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NEW CHARACTERIZATIONS OF LINEAR WEINGARTEN HYPERSURFACES IMMERSED IN THE HYPERBOLIC SPACE

Cícero P. Aquino and Henrique F. de Lima

Abstract. In this paper, we deal with complete *linear Weingarten* hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$, that is, complete hypersurfaces of $\mathbb{H}^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy $R = aH + b$ for some a, $b \in \mathbb{R}$. In this setting, under appropriate restrictions on the mean curvature and on the norm of the traceless part of the second fundamental form, we prove that such a hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder of $\mathbb{H}^{n+1}$. Furthermore, a rigidity result concerning the compact case is also given.

1. Introduction and statements of the main results

Many authors have approached the problem of characterizing hypersurfaces immersed with constant mean curvature or with constant scalar curvature in a real space form Q_c^{n+1} of constant sectional curvature c. For instance, Cheng and Yau [6] classified closed hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in Q_c^{n+1}. Later on, Li [7] extended the results due to Cheng and Yau [6] in terms of the squared norm of the second fundamental form of the hypersurface. In [13], Shu used the so-called generalized maximum principle of Omori-Yau [11, 14] to prove that a complete hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$ with constant normalized scalar curvature and nonnegative sectional curvature must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = -1$. In [8], Li studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed in a unit sphere with scalar curvature proportional to mean curvature.

More recently, Li et al. [9] studied the so-called *linear Weingarten* hypersurfaces immersed in a unit sphere, that is, hypersurfaces of $\mathbb{S}^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy $R = aH + b$, for some $a, b \in \mathbb{R}$. In this setting, they showed that if M^n is a compact linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$, such that $R = aH + b$

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with $(n-1)a^2+4n(b-1)\geq 0$, then M^n is either totally umbilical or isometric to $\mathbb{S}^{n-k}(c_1)\times\mathbb{S}^k(c_2)$, where $1\leq k\leq n-1$, $c_1,c_2>0$ and $\frac{1}{c_1}+\frac{1}{c_2}=1$. Afterwards, the authors [3] investigated the geometry of complete linear Weingarten hypersurfaces with nonnegative sectional curvature immersed in the hyperbolic space. In this setting, under the assumption that the mean curvature attains its maximum, they showed that such a hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder.

Here, motivated by the works described above, we study the geometry of complete linear Weingarten hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$. First we apply a suitable extension of a generalized maximum principle at the infinity of Yau [15] due to Caminha in [4] (cf. Lemma 3.3) in order to obtain the following characterization result:

Theorem 1.1. *Let M^n be a complete linear Weingarten hypersurface immersed in $\mathbb{H}^{n+1}$ such that $R=aH+b$ with $H^2\geq 1$ and $(n-1)a^2+4n(b+1)>0$. If H is bounded, ∇H has integrable norm on M^n and*

$$|\Phi|\leq\mathcal{R}_H^+, \tag{1.1}$$

where Φ stands for the traceless part of the second fundamental form of M^n and

$$\mathcal{R}_H^+=\frac{1}{2}\sqrt{\frac{n}{n-1}}\left(\sqrt{n^2H^2-4(n-1)}-(n-2)H\right),$$

then M^n is either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1)\times\mathbb{H}^1(c_2)$, if $R>0$, or is isometric to $\mathbb{S}^1(c_1)\times\mathbb{H}^{n-1}(c_2)$, if $R<0$, where $c_1>0$, $c_2<0$ and $\dfrac{1}{c_1}+\dfrac{1}{c_2}=-1$.

We want to point out that, from Example (H-5) in Section 4 of [1] it is not difficult to verify that $|\Phi|\equiv\mathcal{R}_H^+$ in the hyperbolic cylinders $\mathbb{S}^{n-1}(c_1)\times\mathbb{H}^1(c_2)$ and $\mathbb{S}^1(c_1)\times\mathbb{H}^{n-1}(c_2)$. In this sense, since $|\Phi|\equiv 0$ in the totally umbilical hypersurfaces, we have that inequality (1.1) is a mild hypothesis and that Theorem 1.1 can be regarded as a gap result.

Afterwards, we also get the following rigidity result related to the compact case:

Theorem 1.2. *Let M^n be a compact linear Weingarten hypersurface immersed in $\mathbb{H}^{n+1}$ such that $R=aH+b$ with $H^2>1$ and $(n-1)a^2+4n(b+1)\geq 0$. If inequality (1.1) is strict, then M^n is isometric to $\mathbb{S}^n$, up to scaling.*

The proofs of Theorems 1.1 and 1.2 are given in Section 3.

2. A Simons-type formula in the hyperbolic space

Let M^n be an orientable and connect n-dimensional hypersurface immersed in the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. We choose a local field of orthonormal frame $\{e_A\}_{1\leq A\leq n+1}$ in $\mathbb{H}^{n+1}$, with dual coframe $\{\omega_A\}_{1\leq A\leq n+1}$, such that, at each point of M^n, $e_1,\dots,e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1\leq A,B,C,\ldots\leq n+1\,,\qquad 1\leq i,j,k,\ldots\leq n\,.$$

Denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{H}^{n+1}$, we have that the structure equations of $\mathbb{H}^{n+1}$ are given by:

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i + \omega_{An+1} \wedge \omega_{n+1}\,, \quad \omega_{AB} + \omega_{BA} = 0\,,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D\,,$$
$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,.$$

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and we can use Cartan's Lemma [5] to write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j,\ h_{ij} = h_{ji}\,.$$

This gives the second fundamental form of M^n, $B = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,,$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,.$$

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk})\,, \tag{2.1}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = -(n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{2.2}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}\,. \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$\begin{aligned} |B|^2 &= n^2H^2 - n(n-1)(R+1) \\ &= nH^2 + n(n-1)(H^2 - H_2)\,, \end{aligned} \tag{2.4}$$

where $|B|^2 = \sum_{i,j} h_{ij}^2$ is the square of the length of the second fundamental form B of M^n, and $H_2 = \frac{2}{n(n-1)}S_2$ denotes the mean value of the second elementary symmetric function S_2 on the eigenvalues of B. In particular, since (from the Cauchy-Schwarz inequality) $H^2 - H_2 \geq 0$, it follows from (2.4) that M^n is totally umbilical if, and only if, $|B|^2 = nH^2$.

The components h_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{jk}\omega_{ki}\,.$$

The *Codazzi equation* and the *Ricci identity* are, respectively, given by

$$h_{ijk} = h_{ikj} \tag{2.5}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}\,, \tag{2.6}$$

where h_{ijk} and h_{ijk} denote the first and the second covariant derivatives of h_{ij}.

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum\limits_k h_{ijkk}$. From equations (2.5) and (2.6), we obtain that

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,l} h_{kl}R_{lijk} + \sum_{k,l} h_{li}R_{lkjk}\,. \tag{2.7}$$

Since $\Delta|B|^2 = 2\left(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2\right)$, from (2.7) we get

$$\begin{aligned}\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 &+ \sum_{i,i,k} h_{ij}h_{kkij} + \sum_{i,j,k,l} h_{ij}h_{lk}R_{lijk}\\ &+ \sum_{i,j,k,l} h_{ij}h_{il}R_{lkjk}\,.\end{aligned} \tag{2.8}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.8) we obtain the following Simons-type formula

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_i \lambda_i(nH)_{,ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2\,. \tag{2.9}$$

3. Proofs of Theorems 1.1 and 1.2

In order to prove our results, we will quote some key lemmas. The first one is a classic algebraic lemma due to M. Okumura in [10], and completed with the equality case proved in [2] by H. Alencar and M. do Carmo.

Lemma 3.1. *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum\limits_i \mu_i = 0$ and $\sum\limits_i \mu_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3\,, \tag{3.1}$$

and equality holds if, and only if, either at least $(n-1)$ of the numbers μ_i are equal.

The next result corresponds to Lemma 3.1 of [3].

Lemma 3.2. *Let M^n be a linear Weingarten hypersurface in $\mathbb{H}^{n+1}$, such that $R = aH + b$ for some a, $b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(b+1) \geq 0\,. \tag{3.2}$$

Then

$$|\nabla B|^2 \geq n^2 |\nabla H|^2\,. \tag{3.3}$$

Moreover, if the inequality (3.2) *is strict and equality holds in* (3.3) *on M^n, then H is constant on M^n.*

In the paper [15], Yau established the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold M^n: *if $\omega \in \Omega^{n-1}(M)$ is an integrable $(n-1)$-differential form on M^n, then there exists a sequence B_i of domains on M^n such that $B_i \subset B_{i+1}$, $M^n = \bigcup_{i \geq 1} B_i$ and*

$$\lim_{i \to +\infty} \int_{B_i} d\omega = 0\,.$$

Suppose that M^n is oriented by the volume element dM. If $\omega = \iota_X dM$ is the contraction of dM in the direction of a smooth vector field X on M^n, then Caminha (see Proposition 2.1 of [4]) obtained a suitable consequence of Yau's result, which is described below. In what follows, $\mathcal{L}^1(M)$ and $\mathrm{div}_M X$ stand, respectively, for the space of Lebesgue integrable functions and the divergence of a smooth vector field X on M^n.

Lemma 3.3. *Let X be a smooth vector field on the n-dimensional complete oriented Riemannian manifold M^n such that* $\mathrm{div}_M X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}_M X = 0$.

Now, we can proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1.
Let $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}\,. \tag{3.4}$$

Following Cheng-Yau [6], we consider an operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}\,. \tag{3.5}$$

Setting $f = nH$ in (3.5) and taking a local frame field $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (2.4) we obtain the following:

$$\begin{aligned}\Box(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{,ii}\,.\end{aligned}$$

Consequently, taking into account equation (2.9), we get

$$\Box(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2\,. \tag{3.6}$$

Now, we will introduce the following Cheng-Yau's modified operator

$$L = \Box - \frac{n-1}{2}a\Delta\,. \tag{3.7}$$

Let us choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$. Since $R = aH + b$, from (3.6) and (3.7) we have that

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2\,. \tag{3.8}$$

Thus, since from (2.1) we have that $R_{ijij} = \lambda_i\lambda_j - 1$, from (3.8) we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + n^2H^2 - n|B|^2 - |B|^4 + nH\sum_i \lambda_i^3\,. \tag{3.9}$$

Now, set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j\,.$$

Let $|\Phi|^2 = \sum_{i,j}\Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and

$$|\Phi|^2 = |B|^2 - nH^2\,. \tag{3.10}$$

With respect to the frame field $\{e_1, \dots, e_n\}$ on M^n, we have that $\Phi_{ij} = \mu_i\delta_{ij}$ and, with a straightforward computation, we verify that

$$\sum_i \mu_i = 0,\ \sum_i \mu_i^2 = |\Phi|^2 \text{ and } \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3\,. \tag{3.11}$$

Thus, using the Gauss equation (2.1) jointly with (3.11) in (3.9), we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 - n)\,. \tag{3.12}$$

By applying Lemmas 3.1 and 3.2, from (3.12) we have

$$L(nH) \geq |\Phi|^2\Big(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 - n\Big) = |\Phi|^2 P_H\big(|\Phi|\big)\,, \tag{3.13}$$

where

$$P_H\left(|\Phi|\right) = -|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 - n\,. \tag{3.14}$$

Since we are supposing that $H^2 \geq 1$, from (3.14) it is easy to verify that $P_H(|\Phi|)$ has two real roots $\mathcal{R}_H^-$ and $\mathcal{R}_H^+$ given by

$$\mathcal{R}_H^- = -\frac{1}{2}\sqrt{\frac{n}{n-1}}\left(\sqrt{n^2H^2 - 4(n-1)} + (n-2)H\right)$$

and

$$\mathcal{R}_H^+ = \frac{1}{2}\sqrt{\frac{n}{n-1}}\left(\sqrt{n^2H^2 - 4(n-1)} - (n-2)H\right).$$

Consequently, we have that

$$P_H\left(|\Phi|\right) = \big(|\Phi| - \mathcal{R}_H^-\big)\big(\mathcal{R}_H^+ - |\Phi|\big)\,. \tag{3.15}$$

Thus, from (1.1) and (3.15) we conclude that $P_H\left(|\Phi|\right) \geq 0$. Hence, from (3.13) we get

$$L(nH) \geq |\Phi|^2 P_H\left(|\Phi|\right) \geq 0\,. \tag{3.16}$$

On the other hand, from (3.5) and (3.7), we have no difficult to verify that

$$L(nH) = \operatorname{div}_M(P(\nabla H)), \tag{3.17}$$

where $P = \big(n^2H + \frac{n(n-1)}{2}a\big)I - nB$ and I denotes the identity in the algebra of smooth vector fields on M^n.

Moreover, since $R = aH + b$ and H is bounded on M^n, from equation (2.4) we have that B is bounded on M^n. Consequently, the operator P is bounded and, since we are also assuming that $|\nabla H| \in \mathcal{L}^1(M)$, we obtain that

$$|P(\nabla H)| \in \mathcal{L}^1(M). \tag{3.18}$$

Thus, from (3.16), (3.17), (3.18), we can apply Lemma 3.3 to obtain that $L(nH) = 0$ on M^n. Consequently, taking into account that all the inequalities that we have obtained are, in fact, equalities, from (3.8) we have that $|\nabla B|^2 = n^2|\nabla H|^2$. Since $(n-1)a^2 + 4n(b+1) > 0$, we can apply once more Lemma 3.2 to get that H is constant on M^n. Thus, it follows that $|\Phi|$ is also constant on M^n.

If $|\Phi| < \mathcal{R}_H^+$, then from (3.16) we have that $|\Phi| = 0$ and, hence, M^n is totally umbilical. If $|\Phi| = \mathcal{R}_H^+$, since equality holds in (3.1) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures, one of which is simple. Therefore, from the classification of the complete isoparametric hypersurfaces having at most two distinct principal curvatures due to Ryan [12] (see also Theorem 5.1 of [1]), we conclude that M^n is either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, if $R > 0$, or is isometric to $\mathbb{S}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, if $R < 0$, where $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = -1$. □

We close our paper by presenting the proof of Theorem 1.2.

Proof of Theorem 1.2.

Since the symmetric tensor ϕ defined in (3.4) is divergence-free, it follows from [6] that the operator $\square$ is self-adjoint relative to the L^2 inner product of M^n, that is,

$$\int_M f\square g = \int_M g\square f\,,$$

for any smooth functions f and g on M^n. Hence, the operator L is also self-adjoint relative to the L^2 inner product of M^n. Thus, from (3.16) we have that

$$0 = \int_M L(nH)dM = \int_M \left\{|\Phi|^2 P_H\left(|\Phi|\right)\right\} dM \geq 0\,. \tag{3.19}$$

Consequently, since we are assuming that $|\Phi| < \mathcal{R}_H^+$, from (3.19)) we have that $|\Phi| = 0$ on M^n. Therefore, M^n is totally umbilical and, hence, from the classification of the totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ we conclude that M^n must be isometric to $\mathbb{S}^n(r)$, for some $r > 0$. $\square$

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C.P. Aquino,
Departamento de Matemática,
Universidade Federal do Piauí,
64049-550 Teresina, Piauí, Brazil
E-mail: `cicero.aquino@ufpi.edu.br`

Corresponding author: H.F. de Lima,
Departamento de Matemática,
Universidade Federal de Campina Grande,
58429-970 Campina Grande, Paraíba, Brazil
E-mail: `henrique@dme.ufcg.edu.br`