F}\,. \tag{30}$$

Here we took into consideration that in the astrophysical approach $\delta\rho_c = \rho_c$ where ρ_c is defined by (15). It is worth noting that in the Poisson equation the Newtonian gravitational constant G_N is replaced by an effective one $G_{eff} = G_N/F$.

Equation (25) follows directly from (29) and consequently, may be dropped, while from (27) we get the following Helmholtz equation with respect to the scalaron function δR:

$$\Delta\delta R + \frac{a^2}{3}\Big(R - \frac{F}{F'}\Big)\delta R = -\frac{a^2}{3F'}\kappa^2\delta\rho\,. \tag{31}$$

On the other hand, it can be easily seen that the substitution of equations (29) and (30) into (28) results in the same equation (31). Therefore, in the case of Minkowski background, the mass squared of the scalaron is

$$M^2 = \frac{a^2}{3}\frac{F}{F'}\,. \tag{32}$$

Now we want to take into consideration cosmological evolution. This means that the background functions may depend on time. In this case, it is hardly possible to solve the system directly. Therefore, first we study the case of very large mass of the scalaron. It should be noted also that we investigate the universe filled with nonrelativistic matter with the rest mass density $\rho \sim \frac{1}{a^3}$. Hence we will drop all terms which decrease (with increasing a) faster than $\frac{1}{a^3}$. This is the accuracy of our approach. Within this approach, $\delta\rho \sim \frac{1}{a^3}$, [5].

4. Large scalaron mass

As we can see from equation (15), the limit of large scalaron mass corresponds to $F^{'} \to 0$. Then δF is also negligible. Therefore, equations (17)–(22) read

$$-\frac{\Delta\Psi}{a^2} + 3H(H\Phi + \dot\Psi) = -\frac{1}{2F}\big[3H\dot F\Phi + 3\dot F(H\Phi + \dot\Psi)\big]\,, \tag{33}$$

$$H\Phi + \dot\Psi = \frac{1}{2F}(-\dot F\Phi)\,, \tag{34}$$

$$\Phi - \Psi = 0\,, \tag{35}$$

$$\begin{aligned} 3(\dot H\Phi + H\dot\Phi + \ddot\Psi) + 6H(H\Phi + \dot\Psi) + 3\dot H\Phi + \frac{\Delta\Phi}{a^2}& \\ = \frac{1}{2F}\big[-3\dot F\dot\Phi - 3\dot F(H\Phi + \dot\Psi) - (3H\dot F + 6\ddot F)\Phi\big]\,,& \end{aligned} \tag{36}$$

$$0 = \dot F(3H\Phi + 3\dot\Psi + \dot\Phi) + 2\ddot F\Phi + 3H\dot F\Phi\,, \tag{37}$$

$$0 = 3(\dot H\Phi + H\dot\Phi + \ddot\Psi) + 12H(H\Phi + \dot\Psi) + \frac{\Delta\Phi}{a^2} + 3\dot H\Phi - 2\frac{\Delta\Psi}{a^2}\,, \tag{38}$$

From (34) and (35) we get

$$\Psi = \Phi = \frac{\varphi}{a\sqrt{F}}\,, \tag{39}$$

where the introduced function φ depends only on spatial coordinates. Substituting (39) into (33), we obtain

$$\frac{1}{a^3\sqrt{F}}\Delta\varphi + \frac{3\dot{F}^2\varphi}{4aF^2\sqrt{F}} = \frac{1}{2F}\kappa^2\delta\rho\,. \tag{40}$$

As we mentioned above, neglecting relativistic matter in the late universe we have $\delta\rho \sim \frac{1}{a^3}$, ([5]). This approximation is getting better and better performed in the limit $a \to \infty$. We assume that this limit corresponds to the final stage of universe evolution. The similar limit with respect to the scalar curvature is $R \to R_\infty$, where the value R_∞ is just finite. Then from (40) we immediately come to the condition

$$F = \text{const.} + o(1)\,, \tag{41}$$

where $o(1)$ is decreasing function of a. This condition holds at the considered late stage. One can naively suppose that in the late universe $\dot{F} \approx \frac{1}{a} + o(\frac{1}{a})$. However this is wrong. Obviously, without loss of generality, we can suppose that const. = 1. From the condition (41) we get

$$F = 1 + o(1) \Rightarrow f = -2\Lambda + R + o(R - R_\infty)\,, \tag{42}$$

where Λ is the cosmological constant. Therefore the limit of the large scalaron mass takes place for models which possess the asymptotic form of (42). For example, R_∞ may correspond to the de Sitter point R_{dS} in future. All three popular models, Starobinsky, Hu-Sawicky and MJWQ, [15], [9], [12], have such stable de-Sitter points in the future (approximately at the redshift $z = -1$), ([11], [10]). The condition of stability is $0 < \frac{RF'}{F} < 1$. Since $F \approx 1$, this condition reads $0 < R < \frac{1}{F'}$, which is fulfilled for the de Sitter points in the above-mentioned models. The reason of it consists in the smallness of F'.

We now return to the remaining equations (36)–(38) to show that they are satisfied within the considered accuracy. First, we study (36) which after the substitution of (39) and (40) and some simple algebra takes the form

$$\frac{\varphi\dot{H}}{a} - \frac{\varphi}{2aF}(H\dot{F} - \ddot{F}) = 0\,. \tag{43}$$

To estimate $\dot{F}$ and $\ddot{F}$, we take into account that in the limit $R \to R_\infty$, $F \approx 1$, $H \approx$ const. $\Rightarrow \dot{H} \approx 0$, and $F'(R_\infty)$ is some finite positive value. Then,

$$\dot{F} = F'\dot{R} \approx F'(R_\infty)\dot{R} \approx \dot{T} \approx d(1/a^3)/dt \approx H(1/a^3) \approx 1/a^3$$

and $\ddot{F} \approx \dot{a}/a^4 \approx \frac{1}{a^3}$. Therefore, the LHS of equation (43) is of order $o(1/a^3)$ and we can put it zero within the accuracy of our approach. Similarly, equations (37) and (38) are satisfied within the considered accuracy. It can be also seen that the second term on the left hand side of equation (40) is of order $O(1/a^7)$ and should be eliminated. Thus, in the case of the large enough scalaron mass we reproduce the linear cosmology from the nonlinear one, as it should be.

5. Quasi-static approximation

Now we do not want to assume a priori that the scalaron mass is large, i.e. $F^{'}$ can have arbitrary values. Hence, we will preserve the δF terms in equations (17)–(22). Moreover, we should keep the time derivatives in these equations. Such a system is very complicated for direct integration. However, we can investigate it in the quasistatic approximation. According to this approximation, the spatial derivatives give the main contribution to equations (17)–(22), [16]. Therefore, first, we should solve "astrophysical" equations (24)–(28), and then check whether the found solutions satisfy (up to the adopted accuracy) the full system of equations. In the other words, in the quasi-static approximation it is naturally supposed that the gravitational potentials (the functions Φ, Ψ) are produced mainly by the spatial distribution of astrophysical/cosmological bodies. As we have seen, equations (24)–(28) result in (29)–(31). Now, we should keep in mind that we have the cosmological background. Moreover, we consider the late universe which is not far from the de Sitter point R_{dS} in future. This means that $\delta\rho = \rho - \bar{\rho}$ in (30), all background quantities are calculated roughly speaking at R_{dS} and the scalaron mass squared (15) reads now

$$M^2 = \frac{a^2}{3}\left(\frac{F}{F'} - R_{dS}\right). \tag{44}$$

Let us consider now equation (31) with the mass squared (44). Taking into account that now $\delta\rho_c = \rho_c - \bar{\rho}_c$, we can rewrite this equation as follows:

$$\Delta\widetilde{\delta R} - M^2\widetilde{\delta R} + \frac{a^2}{3F'}\frac{\kappa^2}{a^3}\sum_i m_i\delta(\vec{r} - \vec{r}_i) = 0\,, \tag{45}$$

where

$$\widetilde{\delta R} = \delta R + \frac{\kappa^2}{(F - F'R_{dS})a^3}\kappa^2\rho_c\,.$$

Then, the general solution for a full system is the sum over all gravitating masses. As a boundary conditions, we require for each gravitating mass the behavior $\delta R \sim \frac{1}{r}$ at small distances r and $\widetilde{R} \to 0$ for $r \to \infty$. Taking all these remarks into consideration, we obtain for the full system

$$\delta R = \frac{\kappa^2}{12\pi a F'}\sum_i \frac{m_i \exp(-M_i|\vec{r} - \vec{r}_i|)}{|\vec{r} - \vec{r}_i|} - \frac{\kappa^2\bar{\rho}_c}{(F - F'R_{dS})a^3}\,. \tag{46}$$

It is worth noting that averaging over the whole co-moving spatial volume V gives the zero value $\overline{\delta R}$. This result is reasonable because the rest mass density fluctuation $\delta\rho$, representing the source of the metric and the scalar curvature fluctuations Φ, Ψ and $\delta\rho$, has a zero average value $\overline{\delta\rho} = 0$. Consequently, all enumerated quantities should also have zero average values: $\bar{\Phi} = \bar{\Psi} = 0$ and $\overline{\delta R} = 0$, in agreement with (46).

From equation (29) we get the scalar perturbation functions Φ and Ψ in the following form:

$$\Psi = \frac{F'}{2F}\Big[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r}-\vec{r}_i|)}{|\vec{r}-\vec{r}_i|} - \frac{\kappa^2}{(F-F'R_{dS})a^3}\bar{\rho}_c\Big] + \frac{\varphi}{a}\,, \tag{47}$$

$$\Phi = \frac{-F'}{2F}\Big[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r}-\vec{r}_i|)}{|\vec{r}-\vec{r}_i|} - \frac{\kappa^2}{(F-F'R_{dS})a^3}\bar{\rho}_c\Big] + \frac{\varphi}{a}\,, \tag{48}$$

where φ satisfies equation (30) with $\delta\rho$ in the form (15) (i.e., $\bar{\rho}_c \neq 0$). Obviously when $F' \to 0$, $M \to \infty$, and we have $\exp(-M|\vec{r}-\vec{r}_i|)/|\vec{r}-\vec{r}_i| \to 4\pi\delta(\vec{r}-\vec{r}_i)/M^2$, so the expression in the square brackets in (47) and (48) is equal to $\kappa^2\delta\rho_c/[(F-F'R_{dS})a^3]$. Therefore, in the considered limit $F' \to 0$ we reproduce the scalar perturbations Φ, Ψ from the previous large scalaron mass case, as it certainly should be.

Thus neglecting for a moment the influence of the cosmological background, but not neglecting the scalaron's contribution, we have found the scalar perturbations. They represent the mix of the standard potential $\frac{\varphi}{a}$ (see the linear case [5]) and the additional Yukawa term which follows from the nonlinearity.

Now we should check that these solutions satisfy the full system (17)–(22). To do it, we substitute (46), (47) and (48) into this system of equations. Obviously the spatial derivatives disappear. Keeping in mind this fact the system (17)–(22) is reduced to the following equations:

$$\begin{aligned} 3H(H\Phi+\dot{\Psi}) = &-\frac{1}{2F}\Big[\Big(3H^2+3\dot{H}+\frac{\Delta}{a^2}\Big)\delta F - 3H\dot{\delta F} + 3H\dot{F}\Phi \\ &+ 3\dot{F}(H\Phi+\dot{\Psi})\Big]\,, \end{aligned} \tag{49}$$

$$H\Phi+\dot{\Psi} = \frac{1}{2F}(\dot{\delta F} - H\delta F - \dot{F}\Phi)\,, \tag{50}$$

$$\begin{aligned} 3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) &+ 6H(H\Phi+\dot{\Psi}) + 3\dot{H}\Phi + \frac{\Delta\Phi}{a^2} \\ &= \frac{1}{2F}\Big[3\ddot{\delta F} + 3H\dot{\delta F} - 6H^2\delta F - \frac{\Delta\delta F}{a^2} \\ &\quad - 3\dot{F}\dot{\Phi} - 3\dot{F}(H\Phi+\dot{\Psi}) - (3H\dot{F}+6\ddot{F})\Phi\Big]\,, \end{aligned} \tag{51}$$

$$\ddot{\delta F} + 3H\dot{\delta F} - \frac{\Delta\delta F}{a^2} = \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) + 2\ddot{F}\Phi + 3H\dot{F}\Phi\,, \tag{52}$$

$$\delta F = F'\delta R\,, \tag{53}$$

$$\begin{aligned} \frac{F'}{F}R_{dS}\delta R = &-2\Big[3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}) + 12H(H\Phi+\dot{\Psi}) \\ &+ \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2}\Big]\,. \end{aligned} \tag{54}$$

Here the term $\frac{F'}{F} R_{dS} \delta R$ in the left hand side of (54) disappear due to the redefinition of the scalaron mass squared (44).
It can be easily seen that all terms in (46), (47) and (48) depend on time, and therefore may contribute to equations (49)–(54). As we wrote above, according to our nonrelativistic approach, we neglect the terms of the order $o(1/a^3)$. On the other hand, exponential functions decrease faster than any power function. Moreover, we can write the exponential term in (46) as follows:

$$\frac{\kappa^2}{12\pi F'} \sum_i \frac{m_i \exp(-\sqrt{\frac{1}{3}(\frac{F}{F'} - R_{dS})} |r_{ph} - r_{ph,i}|)}{|r_{ph} - r_{ph,i}|}, \tag{55}$$

where we introduced the physical distance $r_{ph} = ar$. It is well known that astrophysical tests impose strong restrictions on the non-linearity [1, 13] (the local gravity tests impose even stronger constraints, [1, 13, 6]). According to these constraints, (55) should be small at the astrophysical scales. Consequently, on the cosmological scales it will be even much smaller. So we will not take into account the exponential terms in the above equations. However, in (46), (47) and (48), we have also $\frac{1}{a^3}$ and $\frac{1}{a}$ terms which we should examine. Before performing this, it should be recalled that we consider the late universe which is rather close to the de Sittter point. Therefore, as we already noted in the previous subsection, $F \approx 1$, $H \approx$ const. $\to \dot{H} \approx 0$, $R_{dS} = 12H^2$ and $F'(R_{dS})$ is some finite positive value. Additionally, $\dot{F}, \ddot{F}, \dot{F}' \approx \frac{1}{a^3}$. Hence, all terms of the form of $\dot{F}, \ddot{F}, \dot{F}' \times \Phi, \Psi, \dot{\Phi}, \dot{\Psi}$ are of the order $o(1/a^3)$ and should be dropped. In other words, the functions F and F' can be considered as time independent.
First, let us consider the terms $\Psi = \Phi = \varphi/a$ in equations (47) and (48) and substitute them into equations (49) - (54). Such $1/a$ term is absent in δR. So we should put $\delta R = 0$, $\delta F = 0$. Obviously, this is the linear theory case. It can be easily seen that all equations are satisfied.
Now, we study the terms $\sim 1/a^3$, i.e.,

$$\begin{aligned} \delta R &= -\frac{\kappa^2}{(F - F' R_{dS})} \frac{\bar{\rho}_c}{a^3} \Psi = -\frac{\kappa^2 F'}{2F(F - F' R_{dS})} \frac{\bar{\rho}_c}{a^3} \Phi \\ &= \frac{\kappa^2 F'}{2F(F - F' R_{dS})} \frac{\bar{\rho}_c}{a^3}. \end{aligned} \tag{56}$$

Let us examine, for example equation (49). Keeping in mind that $\delta F = F' \delta R$, one can easily get

$$12H_c^2 \frac{\kappa^2 F'}{2F(F - F' R_{dS})} \frac{\bar{\rho}_c}{a^3} = 12H_c^2 \frac{\kappa^2 F'}{2F(F - F' R_{dS})} \frac{\bar{\rho}_c}{a^3} + o(1/a^3). \tag{57}$$

Therefore, the terms $\sim \frac{1}{a^3}$ exactly cancel each other, and this equation is satisfied up to the adopted accuracy $o(1/a^3)$. One can easily show that the remaining equations are fulfilled with the same accuracy.

Thus we have proved that the scalar perturbation functions Φ and Ψ in the form (47) and (48) satisfy the system of equations (49)–(54) with the required accuracy.

Both of these functions contain the nonlinearity function F and the scale factor a. Therefore both the effects of nonlinearity and the dynamics of the cosmological background are taken into account. The function Φ corresponds to the gravitational potential of the system of inhomogeneities. Hence we can study the dynamical behavior of the inhomogeneities including into consideration their gravitational attraction and cosmological expansion, and also taking into account the effects of nonlinearity. For example, the non-relativistic Lagrange function for a test body of the mass m in the gravitational field described by the metric (16) has the form ([5]):

$$L \approx -m\Phi + \frac{ma^2\vec{v}^2}{2}, \ \vec{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \, .$$

We can use this Lagrange function for analytical and numerical study of mutual motion of galaxies. In the case of the linear theory, such investigation was performed, e.g., in [2].

6. Conclusion

We have been studying scalar perturbations of $f(R)$-theories in the cell of uniformity 150–370 Mpc. We have used three approximations: astrophysical approach, cosmological approach and quasistatic approximation. We obtained the scalar potentials Φ and Ψ in all three cases. Such potentials can be used to numerical simulation of movement of dwarf galaxies in these potentials.

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