ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 347–357

ON METRIZABILITY OF LOCALLY HOMOGENEOUS AFFINE 2-DIMENSIONAL MANIFOLDS

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Abstract. In [19] we proved a theorem which shows how to find, under particular assumptions guaranteeing metrizability (among others, recurrency of the curvature is necessary), all (at least local) pseudo-Riemannian metrics compatible with a given torsion-less linear connection without flat points on a two-dimensional affine manifold. The result has the form of an implication only; if there are flat points, or if curvature is not recurrent, we have no good answer in general, which can be also demonstrated by examples. Note that in higher dimension, the problem is not easy to solve.

Here we try to apply this apparatus to the two main types (A and B from [9], [1]) of torsion-less locally homogeneous connections defined in open domains of 2-manifolds. We prove that in dimension two a symmetric linear connection with constant Christoffels is metrizable if and only if it is locally flat. On the other hand, in the class of connections of type B there are even non-flat metrizable connections.

1. Introduction

To answer the question, under what conditions on initial data, a given linear connection on an n-dimensional manifold coincides with the Levi-Civita connection of some pseudo-Riemannian metric, is a difficult problem in general. Only the case $n = 2$ is easy.

For the Riemannian manifold, probably the most effective method was offered by O. Kowalski in [5], [6], see also [17], [16], [18]. In [19] we developed a method which might be helpful in deciding whether a given linear connection on a 2-dimensional manifold arises as a Riemannian connection of some pseudo-Riemannian metric. In the simplest case, for nowhere flat affine two-manifolds, we formulated necessary and sufficient conditions for local metrizability in [19], and in favourable case all compatible metrics were described in terms of the Ricci tensor. Our aim is to show here application for the class of locally homogeneous connections in affine 2-manifolds, [14], [9], [1].

2010 *Mathematics Subject Classification*: primary 53B05; secondary 53B20.
Key words and phrases: manifold, connection, metric.
The author was supported by the grant from Grant Agency of Czech Republic GAČR no. P201/11/0356 and by the project of specific university research of the Brno University of Technology, No. FAST-S-13-2888.
DOI: 10.5817/AM2013-5-347

We apply simple methods suggested in [19] in the case of the two main classes of locally homogeneous linear connections, called Type A and Type B in [9], [1], [14]. Such connections play the key role in the Classification Theorem for locally homogeneous symmetric linear connections on two-dimensional manifolds. Let us consider the case $n = 2$ in what follows.

1.1. **Affine two-manifolds.** First recall some notation. Let (M, ∇) be an affine manifold with a linear connection ∇, and let R denote the corresponding curvature tensor, $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for X, Y, Z from $\mathcal{X}(M)$.

As well known, in dimension two the curvature tensor R of type $(1,3)$ can be completely recovered from the Ricci tensor, $R(X,Y)Z = \mathrm{Ric}\,(Y,Z)X - \mathrm{Ric}\,(X,Z)Y$ for $X, Y, Z \in \mathcal{X}(M)$. In local coordinates, as far as $R^i_{hjj} = 0$ and $R^i_{hij} = R_{jh}$ for $j \neq i$, we have explicit formulas

$$\begin{aligned} R_{11} &= -R^2_{112} = R^2_{121}, \qquad & R_{21} &= -R^1_{121} = R^1_{112}\,, \\ R_{12} &= -R^2_{212} = R^2_{221}, \qquad & R_{22} &= -R^1_{221} = R^1_{212}\,. \end{aligned} \tag{1}$$

Since $\nabla R = \omega \otimes R$ is equivalent with $\nabla\,\mathrm{Ric} = \omega \otimes \mathrm{Ric}$ provided $n = 2$ we get:

Lemma 1. *The curvature tensor R of (M_2, ∇) is recurrent if and only if the Ricci tensor is recurrent, and $R = 0$ if and only if $\mathrm{Ric} = 0$.*

The induced map $Z \mapsto R(X,Y)Z$, $R(X,Y)\colon T_xM \to T_xM$ (for X, Y, Z from T_xM) is linear and skew-symmetric. The Ricci tensor Ric of type $(0,2)$ is a trace of the endomorphism, $\mathrm{Ric}\,(Y,Z) = \mathrm{Tr}\,\{X \mapsto R(X,Y)Z\}$, $X, Y, Z \in \mathcal{X}(M)$ (hence it carries less information than R in general) and the formula $\mathrm{Tr}\,R(Y,Z) = \mathrm{Ric}\,(Z,Y) - \mathrm{Ric}\,(Y,Z)$ holds, [11, p. 14].

1.2. **The Ricci tensor of a pseudo-Riemannian manifold.** For a pseudo-Riemannian manifold (M, g) consider the curvature tensor of type $(0,4)$ introduced (up to sign) by $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z, W)$; the relations $\tilde{R}(X,Y,Z,W) = \tilde{R}(Z,W,X,Y) = -\tilde{R}(Y,X,Z,W) = -\tilde{R}(X,Y,W,Z)$ hold. In local coordinates components R_{hijk} of $\tilde{R}$ and R^ℓ_{ijk} of R satisfy $R_{hijk} = R_{jkhi} = -R_{ihjk} = -R_{hikj}$, $R_{hijk} = g_{hs}R^s_{ijk}$, and $g^{\ell h}R_{hijk} = R^\ell_{ijk}$. We introduce also the Ricci tensor of type $(1,1)$ with components $R^i_j = g^i_s R_{sj}$. The scalar curvature ϱ is its trace, $\varrho = \mathrm{Tr}\,\mathrm{Ric} = R^s_s = g^{ij}R_{ij}$.

If $n = 2$ the formula $R_{hijk} = K(g_{hj}g_{ik} - g_{hi}g_{jk})$ holds ([3, p. 137]) and $R^i_{hjk} = \delta^i_j R_{kh} - \delta^i_k R_{jh}$ is satisfied ([10], [15]). We then check easily $R^i_{hjk} = K(\delta^i_k g_{hj} - \delta^i_j g_{hk})$, and further $\mathrm{Ric} = \frac{1}{2}\varrho\, g = Kg$ (the Ricci tensor of a 2-dimensional pseudo-Riemannian manifold is proportional to the metric tensor ([10, p. 263], [15, p. 101]) which shows that an affine 2-manifold is an Einstein space, and its Ricci tensor is necessarily *symmetric*.

Obviously for a nowhere flat (M_2, g), the Ricci tensor is *non-degenerate*. Moreover, any nowhere flat pseudo-Riemannian 2-manifold has a *recurrent curvature tensor* provided the sectional curvature does not vanish ([4, I, p. 280]):

Lemma 2. *The Ricci tensor of a nowhere flat 2-dimensional pseudo-Riemannian manifold is recurrent, and the corresponding 1-form is exact* (*gradient*)*.*

Proof. $R \neq 0$ is equivalent with $K(x) \neq 0$ on M (from continuity, K is either positive, or negative); $g = \alpha(x) \cdot \mathrm{Ric}$ with $\alpha(x) = \frac{1}{K(x)} \neq 0$, and $\nabla g = 0$. Obviously $\alpha(x) \cdot \mathrm{Ric}$ is parallel, and $\nabla \mathrm{Ric} = d(-\ln|\alpha|) \otimes \mathrm{Ric}$ holds, [19]. □

Since in the case $R = 0$, the tensor Ric is in fact also recurrent ($\omega = 0$), *recurrency is one of necessary conditions for metrizability of a two-manifold.*

1.3. **Metrizability of affine 2-manifolds.** We obtained two necessary conditions for local metrizability of a symmetric linear connection on a pseudo-Riemannian 2-manifold: the Ricci tensor must be symmetric, and must be also recurrent for some closed 1-form (proof of Lemma 2). Moreover, Ric may be degenerate only if $R = 0$ holds, and if this is the case then $\mathrm{Ric} = 0$. For the sake of global metrizability, the corresponding 1-form must be even exact. As well known, a flat connection is always globally metrizable; we can even prescribe a signature.

Now suppose that the Ricci tensor (or equivalently, the curvature) is non-vanishing in one fixed point $x_0 \in M$, and due to continuity, in some neighborhood of x_0 (note that the subset of non-flat points is open).

Theorem 1. *Let* (M_2, ∇) *be a 2-manifold with a symmetric linear connection such that the Ricci tensor is symmetric, recurrent, i.e.* $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$ *for some 1-form* ω*, and regular* ($|R_{ij}| \neq 0$)*. Then at least locally, there is a metric compatible with the connection.*

Proof. Let $x_0 \in M$. Since $|R_{ij}| \neq 0$ there is a pair (i, j) such that $R_{ij} \neq 0$ about x_0. Due to the recurrency and regularity, $d\omega = 0$. Hence in some neighborhood of x_0, there is a function f such that $\omega = df$. It can be checked that $e^{-f} \cdot \mathrm{Ric}$ is parallel about x_0. Therefore $g = e^{-f} \cdot \mathrm{Ric}$ is a local metric compatible with ∇ in a neighborhood of x_0. □

Theorem 2 ([19])**.** *Let* (M_2, ∇) *be a two-dimensional manifold with a symmetric linear connection and the curvature* R *non-zero everywhere. If the Ricci tensor of* ∇ *is regular, symmetric and satisfies* $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$ *where* $\omega = df$ *is an exact 1-form* (f *being some function*) *then* $g = e^{-f} \cdot \mathrm{Ric}$ *is a global metric tensor such that* $\nabla g = 0$.

Hence a now-here flat symmetric linear connection on M_2 is metrizable if and only if its Ricci tensor is symmetric, regular and recurrent with the corresponding 1-form being exact. If this is the case and $\nabla \mathrm{Ric} = df \otimes \mathrm{Ric}$ holds for some smooth function f then all global metrics corresponding to the connection ∇ form a 1-parameter family of homothetic metrics

$$g_b = \exp(-f + b) \cdot \mathrm{Ric}\,, \qquad b \in \mathbb{R}\,, \tag{2}$$

(i.e. they differ from each other up to a scalar multiple, and each arises from the Ricci tensor as a multiple by a smooth function).

2. Locally homogeneous affine 2-manifolds

As well known, a connection ∇ in a domain U of M_2 is given uniquely by a family of components (Christoffel symbols) which are functions $A(u,v),\dots,F(u,v)$ in two variables u,v where (u,v) are local coordinates in U,

$$\begin{aligned} &\nabla_{\partial_u}\partial_u = A\partial_u + B\partial_v\,, \qquad &&\nabla_{\partial_u}\partial_v = C\partial_u + D\partial_v\,, \\ &\nabla_{\partial v}\partial u = \tilde{C}\partial_u + \tilde{D}\partial_v\,, \qquad &&\nabla_{\partial_v}\partial_v = E\partial_u + F\partial_v\,; \end{aligned} \tag{3}$$

to simplify the usual notation we denoted $\partial_u = \frac{\partial}{\partial u}$, similarly for v, $\Gamma^1_{11} = A$, $\Gamma^2_{11} = B$, $\Gamma^1_{12} = C$, $\Gamma^2_{12} = D$, $\Gamma^1_{21} = \tilde{C}$, $\Gamma^2_{21} = \tilde{D}$, $\Gamma^1_{22} = E$, $\Gamma^2_{22} = F$. An affine manifold (M,∇) is called (affine) *locally homogeneous* if for any points p, q in M there are neighborhoods $U \ni p$, $V \ni q$ and an affine transformation $f\colon (U,\nabla_{|U}) \to (V,\nabla_{|V})$ sending p into q, [13].

A seemingly easy problem, to classify all locally homogeneous torsion-less linear connections in plain domains, was solved only recently, namely by a direct method in [14], by a group-theoretical method in [9], and in a more general setting, for connections with arbitrary torsion, in [1]; the proofs are based on the theory of Lie algebras of vector fields, [12]. Recall the result of B. Opozda from [14] which was a motivation for our contribution.

Theorem 3. *Let M_2 be an affine locally homogeneous 2-manifold with a symmetric connection. Then for any point $p \in M$ there is a neighborhood U_p such that either the restriction of the given connection is the Riemannian connection of a pseudo-Riemannian manifold of constant curvature, or there is a system (u,v) of local coordinates in U_p such that the restriction is ∇ with component expression* (3) *where $A,\dots,F$ are constants* (*called connection of the Type A, with constant Christoffels*)*, or the restriction is one of the symmetric connections $\hat{\nabla}$ given by the formulas*

$$\textbf{TYPE B:} \qquad \begin{aligned} \hat{\nabla}_{\partial_u}\partial_u &= u^{-1}\left(A\partial_u + B\partial_v\right)\,, \\ \hat{\nabla}_{\partial_u}\partial_v &= u^{-1}\left(C\partial_u + D\partial_v\right)\,, \\ \hat{\nabla}_{\partial_v}\partial_v &= u^{-1}\left(E\partial_u + F\partial_v\right) \end{aligned} \tag{4}$$

(*called connections of the Type B*) *where $A,\dots,F$ are real constants, not all of them are zero.*

Formulation of the Theorem 3 might avoke an incorrect impression that it somehow separates metric connections from connections of Type A and Type B. A more general and more precisely formulated version can be found in [1, Theorem 2 (Classification Theorem), pp. 2,3] for connections of arbitrary torsion. Since our aim is to examine metric connections let us consider now only torsion-free (symmetric, $C = \tilde{C}$, $D = \tilde{D}$) connections in what follows. Components of the Ricci tensor of a symmetric connection read:

$$
\begin{aligned}
R_{11} &= \mathrm{Ric}\,(\partial_u, \partial_v) = B(F - C) + D(A - D) + B_v - D_u\,, \\
R_{12} &= \mathrm{Ric}\,(\partial_u, \partial_v) = CD - BE + D_v - F_u\,, \\
R_{21} &= \mathrm{Ric}\,(\partial_v, \partial_u) = CD - BE + C_u - A_v\,, \\
R_{22} &= \mathrm{Ric}\,(\partial_v, \partial_v) = E(A - D) + C(F - C) + E_u - C_v
\end{aligned}
\tag{5}
$$

where the lower index means partial derivative according to the corresponding variable; Ric need not be symmetric, even for a symmetric connection.

2.1. **Type A connections, with constant Christoffels.** Let us examine metrizability of symmetric connections with constant Christoffels in open domains of 2-manifolds. We prove that a complete answer can be given as follows: *In dimension two, a symmetric linear connection with constant Christoffels is metrizable if and only if it is locally flat.* Though the metrizability problem means to solve a system of differential equations in general [2], in our case it is sufficient to solve a system of *algebraic* equations.

In the case of a connection with constant Christoffels defined in $U \subset \mathbb{R}^2[u, v]$, the curvature tensor R as well as the Ricci tensor are constant, the components being

$$
\begin{aligned}
R_{11} &= B(F - C) + D(A - D)\,, \\
R_{12} = R_{21} &= CD - BE \\
R_{22} &= E(A - D) + C(F - C)\,.
\end{aligned}
\tag{6}
$$

Hence Ric is always symmetric, and the shape of R_{12} suggests that vanishing or non-vanishing of one of components B, C, D, E might have consequences. In what follows we distinguish from technical reasons the cases $D = 0$ and $D \neq 0$ (e.g. to be able to control our considerations by computations in Maple, though D is not a preferable coefficient by nature, we could start in fact with something else, e.g. with C or E).

Lemma 3. *A symmetric linear connection ∇ in $M_2 = \mathbb{R}^2[u, v]$ with constant Christoffels is locally flat just in the following cases:*

(a) $D = B = 0$ *and* A, C, E, $F \in \mathbb{R}$ *are related by* $AE - C^2 + CF = 0$*;*

(b) $D \neq 0$, $C = BE/D$, *and* A, B, E, $F \in \mathbb{R}$ *are related by* $D^2A - EB^2 + DBF - D^3 = 0$.

Proof. Suppose Ric $= 0$ holds. Let us distinguish the cases $D = 0$, $D \neq 0$. Let $D = 0$. Then the condition $R_{12} = 0$ implies $BE = 0$. If $E = 0$ as well then we get $B(F - C) = 0 = C(F - C)$ from (6) and in the next step either $B = C = 0$, we have the subcase $B = C = D = E = 0$, A, $F \in \mathbb{R}$, or $F = C$, we obtain the subcase $D = E = 0$, $F = C$, A, B, $C \in \mathbb{R}$; the condition from (a) obviously holds. On the other hand if $B = 0$ then the equation $C(F - C) + AE = 0$ is satisfied, and we have the case (a) again.

Assume $D \neq 0$. We get $C = BE/D$ from $R_{12} = 0$. The other two components of the Ricci tensor vanish if and only if the following holds:

$$B(F - BE/D) + D(A - D) = 0\,, \quad (BE/D)(F - BE/D) + E(A - D) = 0\,.$$

This system is equivalent to the unique third-order algebraic equation $D^2A - EB^2 + DBF - D^3 = 0$, hence (b) is checked. □

The class of linear connections with constant Christoffels contains a three-parameter family of metrizable locally flat connections. To show that besides locally flat connections, among symmetric connections with constant Christoffels on 2-manifolds there are no other metrizable connections we can use Theorem 1. So let us examine recurrency of Ric as a necessary condition for metrizability. Constant components of the covariant derivative $\nabla \operatorname{Ric}$ are

$$\begin{aligned} -R_{11;1} &= 2(AR_{11} + BR_{12})\,, & -R_{11;2} &= 2(CR_{11} + DR_{12})\,,\\ -R_{12;1} &= CR_{11} + (A + D)R_{12} + BR_{22}\,, & -R_{12;2} &= ER_{11} + (C + F)R_{12} + DR_{22}\,,\\ -R_{22;1} &= 2(CR_{12} + DR_{22})\,, & -R_{22;2} &= 2(ER_{12} + FR_{22})\,. \end{aligned}$$

Hence a symmetric connection with constant Christoffels on M_2 is recurrent if and only if there exist constants φ_1, φ_2 such that the following system holds:

$$\begin{aligned} &(\varphi_1 + 2A)R_{11} + 2BR_{12} = 0\,, \quad CR_{11} + (\varphi_1 + A + D)R_{12} + BR_{22} = 0\,,\\ &2CR_{12} + (\varphi_1 + 2D)R_{22} = 0\,, \quad (\varphi_2 + 2C)R_{11} + 2DR_{12} = 0\,,\\ &ER_{11} + (\varphi_2 + C + F)R_{12} + DR_{22} = 0\,, \quad 2ER_{12} + (\varphi_2 + 2F)R_{22} = 0\,. \end{aligned} \tag{7}$$

Lemma 4. *If the Ricci tensor of a symmetric linear connection on M_2 is recurrent, has constant Christoffels and satisfies $R_{12} = 0$ then either* Ric *vanishes or is degenerate.*

Proof. Under our assumptions, the system (7) guaranteeing recurrency reduces to the system of algebraic equations

$$\begin{aligned} (\varphi_1 + 2A)R_{11} &= 0, & (\varphi_2 + 2C)R_{11} &= 0\,,\\ (\varphi_1 + 2D)R_{22} &= 0, & (\varphi_2 + 2F)R_{22} &= 0\,. \end{aligned} \tag{8}$$

Suppose that $R_{11} \cdot R_{22} \neq 0$. Then necessarily $A = D$ and $C = F$, and by (6) $R_{11} = R_{22} = 0$, a contradiction. □

Lemma 5. *If the Ricci tensor of a symmetric linear connection with constant Christoffels on M_2 is recurrent, non-vanishing and non-degenerate then the components must satisfy $R_{11} \cdot R_{22} \neq 0$.*

Proof. If $R_{11} = 0$ then by (7), $BR_{12} = DR_{12} = 0$, hence either $R_{12} = 0$ or $B = D = 0$ which gives $R_{12} = 0$ again, hence the matrix is singular and Ric degenerates. Similarly for $R_{22} = 0$. □

Corollary 1. *If a symmetric connection with constant Christoffels on M_2 and non-vanishing Ricci tensor should have* Ric *recurrent and non-degenerate then necessarily $R_{11} \cdot R_{12} \cdot R_{22} \neq 0$.*

Theorem 4. *If (M,∇) is an affine 2-manifold with constant Christoffels such that the Ricci tensor is recurrent and all its components are non-vanishing on M then* Ric *is degenerate.*

Proof. Since $R_{11}\neq 0$ we get from (7)

$$\varphi_1=-2A-2B\frac{R_{12}}{R_{11}},\qquad \varphi_2=-2C-2D\frac{R_{12}}{R_{11}}\,,$$

$$-2AR_{22}-2B\frac{R_{12}R_{22}}{R_{11}}+2CR_{12}+2DR_{22}=0\,,$$

$$-2CR_{22}-2D\frac{R_{12}R_{22}}{R_{11}}+2ER_{12}+2FR_{22}=0\,.$$

We obtain

$$B\frac{R_{12}R_{22}}{R_{11}}=CR_{12}+(D-A)R_{22}\,,\quad D\frac{R_{12}R_{22}}{R_{11}}=ER_{12}+(F-C)R_{22}=0\,.$$

First suppose that $BD\neq 0$ and show that Ric is degenerate. Indeed, we get

$$\big(-D(D-A)+B(F-C)\big)R_{22}-(CD-BE)R_{12}=0\,.$$

Let us plug for R_{ij} from the expressions: $0=R_{11}\cdot R_{22}-R_{12}^2=\det(R_{ij})$.

Now suppose $B=0$. Then

$$R_{11}=D(D-A)\,,\qquad R_{12}=CD\,,\qquad R_{22}=C(F-C)+E(A-D)$$

and necessarily $D\neq 0$, $C\neq 0$ (otherwise $R_{12}=0$, and the Ricci tensor would vanish by Lemma 4). From the first of the above equations, $(2A+\varphi_1)R_{11}=0$ and due to $R_{11}\neq 0$, $\varphi_1=-2A$. Hence plugging into the second equation we get $CD(D-A+C)=0$ which is equivalent to $C=A-D$, due to $CD\neq 0$. This yields $R_{11}=R_{22}$. Now plugging into the third equation we have $(A-D)(R_{12}-R_{22})=0$. Since $A-D\neq 0$ (otherwise $CD=0$, a contradiction) we obtain $R_{12}=R_{22}=R_{11}$. It follows $\det(R_{ij})=0$.

Finally, if $D=0$ then

$$R_{11}=B(F-C),\qquad R_{12}=-BE\,,\qquad R_{22}=C(F-C)+AE$$

and necessarily $BE\neq 0$, $F\neq C$. We get $\varphi_2=-2C$,

$$(F-C)R_{22}+ER_{12}=0\,,\qquad R_{22}=-\frac{E}{F-C}R_{12}\,,$$

$$(F-C)R_{12}+ER_{11}=0\,,\qquad R_{11}=-\frac{F-C}{E}R_{12}\,.$$

It follows $R_{11}\cdot R_{22}=R_{12}^2$, hence $\det(R_{ij})=0$ again. □

Corollary 2. *In the above notation, exactly the following choices of constants give rise to a metrizable connection ∇ on M_2:*

$B=C=D=E=0,\ \ A,F\in\mathbb{R}$;

$D=E=0,\ F=C,\ \ A,B,C\in\mathbb{R}$;

$B = D = 0$, $A, C, E, F \in \mathbb{R}$ *satisfying* $AE - C^2 + CF = 0$;

$D \neq 0$, $C = BE/D$, $A, B, E, F \in \mathbb{R}$ *satisfying* $D^2A - EB^2 + DBF - D^3 = 0$.

Theorem 5. *Locally flat connections are exactly the only metrizable connections among all symmetric connections with constant Christoffels defined on open domains of affine 2-manifolds.*

Note that the above result corresponds to the results obtained in [1].

2.2. **Connections of the Type B.** Now let us pay attention to metrizability of a symmetric linear connection $\hat{\nabla}$ of the form (4) defined in $\mathbb{R}^2[u,v]\backslash\{(0,v); v \in \mathbb{R}\}$. It appears that the situation is quite different: there exist classes of metrizable non-flat connections which are of Type B.

Let us try the direct method based on Theorem 1. Let $\hat{\Gamma}^k_{ij}$ be components of the connection $\hat{\nabla}$ (i.e. $\tilde{\Gamma}^1_{11} = A/u = \hat{A}$, $\hat{\Gamma}^2_{11} = B/u = \hat{B}$ etc.) and denote by R_{ij} components of the Ricci tensor $\hat{\mathrm{Ric}}$ of $\hat{\nabla}$. We obtain

$$
\begin{aligned}
R_{11} &= u^{-2}[B(F-C) + D(1+A-D)]\,,\\
R_{12} &= u^{-2}[CD - BE + F]\,,\\
R_{21} &= u^{-2}[CD - BE - C]\,,\\
R_{22} &= u^{-2}[E(A-D-1) + C(F-C)]\,.
\end{aligned}
\tag{9}
$$

Components of $\hat{\nabla}\hat{\mathrm{Ric}}$ are

$$
\begin{aligned}
-R_{11;1} &= u^{-1}\big((2A+2)R_{11} + B(R_{12}+R_{21})\big)\,,\\
-R_{12;1} &= u^{-1}\big(CR_{11} + (A+D+2)R_{12} + BR_{22}\big)\,,\\
-R_{21;1} &= u^{-1}\big(CR_{11} + (A+D+2)R_{21} + BR_{22}\big)\,,\\
-R_{22;1} &= u^{-1}\big(C(R_{12}+R_{21}) + (2D+2)R_{22}\big)\,,\\
-R_{11;2} &= u^{-1}(2CR_{11} + 2DR_{12})\,,\\
-R_{12;2} &= u^{-1}\big(ER_{11} + (C+F)R_{12} + DR_{22}\big)\,,\\
-R_{21;2} &= u^{-1}\big(ER_{11} + (C+F)R_{21} + DR_{22}\big)\,,\\
-R_{22;2} &= u^{-1}(2ER_{12} + 2FR_{22})\,.
\end{aligned}
$$

In what follows suppose that the connection $\hat{\nabla}$ has symmetric Ricci tensor, equivalently $F = -C$.

Provided symmetry of Ricci is guaranteed, necessary and sufficient conditions for recurrency of $\hat{\nabla}$ read: the system of algebraic equations

$$
\begin{aligned}
\Big(\varphi_1 + \frac{2A+2}{u}\Big)R_{11} + \frac{2B}{u}R_{12} &= 0\,,\\
\frac{C}{u}R_{11} + \Big(\varphi_1 + \frac{A+D+2}{u}\Big)R_{12} + \frac{B}{u}R_{22} &= 0\,,\\
\frac{2C}{u}R_{12} + \Big(\varphi_1 + \frac{2D+2}{u}\Big)R_{22} &= 0\,,\\
\Big(\varphi_2 + \frac{2C}{u}\Big)R_{11} + \frac{2D}{u}R_{12} &= 0\,,\\
\frac{E}{u}R_{11} + \varphi_2 R_{12} + \frac{D}{u}R_{22} &= 0\,,\\
\frac{2E}{u}R_{12} + \Big(\varphi_2 - \frac{2C}{u}\Big)R_{22} &= 0
\end{aligned}
\tag{10}
$$

must be solvable for functions $\varphi_1(u,v)$, $\varphi_2(u,v)$.

Lemma 6. *Let the Ricci tensor* $\hat{\mathrm{Ric}}$ *of a symmetric connection* $\hat\nabla$ *of Type B in* $M_2 = \mathbb{R}^2[u,v]\backslash\{(0,v); v \in \mathbb{R}\}$ *be symmetric and recurrent. Let* $R_{12} = 0$ *on* M_2 *and* $R_{11} \cdot R_{22} \neq 0$ *on* M_2. *Then* $\hat{\mathrm{Ric}}$ *is non-degenerate,* $B = C = F = 0$, $A = D \neq 0$, $E \neq 0$, $R_{11} = u^{-2}A$, $R_{22} = -u^{-2}E$, *and* $\hat\nabla\hat{\mathrm{Ric}} = \omega \otimes \hat{\mathrm{Ric}}$, $\omega = df$ *where* $\omega = -2u^{-1}(A+1)du$ *and* $f = -2(A+1)\ln|u|^2 + b$, $b \in \mathbb{R}$. *The connection* $\hat\nabla$ *is metrizable.*

Proof. If $R_{12} = 0$ then the system (10) is simplified:

$$
\Big(\varphi_1 + \frac{2A+2}{u}\Big)R_{11} + \frac{2B}{u}R_{12} = 0\,, \quad \frac{2C}{u}R_{12} + \Big(\varphi_1 + \frac{2D+2}{u}\Big)R_{22} = 0\,,
$$
$$
\Big(\varphi_2 + \frac{2C}{u}\Big)R_{11} + \frac{2D}{u}R_{12} = 0\,, \quad \frac{E}{u}R_{11} + \varphi_2 R_{12} + \frac{D}{u}R_{22} = 0\,,
$$
$$
\frac{C}{u}R_{11} + \Big(\varphi_1 + \frac{A+D+2}{u}\Big)R_{12} + \frac{B}{u}R_{22} = 0\,, \quad \frac{2E}{u}R_{12} + \Big(\varphi_2 - \frac{2C}{u}\Big)R_{22} = 0\,.
$$

If $R_{11} \cdot R_{22} \neq 0$ we obtain $\varphi_1 = -2u^{-1}(A+1) = -2u^{-1}(D+1)$, hence $A = D$, moreover, $\varphi_2 = 0$, $C = 0$, $B = 0$, $R_{11} : R_{22} = -D : E$. The rest follows. $\square$

From Theorem 1 we deduce

Theorem 6. *Any choice of real constants* $B = C = F = 0$, $A = D \neq 0$, $E \neq 0$ *defines in* (*a neighborhood* U *of*) $M_2 = \mathbb{R}^2[u,v]\backslash\{(0,v); v \in \mathbb{R}\}$ *a non-flat* (*locally*) *metrizable connection.*

The choice $A = D = -1$, $E = 1$, $B = C = F = 0$ gives a non-flat metrizable connection $\hat\nabla$, with (symmetric, recurrent and non-degenerate) Ricci tensor $\mathrm{Ric} = -u^{-2}(du \otimes du + dv \otimes dv)$ ($\omega = 0$), compatible with the Riemannian metric $g = -\mathrm{Ric} = u^{-2}(du \otimes du + dv \otimes dv)$ of constant negative curvature $K = -1$. Hence (locally) $\hat\nabla$ is the Levi-Civita connection of the standard hyperbolic plane. On the other hand, the constants $A = D = E = 1$, $B = C = F = 0$ define by (4) the Levi-Civita connection of the Lorentzian metric $g = \mathrm{Ric} = u^{-2}(du \otimes du - dv \otimes dv)$ of constant positive curvature $K = 1$. The result is in accordance with [1].

Remark 1. As concerns the 2-dimensional sphere, it is neither of Type A, nor of Type B, see [1].

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