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YAMABE OPERATOR VIA BGG SEQUENCES

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Abstract. We show that the conformally invariant Yamabe operator on a complex conformal manifold can be constructed as a first BGG operator by inducing from certain infinite-dimensional representation.

1. Introduction

The general problem is to find all natural differential operators between sections of natural vector bundles on some geometric category (e.g. on conformal or projective manifolds). The naturality implies that on an open subset of a homogeneous space (which is the canonical model for the 'geometry' under consideration) the natural vector bundles are the homogeneous vector bundles and natural differential operators are invariant differential operators. However not all invariant differential operators are natural. The counterexample is a power of the Laplace operator (see [9] and [8]).

It was proven in [4] that there is a large class of invariant operators which are natural. Among these so-called BGG operators are various interesting 'geometric' operators whose kernels give e.g. conformal Killing tensor fields. Consequently, the BGG operators were much studied – see e.g. [2] for recent applications. The construction of these operators was much simplified by Calderbank and Diemer in [1].

The construction of BGG operators starts with a general parabolic geometry $(\mathcal{G}, \omega)$ over a manifold M modeled on a parabolic pair (G, P) and a finite-dimensional $(\mathfrak{g}, P)$-representation $\mathbb{V}$ and its output is a sequence of differential operators between associated bundles associated to representations whose weight is given by the Kostant formula [11]. In the flat case this actually yields a complex, which computes the sheaf cohomology of the sheaf of constant sections of the bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$. The Cartan connection of the geometry induces an affine connection on $\mathcal{V}$ and one of the crucial features of BGG operators is that the kernel of the first operator in the sequence contains the so called normal solutions, which arise in an explicit way from parallel sections of this affine connection. In particular, in the flat case all solutions are normal and hence we can realize all solutions of first BGG

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operator as parallel sections of $\mathcal{V}$ with respect to the affine connection induced by ω. In this article we show that the BGG construction by Calderbank and Diemmer basically works for a certain infinite-dimensional representation in which case it yields the conformally invariant Yamabe operator. The Yamabe operator is just the Laplace – Beltrami operator with a scalar curvature term

$$Y = \Delta_{LP} - \frac{n-2}{4(n-1)} R \,.$$

Let us denote the Levi part of P by L and the Lie algebra of the nilradical by $\mathfrak{p}_+$. The original construction of Calderbank and Diemer shows actually much more than the existence of BGG operators. In fact, they modify Hodge theory for a finite-dimensional representation $\mathbb{V}$ developed in [11] in order to get a homotopy transfer between twisted deRham sequence on $\mathcal{V}$ and bundles induced from Lie algebra homology of $\mathfrak{p}_+$ with values in $\mathbb{V}$. This homotopy transfer is basically P-equivariant modification of the homotopy transfer data coming from the L-equivariant Hodge decomposition of $\mathbb{V}$. This modification is not possible without introducing differential terms to the algebraic Hodge Laplacian of [11] and hence the resulting BGG operators can have order higher than one. To be a little bit more concrete, one has from [11] that for a certain algebraic operator $\Box$ there is a Hodge decomposition $\mathbb{V} = \ker \Box \oplus \operatorname{im} \Box$ which is L-equivariant and one would like to extend this to a Hodge decomposition of the sections of $\mathcal{V}$. To this end, one introduces in a straightforward way a differential operator $\Box_{\mathfrak{g}}$ and tries to show that $\Gamma(M, \mathcal{V}) = \operatorname{im} \Box_{\mathfrak{g}} \oplus \ker \Box_{\mathfrak{g}}$. In particular, the operator $\Box_{\mathfrak{g}}$ must be invertible on its own image.

In order to get the Yamabe operator as a resulting BGG: operator, it is necessary to consider an infinite-dimensional representation, since it is known that the kernel of the Yamabe operator is inifite-dimensional on $\mathbb{R}^{p,q}$ and the kernel of the first BGG operator has, in the flat case, the same dimension as $\mathbb{V}$. The representation we will consider is a formal globalization of a unitarizable highest weight module. Unitarizable highest weight module is a module which is both a $(\mathfrak{g}, K)$-module and a highest weight module for $\mathfrak{g}$. The unitarizability of the module ensures that the Hodge decomposition of $\mathbb{V}$ is still valid, while the formal globalization enables us to proceed with the proof of invertibility of $\Box_{\mathfrak{g}}$. The original unitarizable highest weight representation is basically a certain subspace of polynomials, whereas its formal globalization is a subspace of formal power series.

2. Modules, homology & formal globalization

The unitarizable highest weight modules occur only for noncompact Hermitian symmetric pairs and hence we have to restrict the signature. Throughout this article we will use symbol G to denote the group $SO_0(2,p)$. The maximal compact subgroup of this group is $SO(2) \times SO(p)$ and we will denote it by K. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras and let $\mathfrak{g}$ and $\mathfrak{k}$ denote their complexifications. The homogeneous manifold G/K is a noncompact Hermitian space. The Cartan decomposition gives us $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{r}$ and upon complexification we get a splitting of $\mathfrak{r}_{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{p}_+$ into eigenspaces of the complex structure that is defined on the tangent space $T_{eK}G/K \simeq \mathfrak{r}$. Both algebras $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}_+$ and $\overline{\mathfrak{p}} = \mathfrak{k} \oplus \mathfrak{p}_-$ are parabolic subalgebras of $\mathfrak{g}$. Moreover their nilradicals $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are not only nilpotent but even abelian. By P and $\overline{P}$ we denote the corresponding parabolic subgroups of $G_{\mathbb{C}} = SO(p+2, \mathbb{C})$. The homogoneous space $G_{\mathbb{C}}/P$ is diffeomorphic to compact Hermitian symmetric space and $\mathfrak{p}_-$ is naturally mapped via composition of exponential map and projection to an open and dense subset of this compact manifold. The so called Harish-Chandra embedding gives us a realization of the noncompact dual G/K as an orbit in this embedded $\mathfrak{p}_-$. This realizes G/K as a bounded Hermitian symmetric domain in $\mathfrak{p}_-$ and shows where does the Hermitian structure on G/K comes from. We will denote by $K_{\mathbb{C}}$ the complexification of K.

We will use notation ϵ^i, $i = 1, \dots, 2p$ for the elements of the basis of the nilradical $\mathfrak{p}_+$ and by e_i we will denote the dual basis defined by the Killing form. The elements e_i then span the nilradical of the opposite parabolic subalgebra.

There is a choice of a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \leq \mathfrak{k}$. Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$ and let Δ_c denote the set of roots of $(\mathfrak{k}, \mathfrak{h})$. We call elements of Δ_c the compact roots and the remaining roots in $\Delta_n = \Delta \setminus \Delta_c$ are called noncompact. We define the positive roots Δ^+ in such a way that elements of $\Delta_n^+ = \Delta^+ \cap \Delta_n$ span $\mathfrak{p}_-$[1]. We denote the positive compact roots by $\Delta_c^+ = \Delta_c \cap \Delta^+$. By ω_i we denote the i-th fundamental weight in the standard ordering.

Let λ be a Δ_c^+ dominant and integral weight and denote by $F(\lambda)$ the finite dimensional irreducible $\mathfrak{k}$-module. We extend any irreducible representation of $K_{\mathbb{C}}$ to P and to $\overline{P}$ by letting $\mathfrak{p}_+$ and $\mathfrak{p}_-$ act trivially. The generalized Verma module $M(\lambda)$ is defined as $M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\overline{\mathfrak{p}})} F(\lambda)$. It is well known and easy to prove that $M(\lambda)$ contains a maximal nontrivial submodule and we denote by $L(\lambda)$ the corresponding irreducible quotient of $M(\lambda)$. Since the nilradical $\mathfrak{p}_-$ of $\overline{\mathfrak{p}}$ is abelian, we have that $M(\lambda) \simeq S(\mathfrak{p}_+) \otimes F(\lambda)$ as $K_{\mathbb{C}}$ representations, where $S(\mathfrak{p}_+)$ is the symmetric algebra over the Lie algebra $\mathfrak{p}_+$.

There is a distinguish element in the center of $\mathfrak{k}$ called *grading element* which acts by zero on $\mathfrak{k}$, by 1 on $\mathfrak{p}_+$ and by -1 on $\mathfrak{p}_-$. This elements acts by a scalar on any irreducible representation of K. We call this scalar the *geometric weight*. In the case of $M(\lambda)$, the geometric weight corresponds to the polynomial degree shifted by the weight of $F(\lambda)$.

The chain space of Lie algebra homology $C_k(\mathfrak{p}_+, \mathbb{V})$ of the algebra $\mathfrak{p}_+$ with values in $\mathbb{V}$ is defined as $\Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$. The Lie algebra homology differential ∂^* : $C_{k+1}(\mathfrak{p}_+, \mathbb{V}) \to C_k(\mathfrak{p}_+, \mathbb{V})$ is defined for a general nilpotent subalgebra $\mathfrak{p}_+$ by

$$\begin{aligned}\partial^*(Z_0 \wedge \cdots Z_k \otimes v) = &\sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge Z_k \otimes Z_i \cdot v + \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge \widehat{Z_j} \wedge \cdots \wedge Z_k \otimes v\,,\end{aligned}$$

[1]One would expect that the positive roots would span $\mathfrak{p}_+$. However, we choose this condition in order to be consistent with cohomological formulas in [7]. Alternatively, one could work with lowest weight modules instead.

where $Z_i \in \mathfrak{p}_+$ for $i = 0, \dots, k$. Since our algebra $\mathfrak{p}_+$ is abelian, the second term in the sum is zero. The Lie algebra homology differential is P-equivariant.

The k^{th} chain space $C^k(\mathfrak{p}_-, \mathbb{V})$ of the Lie algebra cohomology $H^\bullet(\mathfrak{p}_-, \mathbb{V})$ of $\mathfrak{p}_-$ with coefficients in $\mathbb{V}$ is the space of antisymmetric linear mappings from the k^{th} tensor power of $\mathfrak{p}_-$ to $\mathbb{V}$. The Lie algebra cohomology differential is defined by the following general formula

$$
\begin{aligned}
(\partial\,\psi)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \cdot \psi(X_0, \dots, \widehat{X_i}, \dots, X_l) + \\
+ \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k)\,,
\end{aligned}
$$

where $\psi \in \operatorname{Hom}(\Lambda^k \mathfrak{p}_-, \mathbb{V})$ and $X_i \in \mathfrak{p}_-$ for $i = 0, \dots, k$. Again, we can forget the second term in our case. We can identify $C^k(\mathfrak{p}_-, \mathbb{V}) = \operatorname{Hom}(\Lambda^k \mathfrak{p}_-, \mathbb{V})$ with $\Lambda^k \mathfrak{p}_-{}^* \otimes \mathbb{V}$. Since the Killing form induces an isomorphism $\mathfrak{p}_-{}^* \simeq \mathfrak{p}_+$, we can consider the Lie algebra cohomology differential ∂ as an operator on the chain spaces of Lie algebra homology $\partial : \Lambda^k \mathfrak{p}_-{}^* \otimes V \to \Lambda^{k+1} \mathfrak{p}_-{}^* \otimes V$. After these identifications we get the formula

$$
\partial(Z_1 \wedge \dots \wedge Z_k \otimes v) = \sum_{i=1}^{2p} \epsilon^i \wedge Z_1 \wedge \dots \wedge Z_k \otimes e_i \cdot v\,.
$$

The Lie algebra cohomology differential is $\overline{P}$-equivariant. In particular, both ∂ and ∂^* are $K_{\mathbb{C}}$-equivariant and consequently they preserve the geometric weight.

The Kostant Laplacian is defined on each $C_k(\mathfrak{p}_+, \mathbb{V})$ as $\Box = \partial^*\,\partial + \partial\,\partial^*$. It was proven in [11] that for a finite-dimensional $\mathfrak{g}$-representation $\mathbb{V}$ there is a positive definite scalar product on $C_\bullet(\mathfrak{p}_+, \mathbb{V})$ with respect to which are ∂ and ∂^* adjoint. It follows that there is a direct sum *Hodge decomposition* of $K_{\mathbb{C}}$-modules $C_\bullet(\mathfrak{p}_+, \mathbb{V}) = \operatorname{im}\partial \oplus \ker\Box \oplus \operatorname{im}\partial^*$ and moreover $\ker\partial^* = \ker\Box \oplus \operatorname{im}\partial^*$ and $\ker\partial = \ker\Box \oplus \operatorname{im}\partial$. It follows that $H_\bullet(\mathfrak{p}_+, \mathbb{V}) \simeq \ker\Box \simeq H^\bullet(\mathfrak{p}_-, \mathbb{V})$.

The authors of [10] proved that if the representation $\mathbb{V}$ is a unitarizable $(\mathfrak{g}, K)$-module, then the Hodge decomposition of $C_\bullet(\mathfrak{p}_+, \mathbb{V})$ is still valid.

Inspecting the classification of unitarizable highest weight modules in [6], we see that we can take for $\mathbb{V}$ the module $L(\lambda)$ with $\lambda = (2-n)\omega_1$ in the case $p = 2n-2$ and $\lambda = (3/2 - n)\omega_1$ in the case $p = 2n-1$. From now on, when we write $L(\lambda)$ we will mean either $L((2-n)\omega_1)$ or $L((\frac{3}{2} - n)\omega_1)$ depending on the value of $p = \dim G - 2$.

The cohomology groups $H^i(\mathfrak{p}_-, L(\lambda))$ can be computed from the formula given in [7]. We get in the even case

$$
\begin{aligned}
H^0(\mathfrak{p}_-, L((2-n)\omega_1)) &= F((2-n)\omega_1) \\
H^1(\mathfrak{p}_-, L((2-n)\omega_1)) &= F(-n\omega_1) \\
H^i(\mathfrak{p}_-, L((2-n)\omega_1)) &= 0 \text{ for } i \geq 2
\end{aligned}
$$

and in the odd case

$$H^0(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) = F((\frac{3}{2}-n)\omega_1)$$
$$H^1(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) = F((-\frac{1}{2}-n)\omega_1)$$
$$H^i(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) = 0 \quad \text{for } i \geq 2.$$

The problem is that $L(\lambda)$ is not a P-representation. This is not surprising because the generalized Verma module $M(\lambda)$ was induced from a $\overline{P}$-representation. Let $L(\lambda) = \bigoplus_{\mu\in\widehat{K_{\mathbb{C}}}} L_\mu$ be the decomposition of $L(\lambda)$ into $K_{\mathbb{C}}$-types. Each L_μ is contained in some $S^k(\mathfrak{p}_+, F(\lambda))$ modulo the maximal submodule of $M(\lambda)$ and the algebra $\mathfrak{p}_+$ acts as a multiplication by a variable, while $\mathfrak{p}_-$ acts basically as a differentiation. To formalize this write $L(\lambda) = \bigoplus_{\mu\in\widehat{K_{\mathbb{C}}},k\in\mathbb{N}} L_{\mu,k}$ where $L_{\mu,k}$ is the $K_{\mathbb{C}}$-type contained in $S^k(\mathfrak{p}_+, F(\lambda))$ and note that $\mathfrak{p}_+(L_{\mu,k}) \subset L_{\mu,k+1}$.

The *formal globalization* ([13]) of $L(\lambda)$ is defined as $\overline{L(\lambda)} = \prod_{\mu\in\widehat{K_{\mathbb{C}}}} L_\mu$ (product of topological vector spaces). Since each $K_{\mathbb{C}}$-type is finite-dimensional and each $S^k(\mathfrak{p}_+, F(\lambda))$ contains only finitely many irreducible $K_{\mathbb{C}}$-representations, we can write it as

$$\overline{L(\lambda)} = \prod_{k\in\mathbb{N}} L_k\,,$$

where $L_k = \bigoplus_{\mu\in\widehat{K_c}} L_{\mu,k}$ is a finite sum. The action of $\mathfrak{p}_+$ works as a right shift: $\mathfrak{p}_+(L_k) \subset L_{k+1}$.

Now it is easy to see that the formal globalization is a representation of P, because the action of $\mathfrak{p}_+$ on $\overline{L(\lambda)}$ integrates without any problems. The component of degree k of $\exp(X)v$, $X \in \mathfrak{p}_+$ is given by a sum of $k+1$ elements involving components of v of degree $\leq k$. The P-invariant filtration is given by

$$\overline{L(\lambda)}^l = \prod_{k=l}^{\infty} L_k\,.$$

Theorem 2.1. *There is a Hodge decomposition for* $C^\bullet(\mathfrak{p}, \overline{L(\lambda)})$ *and*

$$H^\bullet(\mathfrak{p}_-, L(\lambda)) = H^\bullet(\mathfrak{p}_-, \overline{L(\lambda)})\,.$$

Proof. The article [11] proves that for a Hodge decomposition to exist it is sufficient to have ∂^* and ∂ disjoint, meaning that $\ker\partial^* \cap \operatorname{im}\partial = 0$ and $\ker\partial \cap \operatorname{im}\partial^* = 0$. We claim that the operators ∂^* and ∂ are disjoint even on the level of formal globalization and thus the Hodge decomposition is still valid. The proof is simple — since both ∂^* and ∂ are $K_{\mathbb{C}}$-equivariant, they act element-wise on $\prod_{\mu\in K_C} L_\mu$.

Explicitly, suppose for contradiction that there exists $u \in \ker\partial^* \cap \operatorname{im}\partial$ such that the μ-component u_μ of u is not zero. Then since ∂^* and ∂ act component-wise, we get that $\partial^* u_\mu = 0$ and there must be $v_\mu \in L_\mu$ such that $\partial v_\mu = u_\mu$. But that means that the nonzero element $u_\mu \in L(\lambda)$ is contained in $\ker\partial^* \cap \operatorname{im}\partial = 0$.

Similarly, since $H^\bullet(\mathfrak{p}_-, L(\lambda)) = \ker\partial^* \cap \ker\partial$ is *finite-dimensional* graded vector space, we get that the cohomology remains the same. □

3. Construction

First we need to recall some fundamental notions of parabolic geometries. The canonical reference is the book by Čap and Slovák [3]. For reader's convenience we repeat the Calderbank – Diemer construction for $\mathbb{V}$ finite-dimensional, however in order to get the Yamabe operator we treat also the case when $\mathbb{V} = L(\lambda)$ and the parabolic pair (G, P) is the pair of complex conformal geometry — i.e. $G = SO(p+2, \mathbb{C})$ and P parabolic subgroup with Levi part $K_{\mathbb{C}}$.

Let G be a Lie group and let P be its parabolic subgroup. A *Cartan geometry* modeled on the pair (G, P) is a P-principal bundle $\pi : \mathcal{G} \to M$ with a P-equivariant one-form $\omega : T\mathcal{G} \to \mathfrak{g}$ such that for each $u \in \mathcal{G}$, $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism restricting to the canonical isomorphism between $T_u(\mathcal{G}_{\pi(u)})$ and $\mathfrak{p}$ (the so called *Cartan connection*). The *curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ of a Cartan geometry is defined by

$$\kappa(u)(\xi, \chi) = [\xi, \chi] - \omega([\omega^{-1}(\xi), \omega^{-1}(\chi)])(u)\,,$$

where the first bracket is the bracket of $\mathfrak{g}$ while the second bracket is just the bracket of vector fields on $\mathcal{G}$.

For any (possibly infinite dimensional) continuous representation $\mathbb{V}$ of P we can form an associated topological vector bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V} \to M$. The bundle associated to $\mathfrak{p}_+$ is the cotangent bundle T^*M and the bundle associated to $\mathfrak{g}/\mathfrak{p}$ is the tangent bundle TM. It what follows we identify the P-representation $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{p}_-$ via the Killing form of $\mathfrak{g}$. The bundle associated to the adjoint representation on $\mathfrak{g}$ is called *adjoint tractor bundle* and is denoted by $\mathcal{A}M$.

It can be checked that κ is in fact horizontal and P-equivariant and hence it induces a section of $\Omega^2 M \otimes \mathcal{A}M$ which we will denote by the same symbol.

Sections of associated bundles are in bijective correspondence with P-equivariant functions on the total space $\mathcal{G}$. For an infinite-dimensional representation $\mathbb{V}$ we define smooth sections as smooth P-equivariant functions on the total space with values in $\mathbb{V}$. It directly follows that a smooth section can have values only in the subspace of smooth vectors of $\mathbb{V}$.

The *invariant derivative* on V is defined by

$$\begin{gathered}\nabla^\omega : \mathcal{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}) \\ \nabla^\omega_\xi f = \mathrm{d}f(\omega^{-1}(\xi))\end{gathered}$$

for all $\xi \in \mathfrak{g}$. It is P-equivariant and so maps $\mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P$ into $\mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V})^P$ and thus we get a linear map $\nabla^\omega : \Gamma(M, \mathcal{V}) \to \Gamma(M, \mathcal{A}M \otimes \mathcal{V})$. Note that from the definition of the fundamental derivative it follows that $\nabla^\omega_{e_i} s$ has the same geometric weight as s.

We define the *tractor connection* by

$$\begin{gathered}\nabla^{\mathfrak{g}} : \mathcal{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}) \\ \nabla^{\mathfrak{g}}_\xi f = \nabla^\omega_\xi f + \xi \cdot f\,.\end{gathered}$$

It is easily checked that for P-equivariant f and for any $\xi \in \mathfrak{p}$ we get $\nabla^{\mathfrak{g}}_\xi f = 0$ and hence $\nabla^{\mathfrak{g}}$ induces a covariant derivative on $\mathcal{V}$.

We define the associated twisted deRham differential

$$\mathrm{d}^{\mathfrak{g}} : \Gamma(M, \Omega^k \mathcal{V}) \to \Gamma(M, \Omega^{k+1} \mathcal{V})$$

by the usual formula. We will need the expression for $\mathrm{d}^{\mathfrak{g}}$ in local coordinates. Let ϵ^i be elements of the basis of $\mathfrak{p}_+$, let e_i be the elements of the dual basis and denote the corresponding sections on M by the same symbols. Then

$$(\mathrm{d}^{\mathfrak{g}} s)(u) = \sum_i \epsilon^i \wedge (\nabla^{\omega}_{e_i} s)(u) + \partial\, s(u) - \sum_{i<j} \epsilon^i \wedge \epsilon^j \wedge \kappa(e_i, e_j) \lrcorner\, s(u)\,,$$

where only the first term depends on the one-jet of $s \in \Gamma(M, \mathcal{V})$ and the remaining two terms act algebraically on the values of s. Note that only the $\mathfrak{p}_-$-component of $\kappa(e_i, e_j)$ (the *torsion component*) contributes to the contraction.

The Lie algebra homology differential $\partial^* \colon \Lambda^i \mathfrak{p}_+ \otimes \mathbb{V} \to \Lambda^{i-1} \mathfrak{p}_+ \otimes \mathbb{V}$ is P-equivariant and hence it induces operator (denoted by the same symbol) on the bundles associated to the chain spaces. These bundles are of course exterior forms with values in $\mathcal{V}$ and we will denote them by $\Omega^i \mathcal{V}$. Let $B_i \mathcal{V}$ denote the image of ∂^* on i-forms with values in $\mathcal{V}$, let $Z_i \mathcal{V}$ denote its kernel and let $H_i \mathcal{V}$ denote the corresponding quotients $Z_i \mathcal{V}/B_i \mathcal{V}$. Again, from P-equivariance it follows that there are natural identifications

$$Z_\bullet \mathcal{V} = \mathcal{G} \times_P \ker \partial^* \qquad B_\bullet \mathcal{V} = \mathcal{G} \times_P \operatorname{im} \partial^* \qquad H_i \mathcal{V} = \mathcal{G} \times_P H_i(\mathfrak{p}_+, \mathbb{V})\,.$$

The BGG operators were constructed in [1] by using a family of differential operators $\Pi^{\mathfrak{g}}_k \colon \Gamma(M, \Omega^k \mathcal{V}) \to \Gamma(M, \Omega^{k+1} \mathcal{V})$ which vanish on im ∂^* and map into $\ker \partial^*$. The *BGG operator* $D_k : \Gamma(\mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})) \to \Gamma(\mathcal{H}_{i+1}(\mathfrak{p}_+, \mathbb{V}))$ is then defined as

$$D_k s := \operatorname{proj} \circ \Pi^{\mathfrak{g}}_{k+1} \circ \mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k \circ \operatorname{rep},$$

where proj is the algebraic projection on homology and rep is a choice of representative in the homology class.

The idea for constructing $\Pi^{\mathfrak{g}}_k$ comes from the expression for the algebraic projection onto $\ker \Box$ which is given by $\mathrm{Id} - \Box^{-1} \Box$. Because ∂^* commutes with $\Box^{-1}$ this equals to $\mathrm{Id} - \Box^{-1} \partial^* d - d \Box^{-1} \partial^*$. We need a P-equivariant operator and since the Lie algebra cohomology differential is the only thing that is not P-equivariant in this formula, we can try to restore the P-equivariance by adding a differential term. This reasoning leads to the following definitions

$$\Box_{\mathfrak{g}} = \partial^* \mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}} \partial^*, \quad Q = {\Box_{\mathfrak{g}}}^{-1} \partial^*$$
$$\Pi^{\mathfrak{g}}_k = \mathrm{Id} - Q\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}} Q.$$

Now the problem arises how to compute the inverse of $\Box_{\mathfrak{g}}$ at least on the image of ∂^*. Once this inverse is provided, the desired properties of $\Pi^{\mathfrak{g}}_k$ follow immediately as algebraic consequences.

There always exists a reduction of our P-bundles to its Levi subgroup L, which means that we can construct the sought inverse by using L-equivariant operators. Since the inverse must be unique it follows that it doesn't depend on the choice of reduction from P to L and hence it is P-equivariant.

Lemma 3.1. *Let $\mathbb{V}$ be either a finite-dimensional $(\mathfrak{g}, P)$-module or the representation $\overline{L(\lambda)}$. Then the operator $\Box_{\mathfrak{g}}$ is invertible on $B_i\mathcal{V}$ and the inverse is given by*

$$\Box_{\mathfrak{g}}{}^{-1} = \Big(\sum_{k=0}^{\infty} N^k\Big)\,\Box^{-1}\,,$$

where $N = -\,\Box^{-1}(\Box_{\mathfrak{g}} - \Box)$.

Proof. We need to prove that the infinite sum makes sense for any section $s \in \Gamma(M, B_i\mathcal{V})$. Let us compute the local expression for $N(s)$, where we consider s to have values in some irreducible L-type.

$$\begin{aligned} -\,\Box\, N(s)(u) &= (\Box_{\mathfrak{g}} - \Box)s(u) \\ &= \partial^*(\mathrm{d}^{\mathfrak{g}} - d)s(u) \qquad \text{because} \quad s \in \Gamma(M, B_i\mathcal{V}) \\ &= \partial^*\Big(\sum_i \epsilon^i \wedge \nabla^{\omega}_{e_i} s - \sum_{i<j} \epsilon^i \wedge \epsilon^j \wedge \kappa(e_i, e_j) \lrcorner\, s\Big)(u)\,. \end{aligned}$$

The first term increases the geometric weight, because fundamental derivative doesn't change w, wedging with an element from $\mathfrak{p}_+$ increases it by one and ∂^* preserves the geometric weight.

The second term also increases the weight, because the contraction with $\kappa(e_i, e_j)$ lowers it by one and wedging with two elements from $\mathfrak{p}_+$ increases it by two.

For a finite-dimensional representation $\mathbb{V}$ it follows that the operator N is nilpotent and in the infinite sum there is only finitely many terms nonzero. Thus the operator $\Pi^{\mathfrak{g}}_k$ is a differential operator of finite order.

If we start with $\mathbb{V} = \overline{L(\lambda)}$, then it is sufficient to consider only the case $i = 0$, because all higher homologies are zero. For a section with values in the representation $\overline{L(\lambda)} = \prod_{k=0}^{\infty} L_k$ we get that N works as a component-wise derivation composed with right shift. It follows that the sum is well defined and the components of $\Pi^{\mathfrak{g}}_k$ are differential operators of increasing order — the component corresponding to L_k of the sum $\sum_{k=0}^{\infty} N(s)(u)$ has at most $k+1$ nonzero terms. $\Box$

The following proposition is one of the main results of [1]. Since the (easy) proof was left to the reader there and since the original statement contained some irrelevant sign errors, we write down all the details here.

Proposition 3.2 ([1], Proposition 5.5)**.** *The operator $\Pi^{\mathfrak{g}}_k \colon \Gamma(M, \Omega^k\mathcal{V}) \to \Gamma(M, \Omega^k\mathcal{V})$ has the following properties.*

(1) *The operator $\Pi^{\mathfrak{g}}_k$ vanishes on $\operatorname{im}\partial^*$ and maps into $\ker\partial^*$:*

$$\Pi^{\mathfrak{g}}_k \circ \partial^* = 0 \quad \& \quad \partial^* \circ \Pi^{\mathfrak{g}}_k = 0\,.$$

(2) *The operator $\Pi^{\mathfrak{g}}_k$ induces identity on the homology $\mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})$:*

$$\Pi^{\mathfrak{g}}_k = \mathrm{Id} \mod \operatorname{im}\partial^*\ .$$

(3) *The commutator of $\mathrm{d}^{\mathfrak{g}}$ and $\Pi^{\mathfrak{g}}$ equals to the commutator of Q and R*

$$\mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k - \Pi^{\mathfrak{g}}_{k+1} \circ \mathrm{d}^{\mathfrak{g}} = Q \circ R - R \circ Q\,,$$

where R is the curvature operator defined by $R(s) = (\mathrm{d}^{\mathfrak{g}} \circ \mathrm{d}^{\mathfrak{g}})(s)$.

(4) *For $k = 0$ and in the flat case, the operator is actually a projection:*

$$\left(\Pi^{\mathfrak{g}}_k\right)^2 = \Pi^{\mathfrak{g}}_k + Q \circ R \circ Q\,.$$

(5)

$$\Pi^{\mathfrak{g}}_k \circ \Box_{\mathfrak{g}} = -Q \circ R \circ \partial^* \quad \& \quad \Box_{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k = -\,\partial^* \circ R \circ Q\,.$$

Thus in the flat case we have a projection $\Pi^{\mathfrak{g}}$ onto a subspace of $\ker \partial^*$ complementary to $\operatorname{im} \partial^*$ and moreover, this projection is actually a chain map between twisted deRham complexes $\mathrm{d}^{\mathfrak{g}} \colon \Omega^\bullet \mathcal{V} \to \Omega^{\bullet+1}\mathcal{V}$ which is homotopic to the identity, the chain-homotopy being the operator $Q \colon \Omega^\bullet \mathcal{V} \to \Omega^{\bullet-1}\mathcal{V}$.

Proof. We will prove these point one by one by easy algebraic manipulations.

The first point is proven by the following two calculations:

$$\begin{aligned} \Pi^{\mathfrak{g}}\,\partial^* &= \left(\mathrm{Id} - \Box_{\mathfrak{g}}{}^{-1}\,\partial^*\,\mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\,\Box_{\mathfrak{g}}{}^{-1}\,\partial^*\right)\partial^* \\ &= \partial^* - \Box_{\mathfrak{g}}{}^{-1}\,\partial^*\,\mathrm{d}^{\mathfrak{g}}\,\partial^* \\ &= \partial^* - \Box_{\mathfrak{g}}{}^{-1}\,\Box_{\mathfrak{g}}\,\partial^* && \text{because } \partial^*\,\mathrm{d}^{\mathfrak{g}}\,\partial^* = \Box_{\mathfrak{g}}\,\partial^* \end{aligned}$$

proves the first half and

$$\begin{aligned} \partial^*\,\Pi^{\mathfrak{g}} &= \partial^* - \partial^*\,\Box_{\mathfrak{g}}{}^{-1}\,\partial^*\,\mathrm{d}^{\mathfrak{g}} - \partial^*\,\mathrm{d}^{\mathfrak{g}}\,\Box_{\mathfrak{g}}{}^{-1}\,\partial^* \\ &= \partial^* - \partial^*\,\mathrm{d}^{\mathfrak{g}}\,\Box_{\mathfrak{g}}{}^{-1}\,\partial^* && \text{because } [\partial^*, \Box_{\mathfrak{g}}{}^{-1}] = 0 \text{ on } \operatorname{im}\partial^* \\ &= \partial^* - \Box_{\mathfrak{g}}\,\Box_{\mathfrak{g}}{}^{-1}\,\partial^* && \text{since } \Box_{\mathfrak{g}} = \partial^*\,\mathrm{d}^{\mathfrak{g}} \text{ on } \operatorname{im}\partial^* \end{aligned}$$

proves the second half of the first point.

The second point is a direct consequence of definitions, because for a section s with values in $Z_k\mathcal{V}$ we get $\Pi^{\mathfrak{g}}_k(s) = s - \Box_{\mathfrak{g}}{}^{-1}\,\partial^*\,\mathrm{d}^{\mathfrak{g}}\,s$ and $\Box_{\mathfrak{g}}{}^{-1}$ maps $B_k\mathcal{V}$ to $B_k\mathcal{V}$.

Proof of the next point of the proposition is also just unwinding the definitions and trivial algebra:

$$[\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}] = [\mathrm{d}^{\mathfrak{g}}, -Q\,\mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\,Q] = -\,\mathrm{d}^{\mathfrak{g}}\,Q\,\mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\,\mathrm{d}^{\mathfrak{g}}\,Q + Q\,\mathrm{d}^{\mathfrak{g}}\,\mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}}\,Q\,\mathrm{d}^{\mathfrak{g}}\;.$$

To prove the fourth point, it is good to note first that from the already proven fact $\Pi^{\mathfrak{g}}\,\partial^* = 0$ it follows that also $\Pi^{\mathfrak{g}}Q = 0$. Moreover even Q^2 equals zero. Now we have

$$\begin{aligned} (\Pi^{\mathfrak{g}})^2 &= \Pi^{\mathfrak{g}}(\mathrm{Id} - Q\,\mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\,Q) \\ &= \Pi^{\mathfrak{g}} - \Pi^{\mathfrak{g}}Q\,\mathrm{d}^{\mathfrak{g}} - \Pi^{\mathfrak{g}}\,\mathrm{d}^{\mathfrak{g}}\,Q \\ &= \Pi^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\,\Pi^{\mathfrak{g}}Q + [\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}]Q \\ &= \Pi^{\mathfrak{g}} + [Q, R]Q && \text{by the third point of the proposition} \\ &= \Pi^{\mathfrak{g}} + QRQ + RQQ. \end{aligned}$$

The last point requires two calculations:

$$\begin{aligned}
\Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} &= \partial^* \,\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}}\, \partial^* \,\Pi^{\mathfrak{g}} && \text{the second term here is zero by the first point}\\
&= \partial^* \,\Pi^{\mathfrak{g}}\, \mathrm{d}^{\mathfrak{g}} + \partial^* [\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}] && \text{here the first term is zero by the first point}\\
&= \partial^* [Q, R] && \text{by point three}\\
&= \partial^* \,QR - \partial^* \,RQ\\
&= -\,\partial^* \,RQ && \text{because } \partial^* \,Q = \partial^* \,\Box_{\mathfrak{g}}{}^{-1}\, \partial^* = 0
\end{aligned}$$

and a similar one

$$\begin{aligned}
\Pi^{\mathfrak{g}} \,\Box_{\mathfrak{g}} &= \Pi^{\mathfrak{g}} (\partial^* \,\mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}}\, \partial^*) = \Pi^{\mathfrak{g}}\, \partial^* \,\mathrm{d}^{\mathfrak{g}} + \Pi^{\mathfrak{g}}\, \mathrm{d}^{\mathfrak{g}}\, \partial^*\\
&= \mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}}\, \partial^* + [\Pi^{\mathfrak{g}}, \mathrm{d}^{\mathfrak{g}}]\, \partial^* = -[Q, R]\, \partial^*\\
&= -QR\, \partial^* + RQ\, \partial^* = -QR\, \partial^* \;.
\end{aligned}$$

Finally, we deal with $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}$. By the fourth point of the proposition we get

$$\partial^* \,u = \Pi^{\mathfrak{g}} v = \Pi^{\mathfrak{g}} \Pi^{\mathfrak{g}} v - QRQv = \Pi^{\mathfrak{g}}\, \partial^* \,u - QRQv$$

which by the first point of the proposition implies that $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}} = \operatorname{im} QRQ$. Thus in the flat case we see that $\operatorname{im} \partial^*$ and $\operatorname{im} \Pi^{\mathfrak{g}}$ are complementary. Finally, in the flat case, the statement that the projection operator $\Pi^{\mathfrak{g}}$ is chain-homotopic to the identity via Q is a direct consequence of definitions. $\square$

Because $\Pi^{\mathfrak{g}}$ maps into $\ker \partial^*$ we have that $\Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} = \partial^* \,\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}}$. Combining this equality with the fifth point of the previous proposition gives us $\partial^* \,\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}} = \Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} = -\,\partial^* \,RQ$. Since $Q = 0$ on $\ker \partial^*$, we see see that $\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}}$ maps $\ker \partial^*$ to $\ker \partial^*$. This allows us to write the BGG operator as $D_k = \operatorname{proj} \circ \mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k \circ \operatorname{rep}$.

The operator $\Pi^{\mathfrak{g}} \circ \operatorname{rep} \colon H_\bullet \mathcal{V} \to \Omega^\bullet \mathcal{V}$ gives us the unique representatives of the homology classes in $\ker \Box_{\mathfrak{g}}$. Indeed $\ker \Box_{\mathfrak{g}} \cap \operatorname{im} \partial^* = 0$, because for $u = \partial^* \,v \in \ker \Box_{\mathfrak{g}}$ we get $0 = \Box_{\mathfrak{g}}\, u = \Box_{\mathfrak{g}}\, \partial^* \,v$ and we know that $\Box_{\mathfrak{g}}$ is invertible on $\operatorname{im} \partial^*$.

Proposition 3.3. *The operator* $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ *maps injectively parallel sections of* $\mathcal{V}$ *into the kernel of* D_0*. In the flat case, the operator is even surjective with inverse being* $\Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}$*.*

Proof. The restriction of $\Pi^{\mathfrak{g}}_0$ to $\ker \mathrm{d}^{\mathfrak{g}}$ equals identity and it is easily computed that the operator $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ maps parallel sections into the solution space of D_0. For a parallel section $s \in \ker \mathrm{d}^{\mathfrak{g}} \cap \operatorname{im} \partial^*$ we have $s = \Pi^{\mathfrak{g}}_0(s)$ and thus $s \in \operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}_0$. Since this space equals to the image of QRQ, which is a zero mapping on 0-forms, we see that the map $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ is indeed injective on parallel section of $\mathcal{V}$.

On the other hand, for $u \in \ker D_0$ we get $\big(\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}\big)(u) \in \operatorname{im} \partial^*$ and in the flat case we can commute $\mathrm{d}^{\mathfrak{g}}$ and $\Pi^{\mathfrak{g}}$ according to the third point of the proposition. It follows that $\big(\mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}\big)(u) = \big(\Pi^{\mathfrak{g}}_1\, \mathrm{d}^{\mathfrak{g}} \circ \operatorname{rep}\big)(u)$ and because $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}_1 = 0$ we get that in the flat case the operator $\Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}$ maps $\ker D_0$ to parallel sections of $\mathrm{d}^{\mathfrak{g}}$. $\square$

4. Conclusion

We have constructed a nontrivial differential operator of finite order over a complex parabolic geometry of conformal type. This operator acts between sections of bundles associated to one-dimensional representations, whose geometric weights are precisely those for the conformal Yamabe operator. It is well known that there is essentially unique differential operator acting between such bundles, i.e. the Yamabe operator.

Of course, it would be desirable to investigate this construction in real cases, i.e. for classical (pseudo)conformal structures. However, as the example in [12] shows, one must be extremely careful to draw any conclusions. The article [10] shows that the Hodge decomposition is valid for all unitarizable highest weight representations, which were classified e.g. in [6], and the same trick with formal globalization goes through. Moreover, the results cover not only the cohomology of $\mathfrak{p}_+$ but also of its appropriate subalgebras. The resulting operators are to be identified not as easily as the Yamabe operator, however there is a close connection of unitarizable highest weight modules and $\mathfrak{k}$-invariant differential operators presented in [5]. Consulting the weights of appropriate zeroth and first homology modules, there doesn't appear to be a case which would yield higher GJMS operators as first BGG operators.

All of these matters are currently investigated by the author of this note and are to be part of his dissertation thesis.

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