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PROJECTIVE STRUCTURE, $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ AND THE SYMPLECTIC DIRAC OPERATOR

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Abstract. Inspired by the results on symmetries of the symplectic Dirac operator, we realize symplectic spinor fields and the symplectic Dirac operator in the framework of (the double cover of) homogeneous projective structure in two real dimensions. The symmetry group of the homogeneous model of the double cover of projective geometry in two real dimensions is $\widetilde{\mathrm{SL}}(3,\mathbb{R})$.

Introduction

There are symplectic counterparts of the notions of the spinor field and the Dirac operator on manifolds with metaplectic structure, see e.g. [6], [4]. The aim of the present short article is to describe a realization of symplectic spinor fields and the symplectic Dirac operator D_s in the framework of (the double covering of) the geometry of projective structure in the real dimension two. An inspiration for this development comes from the recent results on symmetries of the symplectic Dirac operator, [2].

We shall briefly comment on the content of our article. In Section 1 we start with a motivation for our work, namely the question of differential symmetries of the symplectic Dirac operator in real dimension two. Some information on this structure already appeared in [2], but its meaning and interpretation was not clear at that time. Here we find a natural explanation in terms of the projective structure associated to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ as an organizing principle for the symmetry algebra of the solution space of D_s. In Section 2 we briefly review the procedure called F-method, which is applied to produce the structure of singular vectors responsible for the realization of the symplectic Dirac operator. In Section 3 we introduce the homogeneous projective structure in the real dimension two and describe its basic geometrical and representation theoretical properties. The techniques of Section 2 are then applied in the last Section 4 to the simple metaplectic components of the Segal-Shale-Weil representation (twisted by a character of the central generator of the Levi factor) as an inducing representation for generalized Verma modules associated to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ and its maximal parabolic subalgebra. In

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this way we produce the symplectic Dirac operator as an $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-equivariant differential operator on the double covering of the real projective space $\mathbb{RP}^2$.

Let us highlight the meaning of the solution space for the symplectic Dirac operator D_s, regarded as an equivariant differential operator acting on the $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-principal series representation induced from the character twisted metaplectic representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The solution space of D_s on the symplectic space $\mathbb{R}^2$ was already determined in [3], as a consequence of the metaplectic Howe duality for the pair $(\mathfrak{mp}(2,\mathbb{R}),\mathfrak{sl}(2,\mathbb{R}))$. The underlying Harish-Chandra module of $\ker D_s$ for the Harish-Chandra pair $(\mathfrak{sl}(3,\mathbb{R}),\mathrm{SU}(2))$ with the maximal compact subgroup $\mathrm{SU}(2)\subset\widetilde{\mathrm{SL}}(3,\mathbb{R})$ is a unitarizable irreducible representation, equivalent to the exceptional representation of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ which is associated with the minimal coadjoint orbit, cf. [10], [12].

As for the notation used throughout the article, we consider the pair (G,P) consisting of a connected real reductive Lie group G and its parabolic subgroup P. In the Levi decomposition $P=LU$, L denotes the Levi subgroup and U the unipotent subgroup of P. We write $\mathfrak{g}(\mathbb{R})$, $\mathfrak{p}(\mathbb{R})$, $\mathfrak{l}(\mathbb{R})$, $\mathfrak{u}(\mathbb{R})$ for the real Lie algebras and $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{u}$ for the complexified Lie algebras of G, P, L, U, respectively. The symbol U applied to a Lie algebra denotes its universal enveloping algebra, and similarly $\widetilde{}$ applied to a Lie group denotes its double cover.

1. Symmetries of the symplectic Dirac operator

In the present section we start with a motivation for our further considerations. Let $(\mathbb{R}^2,\omega)$ be the 2-dimensional symplectic vector space with the canonical symplectic form $\omega=dx\wedge dy$, where (x,y) are the canonical linear coordinate functions on $\mathbb{R}^2$. We denote by ∂_x,∂_y the coordinate vector fields and by e_1,e_2 corresponding symplectic frame fields acting on a symplectic spinor $\varphi\in\mathbb{C}[\mathbb{R}^2]\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{R})$ by

$$e_1\cdot\varphi=\mathrm{i}q\varphi,\qquad e_2\cdot\varphi=\partial_q\varphi\,. \tag{1.1}$$

Here $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing complex functions on $\mathbb{R}$ equipped with the coordinate function q.

The basis elements $\{X,Y,H\}$ of the metaplectic Lie algebra $\mathfrak{mp}(2,\mathbb{R})$, the double cover of the symplectic Lie algebra $\mathfrak{sp}(2,\mathbb{R})\simeq\mathfrak{sl}(2,\mathbb{R})$, act on the function space $\mathbb{C}[\mathbb{R}^2]\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{R})$ by

$$X=-y\partial_x-\tfrac{\mathrm{i}}{2}q^2,\qquad H=-x\partial_x+y\partial_y+q\partial_q+\tfrac{1}{2},\qquad Y=-x\partial_y-\tfrac{\mathrm{i}}{2}\partial_q^2 \tag{1.2}$$

and satisfy the commutation relations

$$[H,X]=2X\,,\qquad [X,Y]=H\,,\qquad [H,Y]=-2Y\,. \tag{1.3}$$

The operators (1.2) preserve homogeneity in the variables x,y and imply the $\mathfrak{mp}(2,\mathbb{R})$-equivariance of

$$X_s=y\partial_q+\mathrm{i}xq\,,\qquad E=x\partial_x+y\partial_y+\tfrac{1}{2}\,,\qquad D_s=\mathrm{i}q\partial_y-\partial_x\partial_q\,. \tag{1.4}$$

Here the key commutation relation reads $[X_s,D_s]=\mathrm{i}(x\partial_x+y\partial_y+1)$, with D_s termed the symplectic Dirac operator on $\mathbb{R}^2$. We also notice here a different notation for the symbol E in the present article when compared to [2]. In [2], E is reserved for the differential operator $x\partial_x+y\partial_y$, while in our article we follow the conventions of parabolic invariant theory so that E denotes $x\partial_x + y\partial_y + \frac{1}{2}$ due to the presence of weights. The subtle difference between these two differential operators comes from their realization in two different algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$, respectively.

A differential operator P is called a symmetry of D_s provided there exists another differential operator P' such that $P'D_s = D_sP$. Consequently, the symmetry differential operators preserve the solution space of D_s. The vector space of first order (in all variables x, y, q) symmetries was described in [2], and there is a Lie algebra structure given by the commutator of two symmetry differential operators. The commutators of elements in (1.2) with the symmetry differential operators classified in [2] yield the following result, whose proof is a straightforward but tedious computation.

Lemma 1.1. The solution space of the symplectic Dirac operator on $\mathbb{R}^2$ is preserved by the following differential operators:

1) The couple of commuting differential operators

$$\mathcal{O}_1 = x^2\partial_x + xy\partial_y - \tfrac{1}{2}xq\partial_q + \tfrac{i}{2}y\partial_q^2 + \tfrac{1}{2}x = -\tfrac{1}{2}xH - yY + \tfrac{1}{2}xE + \tfrac{1}{2}x, \tag{1.5}$$

$$\mathcal{O}_2 = xy\partial_x + y^2\partial_y + \tfrac{1}{2}yq\partial_q + \tfrac{i}{2}xq^2 + y = \tfrac{1}{2}yH - xX + \tfrac{1}{2}yE + \tfrac{1}{2}y \tag{1.6}$$

satisfies

$$[D_s, \mathcal{O}_1] = \tfrac{3}{2}xD_s \qquad \text{and} \qquad [D_s, \mathcal{O}_2] = \tfrac{3}{2}yD_s, \tag{1.7}$$

which is equivalent to $D_s\mathcal{O}_1 = \big(\mathcal{O}_1 + \frac{3}{2}x\big)D_s$ and $D_s\mathcal{O}_2 = \big(\mathcal{O}_2 + \frac{3}{2}y\big)D_s$, respectively. The operators $\mathcal{O}_1, \mathcal{O}_2$ increase the homogeneity in the variables x, y by one.

2) The couple of commuting differential operators

$$\partial_x \quad \text{and} \quad \partial_y \tag{1.8}$$

commutes with D_s and decreases the homogeneity in the variables x, y by one.

The operators $\mathcal{O}_1, \mathcal{O}_2$ turn out to be useful in the light of the following observation, whose proof is again straightforward and left to the reader.

Theorem 1.2. The first order symmetry differential operators

$$\{\partial_x, \partial_y, H, X, Y, E, \mathcal{O}_1, \mathcal{O}_2\} \tag{1.9}$$

of the symplectic Dirac operator D_s fulfill the following non-trivial commutation relations

$$
\begin{aligned}
&[\partial_x, \mathcal{O}_2] = -X, && [\partial_y, \mathcal{O}_1] = -Y,\\
&[\partial_y, \mathcal{O}_2] = \tfrac{1}{2}(3E+H), && [\partial_x, \mathcal{O}_1] = \tfrac{1}{2}(3E-H),\\
&[\mathcal{O}_2, H] = -\mathcal{O}_2, && [\mathcal{O}_1, H] = \mathcal{O}_1,\\
&[\mathcal{O}_2, E] = -\mathcal{O}_2, && [\mathcal{O}_1, E] = -\mathcal{O}_1,\\
&[\mathcal{O}_2, Y] = \mathcal{O}_1, && [\mathcal{O}_1, X] = \mathcal{O}_2,\\
&[\partial_x, H] = -\partial_x, && [\partial_y, H] = \partial_y,\\
&[\partial_x, Y] = -\partial_y, && [\partial_y, X] = -\partial_x,\\
&[\partial_x, E] = \partial_x, && [\partial_y, E] = \partial_y,\\
&[H, X] = 2X, && [H, Y] = -2Y,\\
&[X, Y] = H. &&
\end{aligned}
\tag{1.10}
$$

The homomorphism of Lie algebras

$$\langle \partial_x, \partial_y, H, X, Y, E, \mathcal{O}_1, \mathcal{O}_2 \rangle \to \mathfrak{sl}(3, \mathbb{R}) \tag{1.11}$$

given by

$$
\begin{gathered}
-\partial_x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad -\partial_y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\\
\mathcal{O}_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{O}_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
X \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\\
E \mapsto \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}
\end{gathered}
\tag{1.12}
$$

is an isomorphism of Lie algebras.

As will be proved in [11] by the techniques of tractor calculus, the operators (1.9) in the variables x, y, q are in fact all first order symmetry differential operators of D_s in the base variables x, y.

In the remaining part of our article we interpret the results of the present section in the framework of (the double cover of) the projective structure in real dimension 2.

2. Generalized Verma modules and singular vectors

It is well-known that the G-equivariant differential operators acting on principal series representations for G can be recognized in the study of homomorphisms between generalized Verma modules for the Lie algebra $\mathfrak{g}$. The latter homomorphisms are determined by the image of the highest weight vectors, referred to as the singular vectors and characterized as being annihilated by the positive nilradical $\mathfrak{u}$.

An approach to find precise positions of singular vectors in the representation space can be found in [5], [8], [9]. Let $\mathbb{V}$ denote a complex simple highest weight L-module, extended to P-module by U acting trivially. We denote by $\mathbb{V}^*$ the (restricted) dual P-module to $\mathbb{V}$. Any character $\lambda \in \mathrm{Hom}_P(\mathfrak{p},\mathbb{C})$ yields a 1-dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{p}$ by

$$Xv = \lambda(X)v, \quad X \in \mathfrak{p},\, v \in \mathbb{C}\,. \tag{2.1}$$

Moreover, assuming that $\lambda \in \mathrm{Hom}_P(\mathfrak{p},\mathbb{C})$ defines a group character $e^\lambda\colon P \to \mathrm{GL}(1,\mathbb{C})$ of P and denoting by $\rho \in \mathrm{Hom}_P(\mathfrak{p},\mathbb{C})$ the character

$$\rho(X) = \tfrac{1}{2}\,\mathrm{tr}_{\mathfrak{u}}\,\mathrm{ad}(X), \quad X \in \mathfrak{p}\,, \tag{2.2}$$

we may introduce a twisted P-module $\mathbb{V}_{\lambda+\rho} \simeq \mathbb{V} \otimes_{\mathbb{C}} \mathbb{C}_{\lambda+\rho}$ (with a twist $\lambda+\rho$) where $p \in P$ acts on $v \in \mathbb{V}_{\lambda+\rho} \simeq \mathbb{V}$ (the isomorphism of vector spaces) by $e^{\lambda+\rho}(p)p.v$. In the rest of our article, $\mathbb{V} = \mathbb{S}$ is one of the simple metaplectic submodules of the Segal-Shale-Weil representation twisted by character of $\mathrm{GL}(1,\mathbb{R})_+$.

By abuse of notation, we call the highest weight modules induced from infinite-dimensional simple L-modules generalized Verma modules, though they are not the objects of the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ because of the lack of L-finiteness condition. However, most of structural results required in the present article carry over to this class of modules, cf. [9].

In general, for a chosen principal series representation of G on the vector space $\mathrm{Ind}_P^G(\mathbb{V}_{\lambda+\rho})$ of smooth sections of the homogeneous vector bundle $G \times_P \mathbb{V}_{\lambda+\rho} \to G/P$ associated to a P-module $\mathbb{V}_{\lambda+\rho}$, we compute the infinitesimal action

$$\pi_\lambda\colon \mathfrak{g} \to \mathcal{D}(U_e) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{V}_{\lambda+\rho}. \tag{2.3}$$

Here $\mathcal{D}(U_e)$ denotes the $\mathbb{C}$-algebra of smooth complex linear differential operators on $U_e = \overline{U}P \subset G/P$ ($\overline{U}$ is the Lie group whose Lie algebra is the opposite nilradical $\overline{\mathfrak{u}}(\mathbb{R})$ to $\mathfrak{u}(\mathbb{R})$), on the vector space $\mathcal{C}^\infty(U_e)\otimes_{\mathbb{C}}\mathbb{V}_{\lambda+\rho}$ of $\mathbb{V}_{\lambda+\rho}$-valued smooth functions on U_e in the non-compact picture of the induced representation.

The dual vector space $\mathcal{D}'_o(U_e) \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho}$ of $\mathbb{V}_{\lambda+\rho}$-valued distributions on U_e supported on the unit coset $o = eP \in G/P$ is $\mathcal{D}(U_e) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{V}_{\lambda+\rho}$-module, and there is an $U(\mathfrak{g})$-module isomorphism

$$\Phi_\lambda\colon M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{V}_{\lambda-\rho}) \equiv U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{V}_{\lambda-\rho} \to \mathcal{D}'_o(U_e) \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho} \simeq \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}/I_e \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho}\,. \tag{2.4}$$

The exponential map allows to identify U_e with the nilpotent Lie algebra $\overline{\mathfrak{u}}(\mathbb{R})$. If we denote by $\mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}$ the Weyl algebra of the complex vector space $\overline{\mathfrak{u}}$, then the vector space $\mathcal{D}'_o(U_e)$ can be identified as an $\mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}$-module with the quotient of $\mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}$ by the left ideal I_e generated by all polynomials on $\overline{\mathfrak{u}}$ vanishing at the origin.

Let $(x_1, x_2, \dots, x_n)$ be the linear coordinate functions on $\overline{\mathfrak{u}}$ and $(y_1, y_2, \dots, y_n)$ be the dual linear coordinate functions on $\overline{\mathfrak{u}}^*$. Then the algebraic Fourier transform

$$\mathcal{F}\colon \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}} \to \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}^*} \tag{2.5}$$

is given by

$$\mathcal{F}(x_i) = -\partial_{y_i}, \qquad \mathcal{F}(\partial_{x_i}) = y_i \tag{2.6}$$

for $i = 1, 2, \dots, n$, and leads to a vector space isomorphism

$$\begin{aligned} \tau\colon \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}/I_e \simeq \mathbb{C}[\overline{\mathfrak{u}}^*] &\xrightarrow{\sim} \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}^*}/\mathcal{F}(I_e) \simeq \mathbb{C}[\overline{\mathfrak{u}}^*], \\ Q \bmod I_e &\mapsto \mathcal{F}(Q) \bmod \mathcal{F}(I_e) \end{aligned} \tag{2.7}$$

for $Q \in \mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}$. The composition of (2.4) and (2.7) gives the vector space isomorphism

$$\tau \circ \Phi_\lambda\colon U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{V}_{\lambda-\rho} \xrightarrow{\sim} \mathcal{D}'_o(U_e) \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho} \xrightarrow{\sim} \mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho}\,, \tag{2.8}$$

thereby inducing the $\mathfrak{g}$-module action $\hat{\pi}_\lambda$ on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho}$.

Definition 2.1. Let $\mathbb{V}$ be a complex simple highest weight L-module, extended to a P-module by U acting trivially. We define the L-module

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V})^{\mathfrak{u}} = \{v \in M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V});\, Xv = 0 \quad \text{for all} \quad X \in \mathfrak{u}\}, \tag{2.9}$$

which we call the vector space of singular vectors.

The vector space of singular vectors is for an infinite-dimensional complex simple highest weight P-module $\mathbb{V}$ an infinite-dimensional L-module. In the case when $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V})^{\mathfrak{u}}$ is a completely reducible L-module, we denote by $\mathbb{W}$ one of its simple L-submodule. Then we obtain the $U(\mathfrak{g})$-module homomorphism from $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{W})$ to $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V})$, and moreover we have

$$\mathrm{Hom}_{(\mathfrak{g},P)}(M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{W}), M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V})) \simeq \mathrm{Hom}_L(\mathbb{W}, M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V})^{\mathfrak{u}})\,. \tag{2.10}$$

We introduce the L-module

$$\begin{aligned} &\mathrm{Sol}(\mathfrak{g},\mathfrak{p};\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho})^{\mathcal{F}} = \\ &\qquad \{f \in \mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho};\, \hat{\pi}_\lambda(X)f = 0 \text{ for all } X \in \mathfrak{u}\}, \end{aligned} \tag{2.11}$$

and by (2.8), there is an L-equivariant isomorphism

$$\tau \circ \Phi_\lambda\colon M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}_{\lambda-\rho})^{\mathfrak{u}} \xrightarrow{\sim} \mathrm{Sol}(\mathfrak{g},\mathfrak{p};\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho})^{\mathcal{F}}\,. \tag{2.12}$$

The action of $\hat{\pi}_\lambda(X)$ on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho}$ produces a system of partial differential equations for the elements in $\mathrm{Sol}(\mathfrak{g},\mathfrak{p};\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda+\rho})^{\mathcal{F}}$, which makes it possible to describe its structure completely in particular cases of interest as the solution space of the systems of partial differential equations.

The formulation above has the following classical dual statement (cf. [1] for the standard formulation in the category of finite dimensional inducing P-modules, or [9] for its extension to inducing modules with infinitesimal character), which explains the relationship between the geometrical problem of finding G-equivariant differential operators between induced representations and the algebraic problem of finding homomorphisms between generalized Verma modules. Let $\mathbb{V}$ and $\mathbb{W}$ be two simple highest weight P-modules. Then the vector space of G-equivariant differential operators $\mathrm{Hom}_{\mathrm{Diff}(G)}(\mathrm{Ind}^G_P(\mathbb{V}), \mathrm{Ind}^G_P(\mathbb{W}))$ is isomorphic to the vector space of $(\mathfrak{g},P)$-homomorphisms $\mathrm{Hom}_{(\mathfrak{g},P)}(M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{W}^*), M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}^*))$.

3. The homogeneous projective structure in dimension 2

A projective structure on a smooth manifold M of real dimension $n \geq 2$ is a class $[\nabla]$ of projectively equivalent torsion-free connections, which define the same family of unparametrized geodesics. A connection is projectively flat if and only if it is locally equivalent to a flat connection. Given any non-vanishing volume form ω on M, there exists a unique connection in the projective class such that $\nabla\omega = 0$. In the case $n > 2$ ($n = 2$), the vanishing of the Weyl curvature tensor (the Cotton curvature tensor) for ∇ is equivalent to the projective flatness of $[\nabla]$ and consequently, the existence of a local isomorphism with the flat model of n-dimensional projective geometry on $\mathbb{RP}^n$ equipped with the flat projective structure given by the absolute parallelism.

The homogeneous (flat) model of projective geometry in the real dimension 2 is $\mathbb{RP}^2 \simeq G/P$, where G is the connected simple real Lie group $\mathrm{SL}(3,\mathbb{R})$ and $P \subset G$ the parabolic subgroup stabilizing the line $[v] \in \mathbb{R}^3$ generated by a non-zero vector v in the defining representation $\mathbb{R}^3$ of G. Although our construction of equivariant differential operators is local, the passage to the generalized flag manifold and sections of associated vector bundles induced from half integral modules (e.g., the simple metaplectic submodules of the Segal-Shale-Weil representation) on G/P requires the double (universal) cover $\widetilde{G} = \widetilde{\mathrm{SL}}(3,\mathbb{R})$ and its parabolic subgroup $\widetilde{P}$. The Lie group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ acts transitively on $S^2 \simeq \mathbb{CP}^1$, the double (universal) cover of $\mathbb{RP}^2$, with parabolic stabilizer $\widetilde{P} = (\mathrm{GL}(1,\mathbb{R})_+ \times \widetilde{\mathrm{SL}}(2,\mathbb{R})) \ltimes \mathbb{R}^2$. We notice that the double (universal) cover $\widetilde{\mathrm{SL}}(3,\mathbb{R})/\widetilde{P} \simeq S^2 \simeq \mathbb{CP}^1$ is a symplectic manifold, while $\mathbb{RP}^2$ is non-orientable and hence not symplectic.

The questions discussed in our article can be treated by infinitesimal methods, and so we introduce the complexified Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ of G and the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ by

$$\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, a_3);\ a_1 + a_2 + a_3 = 0,\ a_1, a_2, a_3 \in \mathbb{C}\}. \tag{3.1}$$

For $i = 1, 2, 3$, we define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\mathrm{diag}(a_1, a_2, a_3)) = a_i$. The root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is $\Delta = \{\pm(\varepsilon_i - \varepsilon_j);\ 1 \leq i < j \leq 3\}$, the positive root system is $\Delta^+ = \{\varepsilon_i - \varepsilon_j;\ 1 \leq i < j \leq 3\}$ with the subset of simple roots $\Pi = \{\alpha_1, \alpha_2\}$, $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, and the fundamental weights are $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$. The subset $\Sigma = \{\alpha_2\}$ of Π generates a root subsystem $\Delta_\Sigma \subset \mathfrak{h}^*$, and we associate to Σ the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. The reductive Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha, \tag{3.2}$$

and the nilradical $\mathfrak{u}$ of $\mathfrak{p}$ and the opposite nilradical $\overline{\mathfrak{u}}$ are

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \smallsetminus \Delta^+_\Sigma} \mathfrak{g}_\alpha, \qquad \overline{\mathfrak{u}} = \bigoplus_{\alpha \in \Delta^+ \smallsetminus \Delta^+_\Sigma} \mathfrak{g}_{-\alpha}\,, \tag{3.3}$$

respectively. The Σ-height $\mathrm{ht}_\Sigma(\alpha)$ of a root $\alpha \in \Delta$ is defined by

$$\mathrm{ht}_\Sigma(a_1\alpha_1 + a_2\alpha_2) = a_1\,, \tag{3.4}$$

and $\mathfrak{g}$ is a $|1|$-graded Lie algebra with respect to the grading given by $\mathfrak{g}_i = \bigoplus_{\alpha\in\Delta,\,\mathrm{ht}_\Sigma(\alpha)=i}\mathfrak{g}_\alpha$ for $0 \neq i \in \mathbb{Z}$, and $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta,\,\mathrm{ht}_\Sigma(\alpha)=0}\mathfrak{g}_\alpha$. In particular, we have $\mathfrak{u} = \mathfrak{g}_1 \simeq \mathbb{C}^2$, $\overline{\mathfrak{u}} = \mathfrak{g}_{-1} \simeq \mathbb{C}^2$ and $\mathfrak{l} = \mathfrak{g}_0 \simeq \mathbb{C} \oplus \mathfrak{sl}(2,\mathbb{C})$.

The basis $\{e_1, e_2\}$ of the root spaces in the nilradical $\mathfrak{u}$ is given by

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

the basis $\{f_1, f_2\}$ of the root spaces in the opposite nilradical $\overline{\mathfrak{u}}$ is

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

and finally the basis $\{h_0, h, e, f\}$ of the Levi subalgebra $\mathfrak{l}$ is

$$\begin{gathered} e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ h_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \end{gathered} \tag{3.7}$$

where h_0 generates a basis of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$.

Any character $\chi \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ is given by

$$\chi = \lambda\widetilde{\omega}_1, \quad \lambda \in \mathbb{C}, \tag{3.8}$$

where $\widetilde{\omega}_1 \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ is equal to $\omega_1 \in \mathfrak{h}^*$ regarded as trivially extended to $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{u}$. Throughout the article we use the simplified notation λ for $\lambda\widetilde{\omega}_1$. The vector $\rho \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ defined by the formula (2.2) is then

$$\rho = \tfrac{3}{2}\,\widetilde{\omega}_1. \tag{3.9}$$

4. $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ and the symplectic Dirac operator

In this section we retain the notation in Section 3 and describe the class of representations of $\mathfrak{g}$ on the space of sections of vector bundles on $\widetilde{G}/\widetilde{P}$ associated to the simple metaplectic submodules of the Segal-Shale-Weil representation $\mathbb{S}_{\lambda+\rho}$ of $\widetilde{P}$ twisted by characters $\lambda + \rho \in \mathrm{Hom}_{\widetilde{P}}(\mathfrak{p}, \mathbb{C})$.

The induced representations in question are described in the non-compact picture, given by restricting sections to the open Schubert cell $U_e \subset \widetilde{G}/\widetilde{P}$ which is isomorphic by the exponential map to the opposite nilradical $\overline{\mathfrak{u}}(\mathbb{R})$. We denote by $(\hat{x}, \hat{y})$ the linear coordinate functions on $\overline{\mathfrak{u}}(\mathbb{R})$ with respect to the basis $\{f_1, f_2\}$ of $\overline{\mathfrak{u}}(\mathbb{R})$, and by (x, y) the dual linear coordinate functions on $\overline{\mathfrak{u}}^*(\mathbb{R})$. The Weyl algebra $\mathcal{A}^{\mathfrak{g}}_{\overline{\mathfrak{u}}}$ is generated by

$$\{\hat{x}, \hat{y}, \partial_{\hat{x}}, \partial_{\hat{y}}\} \tag{4.1}$$

and the Weyl algebra $\mathcal{A}^{\mathfrak{g}}_{\mathfrak{u}^*}$ by

$$\{x, y, \partial_x, \partial_y\}. \tag{4.2}$$

For a $\mathfrak{p}$-module $(\sigma, \mathbb{V})$, $\sigma\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{V})$, the twisted $\mathfrak{p}$-module $(\sigma_\lambda, \mathbb{V}_\lambda)$, $\sigma_\lambda\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{V}_\lambda)$, with a twist $\lambda \in \mathrm{Hom}_{\widetilde{P}}(\mathfrak{p}, \mathbb{C})$ is defined by

$$\sigma_\lambda(X)v = \sigma(X)v + \lambda(X)v \tag{4.3}$$

for all $X \in \mathfrak{p}$ and $v \in \mathbb{V}_\lambda \simeq \mathbb{V}$ (the isomorphism of vector spaces).

We use the following realization of the simple $\mathfrak{mp}(2,\mathbb{C})$-submodules of the Segal-Shale-Weil representation. The Fock model is the unitarizable $\mathfrak{mp}(2,\mathbb{C})$-module $\mathbb{S} = \mathbb{C}[q]$. Since the simple part of the Levi algebra $\mathfrak{l}$ is $\mathfrak{l}^{\mathrm{s}} \simeq \mathfrak{mp}(2,\mathbb{C})$, we realize the simple metaplectic submodules of the Segal-Shale-Weil representation as the representations of $\mathfrak{l}^{\mathrm{s}}$ on the subspace of polynomials of even and odd degree, respectively. The generators act as

$$\sigma(e) = \tfrac{\mathrm{i}}{2}\partial_q^2, \qquad \sigma(h) = -q\partial_q - \tfrac{1}{2}, \qquad \sigma(f) = \tfrac{\mathrm{i}}{2}q^2. \tag{4.4}$$

The scalar product $\langle \cdot\,, \cdot \rangle\colon \mathbb{S} \otimes_{\mathbb{C}} \mathbb{S} \to \mathbb{C}$ on $\mathbb{S}$ is defined through the $\mathfrak{l}^{\mathrm{s}}$-equivariant embedding into the space of Schwartz functions $\iota\colon \mathbb{S} \to \mathcal{S}(\mathbb{R})$,

$$\langle p_1, p_2 \rangle = \int_{\mathbb{R}} \iota(\overline{p}_1)\iota(p_2)\,\mathrm{d}q \quad \text{for all} \quad p_1, p_2 \in \mathbb{S}. \tag{4.5}$$

The representation of $\mathfrak{l}^{\mathrm{s}}$ is then extended to a representation of $\mathfrak{p}$ by the trivial action of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ and by the trivial action of the nilradical $\mathfrak{u}$ of $\mathfrak{p}$. We retain the same notation $\sigma\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{S})$ for the extended action of the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. In what follows, we are interested in the twisted $\mathfrak{p}$-module $\sigma_\lambda\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{S}_\lambda)$ with a twist $\lambda \in \mathrm{Hom}_{\widetilde{P}}(\mathfrak{p}, \mathbb{C})$.

Theorem 4.1. Let $\lambda \in \mathrm{Hom}_{\widetilde{P}}(\mathfrak{p}, \mathbb{C})$. Then the embedding of $\mathfrak{g}$ into $\mathcal{A}^{\mathfrak{g}}_{\mathfrak{u}} \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{S}_{\lambda+\rho}$ and $\mathcal{A}^{\mathfrak{g}}_{\mathfrak{u}^*} \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{S}_{\lambda+\rho}$ is given by

1)

$$\pi_\lambda(f_1) = -\partial_{\hat{x}}, \qquad \pi_\lambda(f_2) = -\partial_{\hat{y}}, \tag{4.6}$$

$$\hat{\pi}_\lambda(f_1) = -x, \qquad \hat{\pi}_\lambda(f_2) = -y; \tag{4.7}$$

2)

$$\begin{gathered}\pi_\lambda(e) = -\hat{y}\partial_{\hat{x}} + \tfrac{\mathrm{i}}{2}\partial_q^2\,, \quad \pi_\lambda(h) = -\hat{x}\partial_{\hat{x}} + \hat{y}\partial_{\hat{y}} - q\partial_q - \tfrac{1}{2}\,, \quad \pi_\lambda(f) = -\hat{x}\partial_{\hat{y}} + \tfrac{\mathrm{i}}{2}q^2\,, \\ \pi_\lambda(h_0) = \tfrac{3}{2}(\hat{x}\partial_{\hat{x}} + \hat{y}\partial_{\hat{y}}) + \lambda + \tfrac{3}{2}\,,\end{gathered} \tag{4.8}$$

$$\begin{gathered}\hat{\pi}_\lambda(e) = x\partial_y + \tfrac{\mathrm{i}}{2}\partial_q^2\,, \quad \hat{\pi}_\lambda(h) = x\partial_x - y\partial_y - q\partial_q - \tfrac{1}{2}\,, \quad \hat{\pi}_\lambda(f) = y\partial_x + \tfrac{\mathrm{i}}{2}q^2\,, \\ \hat{\pi}_\lambda(h_0) = -\tfrac{3}{2}(x\partial_x + y\partial_y) + \lambda - \tfrac{3}{2};\end{gathered} \tag{4.9}$$

3)

$$\begin{aligned}\pi_\lambda(e_1) &= \hat{x}(\hat{x}\partial_{\hat{x}} + \hat{y}\partial_{\hat{y}} + \lambda + \tfrac{3}{2}) + \tfrac{1}{2}\hat{x}\big(q\partial_q + \tfrac{1}{2}\big) - \tfrac{\mathrm{i}}{2}\hat{y}q^2, \\ \pi_\lambda(e_2) &= \hat{y}(\hat{x}\partial_{\hat{x}} + \hat{y}\partial_{\hat{y}} + \lambda + \tfrac{3}{2}) - \tfrac{1}{2}\hat{y}\big(q\partial_q + \tfrac{1}{2}\big) - \tfrac{\mathrm{i}}{2}\hat{x}\partial_q^2,\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
\hat{\pi}_\lambda(e_1) &= \partial_x\big(x\partial_x + y\partial_y - \lambda + \tfrac{1}{4}\big) + \tfrac{1}{2}q(\mathrm{i}q\partial_y - \partial_x\partial_q),\\
\hat{\pi}_\lambda(e_2) &= \partial_y\big(x\partial_x + y\partial_y - \lambda + \tfrac{1}{4}\big) - \tfrac{\mathrm{i}}{2}\partial_q(\mathrm{i}q\partial_y - \partial_x\partial_q).
\end{aligned}
\tag{4.11}
$$

Proof. The proof of this claim for the trivial representation of $\mathfrak{p}$ instead of the simple metaplectic submodules of the Segal-Shale-Weil representation follows from Theorem 1.3 in [8]. For the Segal-Shale-Weil representation it follows by a straightforward verification of all commutation relations for the Lie algebra $\mathfrak{g}$.

Theorem 4.2. The space $\mathrm{Sol}(\mathfrak{g},\mathfrak{p};\mathbb{C}[\overline{\mathfrak{u}}^*]\otimes_{\mathbb{C}}\mathbb{S}_{\lambda+\rho})^{\mathcal{F}}$ is for $\lambda = \frac{3}{4}\widetilde{\omega}_1$ non-trivial and contains the L-submodule $X_s\mathbb{S}_{\lambda+\rho}$.

Proof. By Theorem (4.1), the $\mathfrak{sl}(2,\mathbb{C})$-algebra structure for the operators (1.4) implies

$$
\begin{aligned}
\hat{\pi}_\lambda(e_1)X_sv_0 &= \partial_x\big(1-\lambda+\tfrac{1}{4}\big)X_sv_0 + \tfrac{1}{2}q[D_s,X_s]v_0 = \big(\tfrac{5}{4}-\lambda\big)[\partial_x,X_s]v_0 - \tfrac{\mathrm{i}}{2}qv_0\\
&= \big(\tfrac{5}{4}-\lambda-\tfrac{1}{2}\big)\mathrm{i}qv_0 = 0\,,\\
\hat{\pi}_\lambda(e_2)X_sv_0 &= \partial_y\big(1-\lambda+\tfrac{1}{4}\big)X_sv_0 - \tfrac{1}{2}\mathrm{i}\partial_q[D_s,X_s]v_0 = \big(\tfrac{5}{4}-\lambda\big)[\partial_y,X_s]v_0 - \tfrac{1}{2}\partial_qv_0\\
&= \big(\tfrac{5}{4}-\lambda-\tfrac{1}{2}\big)\partial_qv_0 = 0
\end{aligned}
$$

for all $v_0\in\mathbb{S}_{\lambda+\rho}$, provided $\lambda=\frac{3}{4}$. The proof is complete.

Remark. We notice that the vectors X_s^kv for $k>1$ are not singular vectors. For example, a straightforward computation reveals

$$
\hat{\pi}_\lambda(e_1)X_s^kv_0 = \big(k-\lambda+\tfrac{1}{4}\big)\big(\mathrm{i}kqX_s^{k-1} + \mathrm{i}\tfrac{k(k-1)}{2}yX_s^{k-2}\big)v - \mathrm{i}\tfrac{k(k-1)}{4}qX_s^{k-1}v,
\tag{4.12}
$$

which is non-zero for all $\lambda\in\mathbb{C}$.

It is not difficult to exploit the results in [3] in order to classify the complete set of solutions of the system in Theorem 4.1, but the detailed analysis goes beyond the scope of our article.

Theorem 4.2 has the following classical corollary, which explains the relationship between the geometrical problem of finding G-equivariant differential operators between induced representations and the algebraic problem of finding homomorphisms between generalized Verma modules, cf. [1], [9]. There is a double cover $\widetilde{P} = (\mathrm{GL}(1,\mathbb{R})_+\times\widetilde{\mathrm{SL}}(2,\mathbb{R}))\ltimes\mathbb{R}^2$ of the maximal parabolic subgroup P of the Lie group $\mathrm{SL}(3,\mathbb{R})$, which splits over the unipotent subgroup $N\simeq\mathbb{R}^2$ in the Langlands-Iwasawa decomposition of $\widetilde{P}$, [13]. Let us note that the extension cocycle splits over the field of complex numbers.

Theorem 4.3. Let $\widetilde{G} = \widetilde{\mathrm{SL}}(3,\mathbb{R})$ and let $\widetilde{P} = (\mathrm{GL}(1,\mathbb{R})_+\times\widetilde{\mathrm{SL}}(2,\mathbb{R}))\ltimes\mathbb{R}^2$ be the maximal parabolic subgroup of $\widetilde{G}$, whose unipotent subgroup in the Langlands-Iwasawa decomposition of $\widetilde{P}$ is $N\simeq\mathbb{R}^2$. For $\mathbb{V}=\mathbb{S}_\lambda$ we have $\mathbb{V}^*\simeq\mathbb{S}^*_{-\lambda}$. Then the singular vector constructed in Theorem 4.2 corresponds to the $\widetilde{G}$-equivariant differential operator, given in the non-compact picture of the induced representations by

$$
\begin{aligned}
D_s\colon \mathcal{C}^\infty(\overline{\mathfrak{u}}(\mathbb{R}),\mathbb{S}^*_{\frac{3}{4}\widetilde{\omega}_1}) &\to \mathcal{C}^\infty(\overline{\mathfrak{u}}(\mathbb{R}),\mathbb{S}^*_{\frac{9}{4}\widetilde{\omega}_1})\,,\\
\varphi &\mapsto (\mathrm{i}q\partial_{\hat{y}} - \partial_{\hat{x}}\partial_q)\varphi\,.
\end{aligned}
\tag{4.13}
$$

The infinitesimal intertwining property of D_s is

$$D_s\pi^*_{-\frac{3}{4}}(X) = \pi^*_{\frac{3}{4}}(X)D_s \tag{4.14}$$

for all $X \in \mathfrak{sl}(3,\mathbb{R})$. With abuse of notation we used the same symbol D_s and the same terminology "the symplectic Dirac operator" as in (1.4) due to the coincidence of (1.4) and (4.13).

Let us finally explain the notion of the dual representation $\mathbb{S}^*$. Let us define a non-degenerate pairing $(\cdot\,,\cdot)\colon \mathbb{S}\otimes_{\mathbb{C}}\mathbb{S}\to\mathbb{C}$ on $\mathbb{S}$ by the formula

$$\big(p_1(q),p_2(q)\big) = p_1(\partial_q)p_2(q)|_{q=0}. \tag{4.15}$$

Then we can identify the (restricted) dual space to $\mathbb{S}$ with $\mathbb{S}$ and the structure of the dual $\mathfrak{p}$-module on $\mathbb{S}$ is given as follows: the generators of $\mathfrak{mp}(2,\mathbb{C})$ act on $\mathbb{S}^*$ by

$$\sigma^*(e) = -\tfrac{\mathrm{i}}{2}q^2, \qquad \sigma^*(h) = q\partial_q + \tfrac{1}{2}, \qquad \sigma^*(f) = -\tfrac{\mathrm{i}}{2}\partial_q^2, \tag{4.16}$$

while the generator h_0 of the center $\mathfrak{z}(\mathfrak{l})\subset\mathfrak{l}$ and of the nilradical $\mathfrak{u}$ of $\mathfrak{p}$ act trivially. We notice that this representation is compatible with (1.2).

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