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RICCI AND SCALAR CURVATURES OF SUBMANIFOLDS OF A CONFORMAL SASAKIAN SPACE FORM

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Abstract. We introduce a conformal Sasakian manifold and we find the inequality involving Ricci curvature and the squared mean curvature for semi-invariant, almost semi-invariant, θ-slant, invariant and anti-invariant submanifolds tangent to the Reeb vector field and the equality cases are also discussed. Also the inequality involving scalar curvature and the squared mean curvature of some submanifolds of a conformal Sasakian space form are obtained.

1. Introduction

According to B.-Y. Chen [5], to establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of a Riemannian submanifold is one of the fundamental problems in the submanifold theory. For a submanifold of a Riemannian manifold, the main intrinsic invariants include Ricci, scalar and k-Ricci curvature, while the most important extrinsic invariants are the shape operator and the squared mean curvature. In [6], B.Y.Chen found a relationship between the sectional curvature function and the shape operator for submanifolds in real space forms. In [7], he also gave a sharp inequality for a submanifold in a real space form involving the Ricci curvature and the squared mean curvature as follow

Theorem 1.1. *Let M be an m-dimensional submanifold of a real space form $R^n(c)$. Then the following statements are true.*

(a) *For a unit vector $X \in T_pM$, we have*

$$\|H\|^2 \geq \frac{4}{m^2}\{\mathrm{Ric}(X) - (m-1)c\}\,. \tag{1.1}$$

(b) *If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of* (1.1) *if and only if X belongs to the relative null space $\mathcal{N}_p$.*

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(c) *The equality case of* (1.1) *holds for all unit vectors* $X \in T_pM$ *if and only if either* p *is a totally geodesic point or* $m = 2$ *and* p *is a totally umbilical point.*

Following B.-Y. Chen, many researchers, established same kind of inequalities for different kind of submanifolds in various ambient spaces, for example see [9], [10], [11], [12], [13].

On the other hand, I. Vaisman [16] introduced the conformal changes of almost metric structures as follows. Let M be a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g). A conformal change of the metric g leads to a metric which is no more compatible with the almost contact structure (φ, ξ, η). This can be corrected by a convenient change of ξ and η which implies rather strong restrictions. Using this definition, we introduce a new type of almost contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on a $(2n+1)$-dimensional manifold $\overline{M}$ which is said to be a conformal Sasakian structure if the structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is conformal related to a Sasakian structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$.

Motivated by these circumstances, in this paper we study the submanifolds tangent to the structure vector field (Reeb vector field) $\overline{\xi}$ in a conformal Sasakian manifold of a conformally Sasakian space form. and establish a basic inequality between the main intrinsic invariants including scalar curvature and Ricci curvature, and their main extrinsic invariants, namely squared mean curvature of these submanifolds.

The paper is organized as follows. In Section 2, we review the notion of Ricci curvature, Sasakian space form and a brief account of submanifolds. In Section 3, we give some basic results about conformal Sasakian manifolds. In Section 4, we establish the inequality involving Ricci curvature and the squared mean curvature for certain submanifolds of a conformal Sasakian space form, while Section 5 is devoted to establish the inequality involving scalar curvature and the squared mean curvature. The equality cases are also discussed.

2. Preliminaries

2.1. **Ricci curvature.** Let (M, g) be an m-dimensional Riemannian manifold and ∇ the Riemannian connection. The curvature tensor is a $(1,3)$-tensor defined by [14]

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,,$$

on vector fields X, Y, $Z \in TM$. Using the metric g we can change this to a $(0,4)$-tensor as follows

$$R(X,Y,Z,W) = g\big(R(X,Y)Z, W\big)\,.$$

Let $p \in M$, the $(0,2)$-tensor of Ricci is defined by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\left(Z \mapsto R(Z,X)Y\right), \tag{2.1}$$

for X, Y, $Z \in TM$. The Ricci curvature of $X \in TM$ is given by

$$\mathrm{Ric}(X) = \mathrm{Ric}(X,X)\,. \tag{2.2}$$

The sectional curvature of a plane section spanned by the linearly independent vectors $\{X, Y\}$, denoted by $K(X, Y)$, is given by

$$K(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} .$$

If $\{e_1, \ldots, e_m\}$ be any orthonormal basis for T_pM, then

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{m} g\big(R(e_i, X)Y, e_i\big) .$$

If $X \in T_pM$ is a unit vector and we complete it to an orthonormal basis $\{X, e_2, \ldots, e_m\}$ for T_pM, then

$$\mathrm{Ric}(X,X) = \sum_{i=2}^{m} K(X, e_i) .$$

The sectional curvature of a plane section spanned by orthonormal unit vectors e_i and e_j at $p \in M$, denoted K_{ij}, is [10]

$$K_{ij} = R(e_i, e_j, e_j, e_i) .$$

For a fixed $i \in \{1, \ldots, m\}$, the Ricci curvature of e_i, denoted $\mathrm{Ric}(e_i)$, is given by

$$\mathrm{Ric}(e_i) = \sum_{i \neq j}^{m} K_{ij} , \tag{2.3}$$

Moreover, the scalar curvature is the trace of Ric and denoted by τ. τ depends only on $p \in M$ and is therefore a function, $\tau \colon M \to \mathbb{R}$ and defined as follow

$$\tau(p) = \sum_{1 \leq i < j \leq m} K_{ij} = \frac{1}{2} \sum_{i=1}^{m} \mathrm{Ric}(e_i) . \tag{2.4}$$

From (2.3) and (2.4), we have

$$\mathrm{Ric}(e_1) = \tau(p) - \sum_{2 \leq i < j \leq m} K_{ij} = \tau(p) - \frac{1}{2} \sum_{2 \leq i \neq j \leq m} K_{ij} . \tag{2.5}$$

Let L be a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. The k-Ricci curvature $\mathrm{Ric}_L(X)$ is defined by [7]

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k} .$$

Thus for each fixed e_i, $i \in \{1, \ldots, k\}$ we get

$$\mathrm{Ric}_L(e_i) = \sum_{i \neq j}^{k} K_{ij} .$$

2.2. **Sasakian space form.** Let $(\widetilde{M}, g)$ be an odd-dimensional Riemannian manifold. Then $\widetilde{M}$ is said to be an almost contact metric manifold [2] if there exist on $\widetilde{M}$ a tensor φ of type $(1,1)$, a vector field ξ (structure vector field), and a 1-form η satisfying

$$\begin{gathered} \varphi^2 X = -X + \eta(X)\xi\,, \qquad g(X,\xi) = \eta(X), \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \end{gathered} \tag{2.6}$$

for any X, $Y \in T\widetilde{M}$. The 2-form Φ is called the fundamental 2-form in $\widetilde{M}$ and the manifold is said to be a contact metric manifold if $\Phi = d\eta$.

The almost contact structure of $\widetilde{M}$ is said to be normal if $[\varphi,\varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is a normal contact metric manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X\,,$$

for any X, $Y \in T\widetilde{M}$.

A plane section π in $T_p\widetilde{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a φ-section is called a φ-sectional curvature. A Sasakian manifold with constant φ-sectional curvature c is said to be a Sasakian space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor of $\widetilde{M}(c)$ of a Sasakian space form $\widetilde{M}(c)$ is given by [2]

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & - \frac{c-1}{4}\big\{\eta(Z)\,(\eta(Y)X - \eta(X)Y) + \big(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big)\xi \\ & - g(\varphi Y, Z)\varphi X + g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z\big\} \end{aligned}$$

for any tangent vector fields X, Y, Z on $\widetilde{M}(c)$.

2.3. **Submanifolds.** Let (M, g) be a submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ where g is the induced metric on M. Then, the Gauss and Weingarten formulas are given respectively by [10]

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X V = -\,A_V X + \nabla^{\perp}_X V\,,$$

for any X, $Y \in TM$ and $V \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form of M related to the shape operator A by $g(A_V X, Y) = g(h(X,Y), V)$.

The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) = {} & \widetilde{R}(X,Y,Z,W) + \widetilde{g}\big(h(X,W), h(Y,Z)\big) \\ & - \widetilde{g}(h(X,Z), h(Y,W))\,, \end{aligned} \tag{2.7}$$

for all X, Y, Z, $W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively.

The mean curvature vector H is given by $H = \frac{\operatorname{trace}(h)}{m}$, where $m = \dim M$. The submanifold M is totally geodesic in $\widetilde{M}$ if $h = 0$, minimal if $H = 0$, and totally umbilical if $h(X, Y) = g(X, Y)H$ for all X, $Y \in TM$.

The relative null space of M at p is defined by [7]

$$\mathcal{N}_p = \{X \in T_pM : h(X, Y) = 0, \text{ for all } Y \in T_pM\},$$

which is also known as the kernel of the second fundamental form at p [8].

For any $X \in TM$, we write $\varphi X = PX + FX$, where PX (resp. FX) is the tangential component (resp. normal component) of φX. Similarly, for any $V \in T^{\perp}M$, we have $\varphi V = tV + fV$, where tV (resp. fV) is the tangential component (resp. normal component) of φV.

The submanifold M is said to be invariant (anti-invariant) if $\varphi X \in TM$, for any $X \in TM$($\varphi X \in T^{\perp}M$, for any $X \in TM$).

Theorem 2.1 ([10])**.** *Let (M, g) be an m-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. Then the following statements are true:*

(i) *For any unit vector $X \in T_pM$ we have*

$$\operatorname{Ric}(X) \leq \frac{m^2}{4}\|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_pM)}(X), \tag{2.8}$$

where $\widetilde{\operatorname{Ric}}_{(T_pM)}(X)$ is the m-Ricci curvature of T_pM at $X \in T_pM$ with respect to the ambient manifold $\widetilde{M}$.

(ii) *The equality case of* (2.8) *is satisfied by a unit vector $X \in T_pM$ if and only if*

$$h(X, X) = \frac{m}{2}H(p), \quad h(X, Y) = 0,$$

for all $Y \in T_pM$ such that $g(X, Y) = 0$.

(iii) *The equality case of* (2.8) *holds for all unit vectors $X \in T_pM$ if and only if either* (1) *p is a totally geodesic point or* (2) *$m = 2$ and p is a totally umbilical point.*

2.4. **Almost semi-invariant submanifold.** We recall the definition of an almost semi-invariant submanifold as follows (cf. [11], [15]).

A submanifold M of an almost contact metric manifold $\widetilde{M}$ with $\xi \in TM$ is said to be an almost semi-invariant submanifold of $\widetilde{M}$ if there are k distinct functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that TM is decomposed as P-invariant mutually orthogonal differentiable distributions given by

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k} \oplus \{\xi\},$$

where $\mathcal{D}^1_p = \ker(F|_{\{\xi\}^{\perp}})_p$, $\mathcal{D}^0_p = \ker\left(P|_{\{\xi\}^{\perp}}\right)_p$ and

$$\mathcal{D}^{\lambda_i}_p = \ker\left(P^2|_{\{\xi\}^{\perp}} + \lambda_i^2(p)I\right)_p, \qquad i \in \{1, \ldots, k\}.$$

If in addition, each λ_i is constant, then M is called an almost semi-invariant submanifold.

An almost semi-invariant submanifold becomes

(a) A semi-invariant submanifold [1] if $k = 0$.

(b) An invariant submanifold [1] if $k = 0$ and $\mathcal{D}^0 = 0$.

(c) An anti-invariant submanifold [1] if $k = 0$ and $\mathcal{D}^1 = 0$.

(d) A θ-slant submanifold [4] if $\mathcal{D}^1 = 0 = \mathcal{D}^0$, $k = 1$ and λ_1 is constant. In this case, we have $TM = \mathcal{D}^{\lambda_1} \oplus \{\xi\}$ and the slant angle θ is given by $\lambda_1 = \cos\theta$. A slant submanifold which is not invariant nor anti-invariant is called a proper θ-slant submanifold.

If M is an almost semi-invariant submanifold of an almost contact metric manifold $\widetilde{M}$, then for $X \in TM$ we may write [11]

$$X = U^1X + U^0X + U^{\lambda_1}X + \cdots + U^{\lambda_k}X + \eta(X)\xi,$$

where $U^1, U^0, U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^0$, $\mathcal{D}^{\lambda_1}$, $\ldots, \mathcal{D}^{\lambda_k}$ respectively. Then, it follows that

$$\|X\|^2 = \|U^1X\|^2 + \|U^0X\|^2 + \|U^{\lambda_1}X\|^2 + \cdots + \|U^{\lambda_k}X\|^2 + \eta(X)^2. \tag{2.9}$$

We also have

$$P^2X = -U^1X - \lambda_1^2(U^{\lambda_1}X) - \cdots - \lambda_1^2(U^{\lambda_k}X)\,,$$

which implies that

$$\|PX\|^2 = \widetilde{g}(PX, PX) = -\widetilde{g}(P^2X, X) = \sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}} \lambda^2\|U^\lambda X\|^2\,. \tag{2.10}$$

In particular, if M is a m-dimensional θ-slant submanifold, then $\lambda_1^2 = \cos^2\theta$ and we have

$$\|PX\|^2 = \cos^2\theta\|U^{\lambda_1}X\|^2 = \cos^2\theta\left(\|X\|^2 - \eta(X)^2\right)\,. \tag{2.11}$$

3. Conformal Sasakian manifolds

A $(2n+1)$-dimensional Riemannian manifold $\overline{M}$ endowed with the almost contact metric structure $(\overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ is called a conformal Sasakian manifold if for a C^∞ function $f\colon \overline{M} \to \mathbb{R}$, there are

$$\widetilde{g} = \exp(f)\overline{g}, \qquad \widetilde{\xi} = \left(\exp(-f)\right)^{\frac{1}{2}}\overline{\xi}, \qquad \widetilde{\eta} = \left(\exp(f)\right)^{\frac{1}{2}}\overline{\eta}, \qquad \widetilde{\varphi} = \overline{\varphi}\,,$$

such that $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$ is a Sasakian manifold.

Example 3.1. Let $\mathbb{R}^{2n+1}$ be the $(2n+1)$-dimensional Euclidean space endowed with the almost contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ defined by

$$\overline{\varphi}\Big(\sum_{i=1}^n \Big(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\Big) + Z\frac{\partial}{\partial z}\Big) = \sum_{i=1}^n \Big(Y_i\frac{\partial}{\partial x^i} - X_i\Big)\frac{\partial}{\partial y^i} + \sum_{i=1}^n Y_iy^i\frac{\partial}{\partial z}\,,$$

$$\overline{g} = \exp(-f)\Big\{\overline{\eta}\otimes\overline{\eta} + \frac{1}{4}\sum_{i=1}^{n}(dx^i)^2 + (dy^i)^2\Big\},$$
$$\overline{\eta} = \big(\exp(-f)\big)^{\frac{1}{2}}\Big\{\frac{1}{2}\Big(dz - \sum_{i=1}^{n} y^i dx^i\Big)\Big\},$$
$$\overline{\xi} = \big(\exp(f)\big)^{\frac{1}{2}}\Big\{2\frac{\partial}{\partial z}\Big\},$$

where

$$f = \sum_{i=1}^{n}(x^i)^2 + (y^i)^2 + z^2 .$$

It is easy to show that $(\mathbb{R}^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is not a Sasakian manifold, but $\mathbb{R}^{2n+1}$ with the structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ given by

$$\widetilde{\varphi} = \overline{\varphi},$$
$$\widetilde{g} = \overline{\eta}\otimes\overline{\eta} + \frac{1}{4}\sum_{i=1}^{n}\big\{(dx^i)^2 + (dy^i)^2\big\},$$
$$\widetilde{\eta} = \frac{1}{2}\Big(dz - \sum_{i=1}^{n} y^i dx^i\Big),$$
$$\widetilde{\xi} = 2\frac{\partial}{\partial z},$$

is a Sasakian space form with the $\widetilde{\varphi}$-sectional curvature equal to -3.

Let $\widetilde{\nabla}$ and $\overline{\nabla}$ are the Riemannian connections on $\overline{M}$ with respect to the metrics $\widetilde{g}$ and $\overline{g}$, respectively. Using Koszul formula, we derive the following relation between the connections $\widetilde{\nabla}$ and $\overline{\nabla}$

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \frac{1}{2}\big\{\omega(X)Y + \omega(Y)X - \overline{g}(X,Y)\omega^\sharp\big\}, \quad \forall X, Y \in T\overline{M}, \tag{3.1}$$

where $\omega(X) = X(f)$ and $\overline{g}(\omega^\sharp, X) = \omega(X)$.

By using (3.1), we get the relation between the curvature tensors of $(\overline{M}, \overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ and $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$ as follow

$$\begin{aligned}\exp(-f)\widetilde{R}(X,Y,Z,W) = \overline{R}(X,Y,Z,W) &+ \frac{1}{2}\big\{B(X,Z)\overline{g}(Y,W)\\ &- B(Y,Z)\overline{g}(X,W) + B(Y,W)\overline{g}(X,Z)\\ &- B(X,W)\overline{g}(Y,Z)\big\} + \frac{1}{4}\|\omega^\sharp\|^2\big\{\overline{g}(X,Z)\overline{g}(Y,W)\\ &- \overline{g}(Y,Z)\overline{g}(X,W)\big\},\end{aligned} \tag{3.2}$$

for all X, Y, Z, $W \in T\overline{M}$, such that $B = \overline{\nabla}\omega - \frac{1}{2}\omega\otimes\omega$ and $\overline{R}$ and $\widetilde{R}$ are the curvature tensors of $(\overline{M}, \overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ and $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$, respectively.

From (3.1) it follows that

$$\begin{aligned}\overline{\nabla}_X\overline{\xi} &= -\big(\exp(f)\big)^{\frac{1}{2}}\overline{\varphi}X + \frac{1}{2}\{\overline{\eta}(X)\omega^\sharp - \omega(\overline{\xi})X\}\,,\\ (\overline{\nabla}_X\overline{\varphi})Y &= \big(\exp(f)\big)^{\frac{1}{2}}\{\overline{g}(X,Y)\overline{\xi} - \overline{\eta}(Y)X\}\\ &\quad - \frac{1}{2}\big\{\omega(\overline{\varphi}Y)X - \omega(Y)\overline{\varphi}X + \overline{g}(X,Y)\overline{\varphi}\omega^\sharp - \overline{g}(X,\overline{\varphi}Y)\omega^\sharp\big\}\,.\end{aligned}$$

Now, let (M,g) be an m-dimensional submanifold of a $(2n+1)$-dimensional conformal Sasakian manifold $(\overline{M},\overline{g})$, where g is the induced metric on M. Let $\{e_1,\ldots,e_m\}$ and $\{e_{m+1},\ldots,e_{2n+1}\}$ be the orthonormal bases of the tangent space T_pM and the normal space $T_p^\perp M$, respectively. We put

$$\begin{gathered}h_{ij}^r = g(h(e_i,e_j),e_r)\,,\quad i,j\in\{1,\ldots,m\},\quad r\in\{m+1,\ldots,2n+1\}\,,\\ \|h\|^2 = \sum_{i,j=1}^{m} g\big(h(e_i,e_j),h(e_i,e_j)\big)\,.\end{gathered}$$

Let K_{ij} and $\overline{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\overline{M}$, respectively. Thus, K_{ij} and $\overline{K}_{ij}$ are the intrinsic and extrinsic sectional curvature of the equation (2.7),we have [10]

$$K_{ij} = \overline{K}_{ij} + \sum_{r=m+1}^{2n+1}\big(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\big)\,. \tag{3.3}$$

From (3.3) it follows that

$$2\tau(p) = 2\overline{\tau}(T_pM) + m^2\|H\|^2 - \|h\|^2\,, \tag{3.4}$$

where

$$\overline{\tau}(T_pM) = \sum_{1\le i<j\le m}\overline{K}_{ij}\,,$$

denote the scalar curvature of the m-plane section T_pM in the ambient manifold $\overline{M}$. Thus, $\tau(p)$ and $\overline{\tau}(T_pM)$ are the intrinsic and extrinsic scalar curvature of the submanifold at p, respectively.

A $(2n+1)$-dimensional conformal Sasakian manifold with constant sectional curvature c, denoted $\overline{M}(c)$, is called a conformal Sasakian space form, and from (3.2), we get its curvature tensor as follow

$$\begin{aligned}R(X,Y,Z,W) = \exp(f)\Big\{&\frac{c+3}{4}\big(\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W)\big)\\ &+\frac{c-1}{4}\big(\overline{\eta}(X)\overline{\eta}(Z)\overline{g}(Y,W) - \overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(X,W)\end{aligned}$$

$$
\begin{aligned}
&+\overline{g}(X,Z)\overline{g}(\overline{\xi},W)\overline{\eta}(Y)-\overline{g}(Y,Z)\overline{g}(\overline{\xi},W)\overline{\eta}(X)\\
&+\overline{g}(\overline{\varphi}Y,Z)\overline{g}(\overline{\varphi}X,W)-\overline{g}(\overline{\varphi}X,Z)\overline{g}(\overline{\varphi}Y,W))\\
&-2\overline{g}(\overline{\varphi}X,Y)\overline{g}(\overline{\varphi}Z,W)\big)\big\}-\frac{1}{2}\big\{B(X,Z)\overline{g}(Y,W)\\
&-B(Y,Z)\overline{g}(X,W)+B(Y,W)\overline{g}(X,Z)-B(X,W)\overline{g}(Y,Z)\Big\}\\
&-\frac{1}{4}\|\omega^{\sharp}\|^2\big\{\overline{g}(X,Z)\overline{g}(Y,W)-\overline{g}(Y,Z)\overline{g}(X,W)\big\},
\end{aligned}
\tag{3.5}
$$

for all X, Y, Z, W tangent to $\overline{M}(c)$.

Theorem 3.2. *Let M be a submanifold of a conformal Sasakian space form $\overline{M}(c)$ such that $\omega^{\sharp}$, $\overline{\xi} \in TM$. If $p \in M$ is a totally umbilical point, then p is a totally geodesic point and hence $\overline{\varphi}(T_pM) \subseteq T_pM$.*

Proof. For a conformal Sasakian space form we have

$$
\overline{\nabla}_X\overline{\xi} = -\big(\exp(f)\big)^{\frac{1}{2}}\overline{\varphi}X - \frac{1}{2}\big\{\omega(\overline{\xi})X - \overline{\eta}(X)\omega^{\sharp}\big\}\,. \tag{3.6}
$$

By the Gauss formula for the submanifold M of a conformal Sasakian space form $\overline{M}(c)$ such that $\omega^{\sharp}$, $\overline{\xi} \in TM$, and comparing the tangential and the normal part of (3.6), we get

$$
\begin{aligned}
\nabla_X\overline{\xi} &= -\big(\exp(f)\big)^{\frac{1}{2}}PX - \frac{1}{2}\left(\omega(\overline{\xi})X - \overline{\eta}(X)\omega^{\sharp}\right)\\
h(X,\overline{\xi}) &= -\big(\exp(f)\big)^{\frac{1}{2}}FX\,.
\end{aligned}
\tag{3.7}
$$

Now, let $p \in M$ be a totally umbilical point. Then, we get

$$
H = \overline{g}(\overline{\xi},\overline{\xi})H = h(\overline{\xi},\overline{\xi}) = -\big(\exp(f)\big)^{\frac{1}{2}}F\overline{\xi} = 0\,,
$$

which shows that $h(X,Y) = 0$ for all X, $Y \in T_pM$, that p is a totally geodesic point. Since p is a totally geodesic point, therefore we have

$$
h(X,\overline{\xi}) = -\big(\exp(f)\big)^{\frac{1}{2}}FX = 0\,,
$$

for all $X \in T_pM$, which shows that $\overline{\varphi}(T_pM) \subseteq T_pM$. □

Corollary 3.3. *A totally umbilical submanifold M of a conformal Sasakian space form $\overline{M}(c)$ such that $\omega^{\sharp}$, $\overline{\xi} \in TM$, is a totally geodesic invariant submanifold.*

4. Ricci curvature

Theorem 4.1. *Let M be an m-dimensional $(m \geq 2)$ submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$. Then,*

$$
\begin{aligned}
\mathrm{Ric}(X) \leq \frac{1}{4}\big\{&m^2\|H\|^2 + \exp(f)\big\{(m-1)(c+3) + (c-1)(3\|PX\|^2\\
&+(2-m)\overline{\eta}(X)^2 - 1)\big\} + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big)\\
&+(m-1)\|\omega^{\sharp}\|^2\big\}\,,
\end{aligned}
\tag{4.1}
$$

for any unit vector $X \in T_pM$.

Proof. From (3.3), we get

$$\frac{m^2\|H\|^2}{4} = \tau(p) - \overline{\tau}(T_pM) + \frac{1}{4}\sum_{r=m+1}^{2n+1}(h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2$$
$$+ \sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(h_{1j}^r)^2 - \sum_{2\le i<j\le m}(K_{ij} - \overline{K}_{ij})\,. \tag{4.2}$$

From (2.5), (4.2) yields to

$$\frac{m^2\|H\|^2}{4} = \mathrm{Ric}(e_1) - \overline{\mathrm{Ric}}(e_1) + \frac{1}{4}\sum_{r=m+1}^{2n+1}(h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2$$
$$+ \sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(h_{1j}^r)^2\,. \tag{4.3}$$

In view of (3.5), we have

$$\overline{\mathrm{Ric}}(e_1) = \frac{1}{4}\big\{\exp(f)\big\{(m-1)(c+3) + (c-1)(3\|PX\|^2$$
$$+ (2-m)(\overline{\eta}(X))^2 - 1)\big\} - 2\big\{(2-m)((\overline{\nabla}_X\omega)X$$
$$- \frac{1}{2}\omega(X)^2) - \operatorname{tr} B\big\} + (m-1)\|\omega^\sharp\|^2\big\}\,, \tag{4.4}$$

Now by substituting (4.4) in (4.3), we obtain

$$\frac{m^2\|H\|^2}{4} = \mathrm{Ric}(e_1) - \frac{1}{4}\big\{\exp(f)\big\{(m-1)(c+3) + (c-1)(3\|PX\|^2$$
$$+ (2-m)\overline{\eta}(X)^2 - 1)\big\} + 2\big\{(2-m)\big((\overline{\nabla}_X\omega)X - \frac{1}{2}\omega(X)^2\big)$$
$$- \operatorname{tr} B\big\} - (m-1)\|\omega^\sharp\|^2\big\} + \frac{1}{4}\sum_{r=m+1}^{2n+1}(h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2$$
$$+ \sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(h_{1j}^r)^2\,. \tag{4.5}$$

Since, we can choose $e_1 = X$ as any unit vector in T_pM. Therefore the above equation implies (4.1). □

Theorem 4.2. *Let* M *be an* m*-dimensional* $(m \ge 2)$ *submanifold of a conformal Sasakian space form* $\overline{M}(c)$, *tangent to the structure vector field* $\overline{\xi}$. *Then,*

(i) *A unit vector* $X \in T_pM$ *satisfies the equality case of* (4.1) *if and only if either* $m = 2$ *or* $\omega^\sharp$ *be orthogonal to* X *and*

$$h(X,X) = \frac{m}{2}H(p)\,,$$
$$h(X,Y) = 0\,, \quad \forall Y \in \{X\}^\perp. \tag{4.6}$$

(ii) *If M is minimal at p, then a unit vector $X \in T_pM$ satisfies the equality case of* (4.1) *if and only if X lies in the relative null space of M and either $m = 2$ or $\omega^\sharp$ be orthogonal to X.*

Proof. Assuming $X = e_1$, from (4.5) the equality case of (4.1) is valid if and only if the following relations be satisfied.

$$(a) \quad h^r_{1j} = 0, \quad \forall j = 2, \dots, m, \quad r = m+1, \dots, 2n+1, \tag{4.7}$$

$$(b) \quad h^r_{11} = \sum_{i=2}^{m} h^r_{ii}, \quad \forall r = m+1, \dots, 2n+1, \tag{4.8}$$

$$(c) \quad (2-m)\omega(X)^2 = 0\,. \tag{4.9}$$

Satisfying (a), (b) and (c) is equivalent to statement (i).

For proving the statement (ii) we note that minimality at p means $H(p) = 0$. So, in view of (4.7), (4.8) and (4.9), we conclude that X lies in the relative null space of M and either $m = 2$ or $\omega^\sharp$ be orthogonal to X. □

Corollary 4.3. *Let M be an m-dimensional $(m \geq 2)$ submanifold of a conformal Sasakian space form $\overline{M}(c)$. For a unit vector $X \in T_pM$, any three of the following four statements imply the remaining one.*

(i) $\omega(X) = 0$ *(means df in the direction of X is zero).*

(ii) *The mean curvature vector $H(p)$ vanishes.*

(iii) *The unit vector X belongs to the relative null space $\mathcal{N}_p$.*

(iv) *The unit vector X satisfies the following equality case*

$$\begin{aligned} 4\,\mathrm{Ric}(X) = m^2\|H\|^2 &+ \exp(f)\big\{(m-1)(c+3) + (c-1)(3\|PX\|^2 \\ &+ (2-m)\overline{\eta}(X)^2 - 1)\big\} + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) \\ &+ (m-1)\|\omega^\sharp\|^2\big\}\,. \end{aligned}$$

Theorem 4.4. *Let M be an m-dimensional $(m \geq 2)$ submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$. Then, the equality case of* (4.1) *holds for all unit vectors $X \in T_pM$ if and only if either p is a totally geodesic point and $\omega^\sharp \in T_p^\perp M$ or $m = 2$ and p is a totally geodesic point and in this case if $\omega^\sharp \in T_pM$ then $\overline{\varphi}(T_pM) \subseteq T_pM$.*

Proof. Assume that the equality case of (4.1) is satisfied for all unit vectors $X \in T_pM$, in view of (4.7), (4.8) and (4.9), we have

$$h^r_{ij} = 0\,, \quad i \neq j, \quad r = m+2, \dots, 2n+1\,, \tag{4.10}$$

$$2h^r_{ii} = h^r_{11} + h^r_{22} + \cdots + h^r_{mm}, \quad i = 1, \dots, m, \quad r = m+2, \dots, 2n+1, \tag{4.11}$$

$$(2-m)\omega(e_i)^2 = 0, \quad i = 1, \dots, m. \tag{4.12}$$

From (4.12), we have either $m = 2$ or $\omega^\sharp$ be in $T_p^\perp M$.

Assume that $m \neq 2$, so $\omega^\sharp$ be in $T_p^\perp M$ and from (4.11), we get

$$2h_{11}^r = 2h_{22}^r = \cdots = 2h_{mm}^r = h_{11}^r + h_{22}^r + \cdots + h_{mm}^r,$$

which yields to

$$(m-2)(h_{11}^r + h_{22}^r + \cdots + h_{mm}^r) = 0.$$

Thus, $h_{11}^r + h_{22}^r + \cdots + h_{mm}^r = 0$. Then, in view of (4.10) and (4.11), we get $h_{ij}^r = 0$ for any $i, j = 1, \ldots, m$ and $r = m+2, \ldots, 2n+1$, that is, p is a totally geodesic point.

Now, assume that $m = 2$ and $\omega^\sharp \in T_P M$, then from (4.11), we have

$$2h_{11}^r = 2h_{22}^r = (h_{11}^r + h_{22}^r)\,,$$

which shows that p is a totally umbilical point. Now Theorem 3.2 implies that p is a totally geodesic point and $\overline{\varphi}(T_pM) \subseteq T_pM$. The proof of the converse part is straightforward. □

Corollary 4.5. *Every m-dimensional $(m > 2)$ totally geodesic submanifold M of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$ such that $\omega^\sharp \in T_p^\perp M$, satisfies*

$$\begin{aligned}4\operatorname{Ric}(X) = {} & m^2\|H\|^2 + \exp(f)\big\{(m-1)(c+3) + (c-1)\big(3\|PX\|^2 \\ & + (2-m)\overline{\eta}(X)^2 - 1\big)\big\} + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) \\ & + (m-1)\|\omega^\sharp\|^2\,.\end{aligned}$$

Now, we need the following lemma

Lemma 4.6. *Let M be an m-dimensional invariant submanifold of a Conformal Sasakian manifold $\overline{M}$, tangent to the structure vector field $\overline{\xi}$. Then M is minimal if and only if $\omega^\sharp$ is tangent to M.*

Proof. From (3.1) and Gauss formula, we have

$$\begin{aligned}h(X, \overline{\varphi}Y) = {} & \overline{\varphi}h(X,Y) - (\nabla_X\overline{\varphi})Y + (\exp(f))^{\frac{1}{2}}\big(\overline{g}(X,Y)\overline{\xi} - \overline{\eta}(Y)X\big) \\ & - \frac{1}{2}\{\omega(\overline{\varphi}Y)X - \omega(Y)\overline{\varphi}X - \overline{g}(X,\overline{\varphi}Y)\omega^\sharp + \overline{g}(X,Y)\overline{\varphi}\omega^\sharp\}\,.\end{aligned}$$

Now by comparing the tangential part and the normal part, we get

$$h(X,\overline{\varphi}Y) = \overline{\varphi}h(X,Y) - \frac{1}{2}\{\overline{g}(X,Y)\overline{\varphi}{\omega^\sharp}^\perp - \overline{g}(X,\overline{\varphi}Y){\omega^\sharp}^\perp\}\,.$$

We note that $\omega^\sharp$ is tangent to M, so we obtain

$$h(X,\overline{\varphi}Y) = \overline{\varphi}h(X,Y)\,.$$

Let $\{e_i, \overline{\varphi}e_i\}, i = 1, \ldots, \frac{m}{2}$ be an othonormal basis on M. Then,

$$H = \frac{1}{m}\sum_{i=1}^{\frac{m}{2}} h(e_i,e_i) + h(\overline{\varphi}e_i, \overline{\varphi}e_i) = 0\,.$$

This completes the proof. □

As a result of Lemma 4.6 and Theorem 4.4, we have

Corollary 4.7. *Every 2-dimensional totally geodesic invariant submanifold M of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$, such that $\omega^\sharp \in TM$, satisfies*

$$4\operatorname{Ric}(X) = \exp(f)\big\{(m-1)(c+3) + (c-1)\big(3\|PX\|^2 + (2-m)\overline{\eta}(X)^2 - 1\big)\big\} \\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2 .$$

Theorem 4.8. *Let M be an m-submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$. Then,*

(i) *For each unit vector $X \in \{\overline{\xi}\}_p^\perp$, we have*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\big\{m^2\|H\| + \exp(f)\big\{(m-1)(c+3) + (c-1)\big(3\|PX\|^2 - 1\big)\big\} \\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\} . \tag{4.13}$$

(ii) *The equality case of (4.13) holds for all $X \in \{\overline{\xi}\}_p^\perp$ if and only if either p is a totally geodesic point and $\omega^\sharp \in T_p^\perp M$ or $m = 2$ and p is a totally geodesic point and in this case if $\omega^\sharp \in T_pM$ then $\overline{\varphi}(T_pM) \subseteq T_pM$.*

Proof. Put $\overline{\eta}(X) = 0$ in (4.1) to get (4.13). Rest of the proof is straightforward. □

Theorem 4.9. *Let M be an m-dimensional ($m \geq 2$) semi-invariant submanifold of a conformal Sasakian space form $\overline{M}(c)$ such that $T_pM = \mathcal{D}_p \oplus \mathcal{D}_p^\perp \oplus \langle\overline{\xi}\rangle$ then,*

(i) *For each unit vector $X \in \mathcal{D}_p$, we have*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\big\{m^2\|H\|^2 + \exp(f)\big((m-1)(c+3) + 2(c-1)\big) \\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\} .$$

(ii) *For each unit vector $X \in \mathcal{D}_p^\perp$, we have*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\big\{m^2\|H\|^2 + \exp(f)\big((m-1)(c+3) - (c-1)\big) \\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (n-1)\|\omega^\sharp\|^2\big\} .$$

Proof. If M is a semi-invariant submanifold, then $\overline{\varphi}(\mathcal{D}_p) \subseteq \mathcal{D}_p$ and $\overline{\varphi}(\mathcal{D}_p^\perp) \subseteq T_p^\perp M$. If $X \in \mathcal{D}_p$, then $\overline{\eta}(X) = 0$ and $\|PX\|^2 = 1$. Now by using the inequality (4.1) we prove (i). For proving (ii), we note that in this case $P = 0$, rest of the proof is similar to (i). □

Theorem 4.10. *Let M be an m-dimensional ($m \geq 2$) submanifold of a conformally Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$. Then,*

(i) *If M is an almost semi-invariant submanifold, then for each unit vector $X \in T_pM$ we have*

$$\begin{aligned}\operatorname{Ric}(X) \leq \frac{1}{4}\Big\{ & m^2\|H\|^2 + \exp(f)\Big\{(m-1)(c+3) \\ & + (c-1)\Big(3\sum_{\lambda\in\{1,\lambda_1,\dots,\lambda_k\}} \lambda^2\|U_p^\lambda X\|^2 + (2-m)(\overline{\eta}(X)^2-1)\Big)\Big\} \\ & + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\Big\}, \end{aligned} \tag{4.14}$$

where $U_p^1, U_p^{\lambda_1}, \dots, U_p^{\lambda_k}$ are orthogonal projection operators of T_pM on $\mathcal{D}_p^1, \mathcal{D}_p^{\lambda_1}, \dots, \mathcal{D}_p^{\lambda_k}$, respectively.

(ii) *If M is a θ-slant submanifold, then for each unit vector $X \in T_pM$ we have*

$$\begin{aligned}\operatorname{Ric}(X) \leq \frac{1}{4}\big\{ & m^2\|H\|^2 + \exp(f)\big\{(m-1)(c+3) \\ & + (c-1)\big(3\cos^2\theta(1-\overline{\eta}(X)^2\big) + (2-m)\overline{\eta}(X)^2 - 1\big\} \\ & + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}. \end{aligned} \tag{4.15}$$

(iii) *If M is an anti-invariant submanifold, then for each unit vector $X \in T_pM$ we have*

$$\begin{aligned}\operatorname{Ric}(X) \leq \frac{1}{4}\big\{ & m^2\|H\|^2 + \exp(f)\big\{(m-1)(c+3) \\ & + (c-1)\big((2-m)\overline{\eta}(X)^2 - 1\big)\big\} \\ & + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}. \end{aligned} \tag{4.16}$$

Proof. Using (2.10) in (4.1) we get (4.14). Next, using (2.11) in (4.1) we get (4.15). For the inequality (4.16), we note that $T_pM = \mathcal{D}_p^0 + \{\overline{\xi}\}$, so $\theta = \frac{\pi}{2}$, therefore we put $\theta = \frac{\pi}{2}$ in (4.15)and then we find (4.16). □

Theorem 4.11. *Let M be an m-dimensional ($m \geq 2$) almost semi-invariant submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$, then for a unit vector $X \in \{\overline{\xi}\}_p^\perp$ we have*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\big\{m^2\|H\|^2 + \exp(f)\Big\{(m-1)(c+3) \tag{4.17}$$

$$+ (c-1)\Big(3\sum_{\lambda\in\{1,\lambda_1,\dots,\lambda_k\}} \lambda^2\|U_p^\lambda X\|^2 - 1\Big)\Big\}$$

$$+ 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}, \tag{4.18}$$

where $U_p^1, U_p^{\lambda_1}, \dots, U_p^{\lambda_k}$ are orthogonal projection operators of T_pM on $\mathcal{D}_p^1$, $\mathcal{D}_p^{\lambda_1}$, $\dots, \mathcal{D}_p^{\lambda_k}$, respectively.

Proof. Put (2.10) in (4.13) and note that in this case $\overline{\eta}(X) = 0$ to get (4.18). □

Theorem 4.12. *Let M be an m-dimensional ($m \geq 2$) submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$, and let $X \in \{\overline{\xi}\}_p^\perp$ is a unit vector. Then, the following statements are true*

(i) *If M is a proper θ-slant submanifold, then*

$$\begin{aligned}\mathrm{Ric}(X) < \frac{1}{4}\big\{m^2\|H\| + \exp(f)\big\{(m-1)(c+3) + (c-1)(3\cos^2\theta - 1)\big\}\\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}\,.\end{aligned} \tag{4.19}$$

(ii) *If M is anti-invariant, then*

$$\begin{aligned}\mathrm{Ric}(X) < \frac{1}{4}\big\{m^2\|H\| + \exp(f)\big\{(m-1)(c+3) - (c-1)\big\}\\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}\,.\end{aligned} \tag{4.20}$$

(iii) *If M is invariant, then*

$$\begin{aligned}\mathrm{Ric}(X) \le \frac{1}{4}\{m^2\|H\| + \exp(f)\big\{(m-1)(c+3) + 2(c-1)\big\}\\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}\,.\end{aligned} \tag{4.21}$$

Proof. Put (2.11) in (4.13) and note that in this case $\overline{\eta}(X) = 0$ to get

$$\begin{aligned}\mathrm{Ric}(X) \le \frac{1}{4}\big\{m^2\|H\| + \exp(f)\big((m-1)(c+3) + (c-1)(3\cos^2\theta - 1)\big)\\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}\,.\end{aligned} \tag{4.22}$$

By putting $\theta = \pi/2$ in (4.22), we obtain

$$\begin{aligned}\mathrm{Ric}(X) \le \frac{1}{4}\big\{m^2\|H\| + \exp(f)\big((m-1)(c+3) - (c-1)\big)\\ + 2\big(\operatorname{tr} B + (m-2)(\overline{\nabla}_X\omega)X\big) + (m-1)\|\omega^\sharp\|^2\big\}.\end{aligned} \tag{4.23}$$

By putting $\theta = 0$ (4.22), we get (4.21). If possible, let equality case of (4.22) or (4.23) is satisfied by a unit vector $X \in \{\overline{\xi}\}_p^\perp$, then it follows that $h(X, \overline{\xi}) = 0$, which in view of $h(X, \overline{\xi}) = -(\exp(f))^{\frac{1}{2}} FX$, is a contradiction. Thus, (4.19) and (4.20) are proved. ☐

5. Scalar curvature

We recall the following theorem and proposition from [12].

Theorem 5.1. *For an m-dimensional submanifold M in an n-dimensional Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \le \frac{m(m-1)}{2}\|H\|^2 + \widetilde{\tau}(T_pM)\,, \tag{5.1}$$

with equality if and only if p is a totally umbilical point.

Proposition 5.2. *For an m-dimensional submanifold M of a Riemannian manifold at each point $p \in M$, we have*

$$\tau(p) \le \frac{1}{2}m^2\|H\|^2 + \widetilde{\tau}(T_pM)\,, \tag{5.2}$$

with equality if and only if p is a totally geodesic point.

Theorem 5.3. *Let M be an m-dimensional submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$, then at each point $p \in M$, we have*

$$\begin{aligned}\tau(p) \leq \frac{1}{8}\{&4m(m-1)\|H\|^2 + \exp(f)\{m(m-1)(c+3)\\ &-(c-1)\big(2m-2-3\|P\|^2\big)\} + 4(m-1)\operatorname{tr} B\\ &+ m(m-1)\|\omega^\sharp\|^2\},\end{aligned} \tag{5.3}$$

where

$$\|P\|^2 = \sum_{i,j=1}^{m} \overline{g}(Pe_i, e_j)^2$$

and the equality case of (5.3) *is satisfied if and only if p is a totally umbilical point.*

Proof. From (2.4), we obtain

$$\begin{aligned}\widetilde{\tau}(T_pM) = \frac{1}{8}\exp(f)\big\{&m(m-1)(c+3) + (c-1)\big(2m-2-3\|P\|^2\big)\\ &-4(m-1)\operatorname{tr} B - m(m-1)\|\omega^\sharp\|^2\big\}.\end{aligned} \tag{5.4}$$

Now, by substituting (5.4) in (5.1) we get (5.3). Rest of the proof is straightforward. □

Theorem 5.4. *Let M be an m-dimensional submanifold of a conformal Sasakian space form $\overline{M}(c)$, tangent to the structure vector field $\overline{\xi}$, then at each point $p \in M$, we have*

$$\begin{aligned}\tau(p) \leq \frac{1}{8}\big\{&4m^2\|H\|^2 + \exp(f)\big\{m(m-1)(c+3)\\ &-(c-1)\big(2m-2-3\|P\|^2\big)\big\} + 4(m-1)\operatorname{tr} B\\ &+ m(m-1)\|\omega^\sharp\|^2\big\},\end{aligned} \tag{5.5}$$

with equality if and only if p is a totally geodesic point.

Proof. By substituting (5.4) in (5.2) we get (5.5). Rest of the proof is straightforward. □

Theorem 5.5. *Let M be an m-dimensional submanifold of a conformal Sasakian space form $\overline{M}(c)$, such that $\omega^\sharp$ and $\overline{\xi}$ are tangent to M. Then, the following statements are true*

(i) *If M is anti-invariant, then*

$$\begin{aligned}\tau(p) \leq \frac{1}{8}\big\{&4m(m-1)\|H\|^2 + \exp(f)\big(m(m-1)(c+3) - 2(m-1)(c-1)\big)\\ &+ 4(m-1)\operatorname{tr} B + m(m-1)\|\omega^\sharp\|^2\big\}.\end{aligned} \tag{5.6}$$

(ii) *If M is invariant, then*

$$\begin{aligned}\tau(p) \leq \frac{1}{8}\big\{&\exp(f)\big(m(m-1)(c+3) - (c-1(2m-2-3\|P\|^2)\big)\\ &+ 4(m-1)\operatorname{tr} B + m(m-1)\|\omega^\sharp\|^2\big\}.\end{aligned} \tag{5.7}$$

Proof. By putting $P = 0$ in (5.3), we get (5.6). For proving (ii), Let M is invariant and $\omega^\sharp \in TM$ so from Lemma 4.6, it follows that M is minimal at p. The rest of the prove is straightforward. $\square$

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