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SOLUTIONS TO A CLASS OF POLYNOMIALLY GENERALIZED BERS–VEKUA EQUATIONS USING CLIFFORD ANALYSIS

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Abstract. In this paper a class of polynomially generalized Vekua–type equations and of polynomially generalized Bers–Vekua equations with variable coefficients defined in a domain of Euclidean space are discussed. Using the methods of Clifford analysis, first the Fischer–type decomposition theorems for null solutions to these equations are obtained. Then we give, under some conditions, the solutions to the polynomially generalized Bers–Vekua equation with variable coefficients. Finally, we present the structure of the solutions to the inhomogeneous polynomially generalized Bers–Vekua equation.

1. Introduction

As an elegant higher dimensional analogue of the classical analytic functions, Clifford analysis focuses on the study of the so–called monogenic functions (see e.g. [4, 6, 7]), i.e., null solutions to the Dirac operator or the generalized Cauchy–Riemann operator. It has been applied successfully to solve different kinds of Vekua–type equations and the related boundary value problems in domains of higher dimensional Euclidean space (see e.g. [7]–[9, 13]–[12]). In [7, 14] Sprössig and his coauthors studied a special generalized Vekua–type problem with quaternion parameter defined in $\mathbb{R}^3$ and the corresponding boundary value problems. In [5]–[9], [13]–[12] Delanghe, Brackx and others studied the polynomially generalized Cauchy–Riemann equations in $\mathbb{R}^{n+1}$, and obtained the solutions to the polynomially generalized Cauchy–Riemann equations and to their Riemann boundary value problems, by means of integral formulas and Fischer–type decomposition theorems for null solutions to the polynomially generalized Cauchy–Riemann equations, respectively. Recently, Berglez (see [1, 2]) discussed a class of iterated generalized Bers–Vekua equations in Clifford analysis, which is a generalization of a special Vekua–type equation in the complex plane (see e.g. [3, 15]) to higher dimensions, and obtained their solutions under some conditions. In [10], a polynomially generalized Vekua–type equation and a polynomially generalized Bers–Vekua equation in a domain Ω of $\mathbb{R}^{n+1}$ are studied in the framework of Clifford analysis. In this setting, based on ideas contained in [16, 1, 10], we will consider in this paper a class of polynomially

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generalized Vekua–type equations $p(\mathcal{D})w = 0$ and of polynomially generalized Bers–Vekua equations $p(\underline{\mathcal{D}})w = 0$ (see Section 3) with variable coefficients defined in $\Omega \subset \mathbb{R}^{n+1}$ with $\mathcal{D}$ and $\underline{\mathcal{D}}$ meaning the generalized Vekua–type operator and the generalized Bers–Vekua operator (see Section 2), respectively. We will obtain the Fischer–type decomposition theorems for the solutions to these equations including $(\mathcal{D} - a(x))^k w = 0$, $(\underline{\mathcal{D}} - a(x))^k w = 0$ $(k \in \mathbb{N})$ (see Section 3) with $a(x) \in \mathcal{C}^k(\Omega, \mathbb{C})$ as special cases, which imply the Almansi–type decomposition theorems for the iterated generalized Bers–Vekua equation of [10] and the polynomially generalized Cauchy–Riemann equation [16, 13] defined in $\Omega \subset \mathbb{R}^{n+1}$. Making use of these decomposition theorems, we will give, under some conditions, the solutions to a class of polynomially generalized Bers–Vekua equations with variable coefficients defined in $\Omega \subset \mathbb{R}^{n+1}$. Finally, we will discuss the structure of the solutions to the inhomogeneous polynomially generalized Bers–Vekua equation $p(\underline{\mathcal{D}})w = v$ defined in $\Omega \subset \mathbb{R}^{n+1}$.

The paper is organized as follows. In Section 2, we recall some basic facts about Clifford analysis which will be needed in the sequel. In Section 3, we obtain the Fischer–type decomposition theorems for the solutions to a class of polynomially generalized Vekua–type equations $p(\mathcal{D})f = 0$ and of polynomially generalized Bers–Vekua equations $p(\underline{\mathcal{D}})f = 0$ with variable coefficients, including $(\mathcal{D} - a(x))^k w = 0$, $(\underline{\mathcal{D}} - a(x))^k w = 0$ $(k \in \mathbb{N})$ as special cases, in domains of $\mathbb{R}^{n+1}$. In Section 4, under the assumption of the existence of a Bauer–type differential operator for the solutions to the generalized Bers–Vekua equation $\underline{\mathcal{D}}w(x) = 0$, we will give the solutions to the polynomially generalized Bers–Vekua equation (i.e., $p(\underline{\mathcal{D}})w = 0$) with variable coefficients in domains of $\mathbb{R}^{n+1}$. In the last section we will discuss the structure of the solutions to the equation $p(\underline{\mathcal{D}})w = v$ with variable coefficients in domains of $\mathbb{R}^{n+1}$.

2. Preliminaries and notations

In this section we recall some basic facts about Clifford algebra and Clifford analysis which will be needed in the sequel. For more details we refer the reader to e.g. [4]–[7], [11]–[8].

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of Euclidean space $\mathbb{R}^n$ and let $\mathbb{R}_{0,n}$ be the 2^n-dimensional real Clifford algebra with basis $\{e_{\mathcal{A}} : \mathcal{A} = \{h_1, \ldots, h_r\} \in \mathcal{PN}\}$, where $\mathcal{N}$ stands for the set $\{1, 2, \ldots, n\}$ and $\mathcal{PN}$ denotes for the family of all order-preserving subsets of $\mathcal{N}$. We denote $e_\emptyset$ as e_0 and $e_{\mathcal{A}}$ as $e_{h_1 \ldots h_r}$ for $\mathcal{A} = \{h_1, \ldots, h_r\} \in \mathcal{PN}$. The product in $\mathbb{R}_{0,n}$ is defined by

$$\begin{cases} e_{\mathcal{A}} e_{\mathcal{B}} = (-1)^{N(\mathcal{A} \cap \mathcal{B})} (-1)^{P(\mathcal{A}, \mathcal{B})} e_{\mathcal{A} \Delta \mathcal{B}}\,, & \text{if} \quad \mathcal{A}, \mathcal{B} \in \mathcal{PN}\,, \\ \lambda\mu = \sum\limits_{\mathcal{A}, \mathcal{B} \in \mathcal{PN}} \lambda_{\mathcal{A}} \mu_{\mathcal{B}} e_{\mathcal{A}} e_{\mathcal{B}}\,, & \text{if} \quad \lambda = \sum\limits_{\mathcal{A} \in \mathcal{PN}} \lambda_{\mathcal{A}} e_{\mathcal{A}}, \mu = \sum\limits_{\mathcal{B} \in \mathcal{PN}} \mu_{\mathcal{B}} e_{\mathcal{B}}\,, \end{cases}$$

where $N(\mathcal{A})$ is the cardinal number of the set $\mathcal{A}$, and $P(\mathcal{A}, \mathcal{B}) = \sum\limits_{j \in \mathcal{B}} P(\mathcal{A}, j)$, with $P(\mathcal{A}, j) = N(\mathcal{Z})$ and $\mathcal{Z} = \{i : i \in \mathcal{A}, i > j\}$. It follows that e_0 is the identity element, now written as 1 and that in particular $e_i^2 = -1$, if $i = 1, 2, \ldots, n$, $e_i e_j + e_j e_i = 0$, if $1 \leqslant i < j \leqslant n$, $e_{h_1} e_{h_2} \ldots e_{h_r} = e_{h_1 h_2 \ldots h_r}$, if $1 \leqslant h_1 < h_2 < \cdots < h_r \leqslant n$.

The complex Clifford algebra $\mathbb{C}_n = \mathbb{C} \otimes \mathbb{R}_{0,n}$ is a complex linear, associative, but non–commutative algebra.

For $k \in \{0, 1, 2, \dots, n\}$ fixed, we call $\mathbb{C}_n^{(k)} = \{a \in \mathbb{C}_n : a = \sum\limits_{N(\mathcal{A})=k} a_{\mathcal{A}} e_{\mathcal{A}}\}$ the subspace of k-vectors. In this way we obtain the decomposition $\mathbb{C}_n = \bigoplus\limits_{k=0}^{n} \mathbb{C}_n^{(k)}$ and hence for an arbitrary $a \in \mathbb{C}_n$, $a = \sum\limits_{k=0}^{n} [a]_k$, $|a| = (\sum\limits_{\mathcal{A}} |a_{\mathcal{A}}|^2)^{\frac{1}{2}}$, where $[a]_k$ is the projection of a on $\mathbb{C}_n^{(k)}$. This leads to the identification of $\mathbb{C}$ with the subspace of complex scalars $\mathbb{C}_n^{(0)}$ and of $\mathbb{R}^{n+1}$ with the subspace of real Clifford vectors $\mathbb{R}_{0,n}^{(1)} = \{a = \sum\limits_{j=0}^{n} e_j a_j : a_j \in \mathbb{R}\} \subset \mathbb{C}_n^{(1)}$. The typical element of $\mathbb{R}^n$ is denoted by $\underline{x} = x_1 e_1 + \cdots + x_n e_n, x_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$.

Define $\mathbb{R}^{n+1} = \{x = x_0 + \underline{x} | x_0 \in \mathbb{R}, \underline{x} \in \mathbb{R}_{0,n} \subset \mathbb{C}_n\}$, where $\mathbb{R}_{0,n}$ now is a real subalgebra of $\mathbb{C}_n$. The conjugation is defined by $\bar{a} = \sum\limits_{\mathcal{A}} \bar{a}_{\mathcal{A}} \bar{e}_{\mathcal{A}}, \bar{e}_{\mathcal{A}} = (-1)^{\frac{s(s+1)}{2}} e_{\mathcal{A}}, N(\mathcal{A}) = s$, $a_{\mathcal{A}} \in \mathbb{C}$, where $\bar{a}_{\mathcal{A}}$ means the complex conjugate and $\overline{ab} = \bar{b}\bar{a}$. The inner product $(\cdot, \cdot)$ in $\mathbb{C}_n$ is defined by putting for arbitrary b, $a \in \mathbb{C}_n$, $(b, a) = [b\bar{a}]_0$. It is easy to see that $(b, a) = \sum\limits_{\mathcal{A} \in \mathcal{PN}} b_{\mathcal{A}} \bar{a}_{\mathcal{A}}$ with $a = \sum\limits_{\mathcal{A} \in \mathcal{PN}} a_{\mathcal{A}} e_{\mathcal{A}}, b = \sum\limits_{\mathcal{A} \in \mathcal{PN}} b_{\mathcal{A}} e_{\mathcal{A}}, a_{\mathcal{A}}, b_{\mathcal{A}} \in \mathbb{C}$. Hence the corresponding norm on $\mathbb{C}_n$ reads $|a| = \big(\sum\limits_{\mathcal{A}} |a_{\mathcal{A}}|^2\big)^{\frac{1}{2}} = \sqrt{(a, a)}$. In the particular case of $x = \sum\limits_{i=0}^{n} e_i x_i \in \mathbb{R}^{n+1} \subset \mathbb{C}_n$ as above, $|x|^2 = (\sum\limits_{i=0}^{n} x_i^2)^{\frac{1}{2}} = (x, x)$.

Now we introduce the generalized Cauchy–Riemann operator $\partial = \sum\limits_{j=0}^{n} e_j \frac{\partial}{\partial_{x_j}}$, the generalized Vekua–type operator $\mathcal{D}w(x) \triangleq \partial w(x) + c_1(x_0) w(x) + c_2(x_0) \overline{w}(x)$, and the generalized Bers–Vekua operator $\underline{\mathcal{D}} w(x) \triangleq \partial w(x) + c_2(x_0) \overline{w}(x)$, where $c_i(x_0)$ $(i = 1, 2)$ are both complex–valued functions of the variable x_0. It is clear that $\overline{\partial}\partial = \partial\overline{\partial} = \sum\limits_{j=0}^{n} \partial_{x_j}^2$, which is the Laplace operator in $\mathbb{R}^{n+1}$. For arbitrary $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, $\mathcal{D}^k w(x) \triangleq \mathcal{D}(\mathcal{D}^{k-1} w(x)), \underline{\mathcal{D}}^k w(x) \triangleq \underline{\mathcal{D}}(\underline{\mathcal{D}}^{k-1} w(x))$ and $\mathcal{D}^0 w(x) = w(x)$, $\underline{\mathcal{D}}^0 w(x) = w(x)$.

Let Ω be a domain in $\mathbb{R}^{n+1}$. The continuity, continuously differentiability and the like of the function $f = \sum\limits_{\mathcal{A}} f_{\mathcal{A}} e_{\mathcal{A}} \colon \Omega (\subset \mathbb{R}^{n+1}) \to \mathbb{C}_n$ are ascribed to each of its components $f_{\mathcal{A}} \colon \Omega \to \mathbb{C}$. Let $\mathcal{C}(\Omega, \mathbb{C}_n)$, $\mathcal{C}^1(\Omega, \mathbb{C}_n)$ and the like, denote the set of all continuous functions, continuously differentiable functions and the like defined in Ω, respectively. The null solutions to the operators ∂ and $\mathcal{D}$, that is, functions f such that $\partial f = 0$ (where f is called monogenic) and $\mathcal{D}w = 0$, are denoted by $\mathbb{M}(\Omega, \mathbb{C}_n)$ and $\ker \mathcal{D}$ respectively.

3. Fischer–type decomposition theorems

In this section we will consider $\mathbb{C}_n$–valued functions defined in Ω. We will give the Fischer–type decomposition theorems for the solutions to a class of polynomially generalized Vekua–type equations $p(\mathcal{D})f = 0$ and of polynomially generalized Bers–Vekua equations $p(\underline{\mathcal{D}})f = 0$ with variable coefficients in the domain Ω of $\mathbb{R}^{n+1}$.

Lemma 3.1 ([10]). *If the complex–valued continuously differentiable function $\varphi(x_0)$ of the variable x_0 is defined in $\Omega, \lambda, d_i \in \mathbb{C}$ $(i = 1, 2)$ and $w, v \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$, then*

(i) $\mathcal{D}(\varphi(x_0)w(x)) = \varphi'(x_0)w(x) + \varphi(x_0)(\mathcal{D}w(x))$,

(ii) $\ker(\mathcal{D} - \lambda) = e^{\lambda x_0} \ker \mathcal{D}$,

(iii) $\mathcal{D}(d_1w(x) + d_2v(x)) = d_1\mathcal{D}w(x) + d_2\mathcal{D}v(x)$,

where λ stands for λI, I denoting the identity operator.

Theorem 3.1. *Suppose that $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\mathcal{D} - a(x))^k w(x) = 0$ where $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\mathcal{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$, then there exist unique functions $w_j \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$ satisfying $(\mathcal{D} - a(x))w_j(x) = 0$ $(j = 0, 1, \ldots, k-1)$ such that*

$$w(x) = e^{\gamma(x)}w_0(x) + x_0e^{\gamma(x)}w_1(x) + \cdots + x_0^{k-1}e^{\gamma(x)}w_{k-1}(x)\,, \tag{1}$$

where

$$\begin{cases} w_{k-1}(x) = \frac{1}{(k-1)!}\mathcal{D}^{k-1}w(x)\,, \\ w_{k-2}(x) = \frac{1}{(k-2)!}\mathcal{D}^{k-2}(I - \frac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1})w(x)\,, \\ \vdots \qquad\qquad\qquad \vdots \\ w_1(x) = \mathcal{D}(I - \frac{1}{2}x_0^2\mathcal{D}^2)\ldots(I - \frac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1})w(x)\,, \\ w_0(x) = (I - x_0\mathcal{D})(I - \frac{1}{2}x_0^2\mathcal{D}^2)\ldots(I - \frac{1}{(k-1)!}x_0^{k-1}\mathcal{D}^{k-1})w(x) \end{cases} \tag{2}$$

Here $(\mathcal{D} - a(x))^k w(x) \triangleq (\mathcal{D} - a(x))((\mathcal{D} - a(x))^{k-1}w(x))$ with $(\mathcal{D} - a(x))w(x) \triangleq \mathcal{D}w(x) - a(x)Iw(x)$. Moreover, when $a(x) \equiv \lambda(\lambda \in \mathbb{C})$ for arbitrary $x \in \Omega$, $\mathcal{D}_\lambda^k w(x) \triangleq \mathcal{D}_\lambda(\mathcal{D}_\lambda^{k-1}w(x))$ with $\mathcal{D}_\lambda w(x) \triangleq \mathcal{D}w(x) - \lambda I w(x)$.

Proof. Since $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$, by applying Lemma 3.1 we have

$$(\mathcal{D} - a(x))w(x) = (\mathcal{D} - a(x))e^{\gamma(x)}(e^{-\gamma(x)}w(x)) = e^{\gamma(x)}\mathcal{D}(e^{-\gamma(x)}w(x))\,,$$

where $a(x)$ satisfies $\mathcal{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$. By direct calculations we get

$$(\mathcal{D} - a(x))^k w(x) = e^{\gamma(x)}\mathcal{D}^k(e^{-\gamma(x)}w(x))\,, \quad \text{i.e., } \ker(\mathcal{D} - a(x))^k = \ker \mathcal{D}^k\,. \tag{3}$$

It is clear that $\ker \mathcal{D} \subset \ker \mathcal{D}^2 \subset \ker \mathcal{D}^3 \subset \cdots \subset \ker \mathcal{D}^{k+1}$.

By recurrent calculation we obtain

$$\mathcal{D}^k(x_0^{k-1} \ker \mathcal{D}) = x_0^{k-1} \ker \mathcal{D}^{k+1}\,.$$

Hence,

$$\ker \mathcal{D}^{k-1} + x_0^{k-1} \ker \mathcal{D} \subset \ker \mathcal{D}^k \,. \tag{4}$$

Conversely, if $w \in \ker \mathcal{D}^k$, then there exists the following decomposition

$$w(x) = \Big(I - \frac{1}{(k-1)!} x_0^{k-1} \mathcal{D}^{k-1}\Big)\, w(x) + x_0^{k-1} \frac{1}{(k-1)!} \mathcal{D}^{k-1} w(x) \,. \tag{5}$$

Moreover, we have

$$\mathcal{D}\Big(\frac{1}{(k-1)!}\mathcal{D}^{k-1}\Big) w(x) = 0 \quad \text{and} \quad \mathcal{D}^{k-1}\Big(I - \frac{1}{(k-1)!} x_0^{k-1}\mathcal{D}^{k-1}\Big) w(x) = 0 \,.$$

Therefore, we obtain

$$\ker \mathcal{D}^k \subset \ker \mathcal{D}^{k-1} + x_0^{k-1} \ker \mathcal{D} \,. \tag{6}$$

Taking into account (4) and (6), we get

$$\ker \mathcal{D}^k = \ker \mathcal{D}^{k-1} + x_0^{k-1} \ker \mathcal{D} \,.$$

By induction, we can easily deduce that

$$\ker \mathcal{D}^k = \ker \mathcal{D} + x_0 \ker \mathcal{D} + \cdots + x_0^{k-1} \ker \mathcal{D} \,.$$

Finally, for any $w \in \ker \mathcal{D}^k$, suppose that $\varphi \in \ker \mathcal{D}^{k-1}$ and $\varphi_{k-1} \in \ker \mathcal{D}$ such that

$$w(x) = \varphi(x) + x_0^{k-1} \varphi_{k-1}(x) \,. \tag{7}$$

Applying $\mathcal{D}^{k-1}$ to both sides of (7), we get

$$\mathcal{D}^{k-1} w(x) = \mathcal{D}^{k-1}\varphi(x) + \mathcal{D}^{k-1}(x_0^{k-1}\varphi_{k-1}(x)) = (k-1)!\varphi_{k-1}(x) \,.$$

That is,

$$\varphi_{k-1}(x) = \frac{1}{(k-1)!}\mathcal{D}^{k-1} w(x) \quad \text{and} \quad \varphi(x) = \Big(I - \frac{1}{(k-1)!}\mathcal{D}^{k-1}\Big) w(x) \,.$$

It follows that decomposition (5) is unique. By induction, applying (7), we obtain the result. □

Corollary 3.1. *If $c_1(x_0) \equiv 0$, the equation $(\mathcal{D} - a(x))^k w(x) = 0$ where $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\mathcal{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ reduces to the case $(\underline{\mathcal{D}} - a(x))^k w(x) = 0$. If the function $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{\mathcal{D}} - a(x))^k w(x) = 0$ where $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\mathcal{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$, then there exist unique functions $w_j \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$ satisfying $\underline{\mathcal{D}} w_j(x) = 0$ $(j = 0, 1, \ldots, k-1)$ such that*

$$w(x) = e^{\gamma(x)} w_0(x) + x_0 e^{\gamma(x)} w_1(x) + \cdots + x_0^{k-1} e^{\gamma(x)} w_{k-1}(x) \,, \tag{8}$$

where

$$
(9)\qquad \begin{cases} llw_{k-1}(x) = \frac{1}{(k-1)!}\underline{\mathcal{D}}^{k-1}w(x) \\ w_{k-2}(x) = \frac{1}{(k-2)!}\underline{\mathcal{D}}^{k-2}\Big(I - \frac{1}{(k-1)!}x_0^{k-1}\underline{\mathcal{D}}^{k-1}\Big)w(x) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ w_1(x) = \underline{\mathcal{D}}\Big(I - \frac{1}{2}x_0^2\underline{\mathcal{D}}^2\Big)\dots\Big(I - \frac{1}{(k-1)!}x_0^{k-1}\underline{\mathcal{D}}^{k-1}\Big)w(x) \\ w_0(x) = (I - x_0\underline{\mathcal{D}})\Big(I - \frac{1}{2}x_0^2\underline{\mathcal{D}}^2\Big)\dots\Big(I - \frac{1}{(k-1)!}x_0^{k-1}\underline{\mathcal{D}}^{k-1}\Big)w(x) \end{cases}
$$

That is,

$$
\begin{aligned}
\ker(\underline{\mathcal{D}} - \lambda)^k &= e^{\gamma(x)}\ker\underline{\mathcal{D}} \oplus x_0 e^{\gamma(x)}\ker\underline{\mathcal{D}} \oplus \cdots \oplus x_0^{k-1}e^{\gamma(x)}\ker\underline{\mathcal{D}}\,, \\
\ker\underline{\mathcal{D}}^k &= \ker\underline{\mathcal{D}} \oplus x_0\ker\underline{\mathcal{D}} \oplus \cdots \oplus x_0^{k-1}\ker\underline{\mathcal{D}}\,.
\end{aligned}
$$

Remark 1. For an appropriately chosen function, for instance, $a(x) = \lambda x_0^2 \in \mathbb{C}$, there exists a function $\gamma(x) = \frac{1}{2}\lambda x_0$ satisfying $\mathcal{D}\gamma(x) = a(x)$.

When $a(x) \equiv \lambda \in \mathbb{C}$, the equation $(\mathcal{D} - a(x))^k w(x) = 0$ reduces to the case $(\mathcal{D} - \lambda)^k w(x) = 0$. Hence expression (1) in Theorem 3.1 gives the decomposition of the solution to $\underline{\mathcal{D}}^k w(x) = 0$ as in [10]. Moreover, when $a(x) \equiv 0$, expression (1) in Theorem 3.1 gives the decomposition of the solution to $\underline{\mathcal{D}}^k w(x) = 0$ as in reference [2].

Remark 2. When $a(x) \equiv \lambda \in \mathbb{C}$, if $c_1(x_0) \equiv 0$, the equation $(\mathcal{D} - a(x))^k w(x) = 0$ reduces to the case $(\underline{\mathcal{D}} - \lambda)^k w(x) = 0$. If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $\underline{\mathcal{D}}^k w(x) = 0$, expression (8) in Corollary 3.1 gives the decomposition for the solution to $\underline{\mathcal{D}}^k w(x) = 0$ as in [10]. Further, when $a(x) \equiv 0$, decomposition (8) in Corollary 3.1 corresponds exactly to that in [2].

Remark 3. When $a(x) \equiv \lambda \in \mathbb{C}$, if $c_1(x_0) \equiv c_1$ and $c_2(x_0) \equiv c_2$ with $c_i \in \mathbb{C}(i = 1, 2)$, the equation $(\mathcal{D} - \lambda)^k w(x) = 0$ reduces to the case $(\partial - \widetilde{\lambda})^k w(x) = 0(\widetilde{\lambda} \in \mathbb{C})$. Then the decomposition (1) in Theorem 3.1 corresponds exactly to the one in [16, 9, 13].

In what follows, we introduce the polynomials

$$
p(\lambda) = \big(\lambda - a_1(x)\big)\big(\lambda - a_2(x)\big)\dots\big(\lambda - a_k(x)\big)\,,
$$

where functions $a_j \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ are chosen such that $\mathcal{D}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for arbitrary $i = 1, 2, 3, \dots, k$ and $a_i(x) \neq a_j(x)$, $x \in \Omega$ $(i, j = 1, 2, \dots, k, i \neq j, k \in \mathbb{N})$. Then the associated polynomially generalized Vekua–type operator $p(\mathcal{D})$ and the polynomially generalized Bers–Vekua operator $p(\underline{\mathcal{D}})$ are defined as

$$
\begin{aligned}
p(\mathcal{D}) &= \big(\mathcal{D} - a_1(x)\big)\big(\mathcal{D} - a_2(x)\big)\dots\big(\mathcal{D} - a_k(x)\big)\,, \\
p(\underline{\mathcal{D}}) &= \big(\underline{\mathcal{D}} - a_1(x)\big)\big(\underline{\mathcal{D}} - a_2(x)\big)\dots\big(\underline{\mathcal{D}} - a_k(x)\big)\,,
\end{aligned}
$$

where a_j stands for $a_j I$ $(j = 1, 2, 3, \dots, k)$. It is obvious that the operators $\mathcal{D} - a_j(x)$ and $\underline{\mathcal{D}} - a_j(x)$ $(j = 1, 2, 3, \dots, k)$ are commutative. In the sequel we use the notations

$$\ker\, p(\mathcal{D}) = \left\{\phi\colon \Omega \subset \mathbb{R}^{n+1} \to \mathbb{C}_n \middle| p(\mathcal{D})\phi = 0\right\},$$
$$\ker\, p(\underline{\mathcal{D}}) = \left\{\phi\colon \Omega \subset \mathbb{R}^{n+1} \to \mathbb{C}_n \middle| p(\underline{\mathcal{D}})\phi = 0\right\}.$$

Theorem 3.2. *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $p(\mathcal{D})w(x) = (\mathcal{D} - a_1(x))(\mathcal{D} - a_2(x)) \dots (\mathcal{D} - a_k(x))w(x) = 0$ where $a_i \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function such that $\mathcal{D}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for arbitrary $i = 1, 2, 3, \dots, k$ and $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i, j = 1, 2, \dots, k,$ $i \neq j, k \in \mathbb{N})$, then there exist unique functions $w_j \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$ satisfying $(\mathcal{D} - a_j(x))w_j(x) = 0$ $(j = 1, 2, \dots, k)$ such that*

$$w(x) = w_1(x) + w_2(x) + \dots + w_k(x)\,, \tag{10}$$

where $w_j(x) = \prod\limits_{i \neq j}^{k} \frac{\mathcal{D} - a_i(x)}{a_j(x) - a_i(x)} w(x) (j = 1, 2, \dots, k)$. That is,

$$\begin{aligned} \ker\, p(\mathcal{D}) &= \ker\big(\mathcal{D} - a_1(x)\big) \oplus \ker\big(\mathcal{D} - a_2(x)\big) \oplus \dots \oplus \ker\big(\mathcal{D} - a_k(x)\big) \\ &= e^{\gamma_1(x)} \ker \mathcal{D} \oplus e^{\gamma_2(x)} \ker \mathcal{D} \oplus \dots \oplus e^{\gamma_k(x)} \ker \mathcal{D}\,. \end{aligned} \tag{11}$$

Proof. By Lagrange's interpolation formula, under the condition $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i, j = 1, 2, \dots, k,\ i \neq j)$, we have

$$w(x) = \sum_{j=1}^{k} \prod_{i \neq j}^{k} \frac{\mathcal{D} - a_i(x)}{a_j(x) - a_i(x)} w(x) \triangleq w_1(x) + w_2(x) + \dots w_k(x)\,, \tag{12}$$

where $w_j(x) = \prod\limits_{i \neq j}^{k} \frac{\mathcal{D} - a_i(x)}{a_j(x) - a_i(x)} w(x)$ $(j = 1, 2, \dots, k)$. Moreover, if $w \in \ker\, p(\mathcal{D})$, then

$$\prod_{i \neq j}^{k} \frac{\mathcal{D} - a_i(x)}{a_j - a_i(x)} w \in \ker(\mathcal{D} - a_j) \quad (j = 1, 2, \dots, k)\,.$$

Next we prove the uniqueness.

When the degree k in the variable λ of the polynomial $p(\lambda)$ is equal to 1, it is trivial.

When the degree k in the variable λ of the polynomial $p(\lambda)$ is equal to 2, and if there exist functions $\widetilde{w}_j \in \ker(\mathcal{D} - a_j(x))(j = 1, 2)$ such that $0 = \widetilde{w}_1(x) + \widetilde{w}_2(x)$, then $\widetilde{w}_2 \in \ker(\mathcal{D} - a_1(x))$. In expression (12), for the function $\widetilde{w}_2(x)$ we get

$$\widetilde{w}_2(x) = \frac{\mathcal{D} - a_1(x)}{a_2 - a_1(x)} \widetilde{w}_2(x) + \frac{\mathcal{D} - a_2(x)}{a_1(x) - a_2(x)} \widetilde{w}_2(x) \equiv 0\,, \quad \widetilde{w}_1(x) \equiv 0\,.$$

Then the decomposition

$$w(x) = w_1(x) + w_2(x)\,, \quad w_j(x) \in \ker\big(\mathcal{D} - a_j(x)\big) \quad (j = 1, 2)$$

is unique.

Suppose that the result holds for the degree of the polynomial $p(\lambda)$ in the variable λ, up to $k - 1 (k \geqslant 2)$. For degree k, if there exist functions $w_j \in \ker(\mathcal{D} - \lambda_j)$

$(j=1,2,\ldots,k)$ such that $0 = w_1(x) + w_2(x) + \cdots + w_k(x) \triangleq w_1(x) + \widetilde{W_1}(x)$, then it is clear that $\widetilde{W_1} \in \ker\, m(\mathcal{D})$ with $m(\lambda) = \frac{p(\lambda)}{\lambda - a_1(x)}$. For the function $w_1(x)$, by (12), we have

$$w_1(x) = \sum_{j=1}^{k} \prod_{i \neq j}^{k} \frac{\mathcal{D} - a_i(x)}{a_j(x) - a_i(x)} w_1(x) \equiv 0\,, \quad \widetilde{W_1}(x) \equiv 0\,.$$

Hence the decomposition

$$w(x) = w_1(x) + \widetilde{W}(x)\,, \quad w_1 \in \ker\big(\mathcal{D} - a_1(x)\big)\,, \quad \widetilde{W} \in \ker\, m(\mathcal{D})$$

is unique. Using the induction hypothesis, the result follows. $\square$

Corollary 3.2. *When $c_1(x_0) \equiv 0$, if the function $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $p(\underline{\mathcal{D}})w(x) = (\underline{\mathcal{D}} - a_1(x))(\underline{\mathcal{D}} - a_2(x)) \ldots (\underline{\mathcal{D}} - a_k(x))w(x) = 0$ where $a_i \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\underline{\mathcal{D}}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for $i = 1,2,3,\ldots,k$ and $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i,j = 1,2,\ldots,k,\ i \neq j,\ k \in \mathbb{N})$, then there exist unique functions $w_j \in \mathcal{C}^1(\Omega, \mathbb{C}_n)$ satisfying $(\underline{\mathcal{D}} - a_j(x))w_j(x) = 0 \ \ (j = 1,2,\ldots,k)$ such that*

$$w(x) = w_1(x) + w_2(x) + \cdots + w_k(x)\,, \tag{13}$$

where $w_j(x) = \prod_{i \neq j}^{k} \frac{\underline{\mathcal{D}} - a_i(x)}{a_j(x) - a_i(x)} w(x)$ $(j = 1,2,\ldots,k)$. That is,

$$\begin{aligned} \ker\, p(\underline{\mathcal{D}}) &= \ker\big(\underline{\mathcal{D}} - a_1(x)\big) \oplus \ker\big(\underline{\mathcal{D}} - a_2(x)\big) \oplus \cdots \oplus \ker\big(\underline{\mathcal{D}} - a_k(x)\big) \\ &= e^{\gamma_1(x)} \ker \underline{\mathcal{D}} \oplus e^{\gamma_2(x)} \ker \underline{\mathcal{D}} \oplus \cdots \oplus e^{\gamma_k(x)} \ker \underline{\mathcal{D}} \end{aligned} \tag{14}$$

Theorem 3.3. *If the function $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $p(\mathcal{D})w(x) = (\mathcal{D} - a_1(x))^{n_1}(\mathcal{D} - a_2(x))^{n_2} \ldots (\mathcal{D} - a_r(x))^{n_r} w(x) = 0$ where $a_i \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function such that $\mathcal{D}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for $i = 1,2,3,\ldots,r$ and $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i,j = 1,2,\ldots,r,$ $i \neq j,\ n_1 + n_2 + \cdots + n_r = k,\ n_r, r, k \in \mathbb{N})$, then there exist unique functions $w_{n_j} \in \mathcal{C}^{n_j}(\Omega, \mathbb{C}_n)$ satisfying $(\mathcal{D} - a_j(x))^{n_j} w_{n_j}(x) = 0$ $(j = 1,2,\ldots,r)$ such that*

$$w(x) = w_{n_1}(x) + w_{n_2}(x) + \cdots + w_{n_r}(x)\,, \tag{15}$$

where

$$w_{n_j}(x) = \sum_{i=1}^{n_j} \frac{1}{(n_j - i)!} \Big[\frac{d^{n_j - i}}{d\lambda^{n_j - i}} \frac{(\lambda - a_j(x))^{n_j}}{l(\lambda)}\Big]\Big|_{\lambda = a_j(x)} l_j(\mathcal{D}) w(x)\,,$$

$l(\lambda) = \prod_{j=1}^{r} (\lambda - a_j(x))^{n_j}$ and $l_j(\lambda) = \frac{l(\lambda)}{(\lambda - a_j(x))^{n_j}}$. That is,

$$\ker\, p(\mathcal{D}) = \ker\big(\mathcal{D} - a_1(x)\big)^{n_1} \oplus \ker\big(\mathcal{D} - a_2(x)\big)^{n_2} \oplus \cdots \oplus \ker\big(\mathcal{D} - a_k(x)\big)^{n_r} \tag{16}$$

Moreover,

$$w(x) = \sum_{j=1}^{r} \sum_{i=0}^{n_j - 1} x_0^i e^{\gamma_j(x)} w_{i,j}(x)\,, \tag{17}$$

where the functions $w_{i,j}(x)$ *satisfying* $\mathcal{D}w_{i,j}(x)=0$ $(j=1,2,\ldots,r; i=0,1,\ldots,n_j-1)$ *are given similarly to* (2)*. That is,*

$$\ker p(\mathcal{D})=e^{\gamma_1(x)}\ker\mathcal{D}^{n_1}\oplus e^{\gamma_2(x)}\ker\mathcal{D}^{n_2}\oplus\cdots\oplus e^{\gamma_r(x)}\ker\mathcal{D}^{n_r} \tag{18}$$

Proof. Similarly to Lemma 4 in [13] or [1], we have

$$w(x)=\sum_{j=1}^{r}\sum_{i=1}^{n_j}\frac{1}{(n_j-i)!}\Big[\frac{d^{n_j-i}}{d\lambda^{n_j-i}}\frac{(\lambda-a_j(x))^{n_j}}{l(\lambda)}\Big]\Big|_{\lambda=a_j(x)}l_j(\mathcal{D})w(x)\,. \tag{19}$$

Moreover, if $w(x)\in\ker(\mathcal{D}-a_1(x))^{n_1}(\mathcal{D}-a_2(x))^{n_2}\ldots(\mathcal{D}-a_r(x))^{n_r}$, then $l_j(\mathcal{D})w(x)\in\ker(\mathcal{D}-a_j(x))^{n_j}$ $(j=1,2,\ldots,r)$. Applying (19) and by induction on $j=1,2,\ldots,r$, the proof is completed. ☐

Corollary 3.3. *When* $c_1(x_0)\equiv 0$*, if function* $w\in\mathcal{C}^k(\Omega,\mathbb{C}_n)$ *is a solution to the equation* $p(\underline{\mathcal{D}})w(x)=(\underline{\mathcal{D}}-a_1(x))^{n_1}(\underline{\mathcal{D}}-a_2(x))^{n_2}\ldots(\underline{\mathcal{D}}-a_r(x))^{n_r}w(x)=0$ *where* $a_i\in\mathcal{C}^k(\Omega,\mathbb{C}^{n+1})$ *is the chosen function satisfying* $\underline{\mathcal{D}}\gamma_i(x)=a_i(x)$ *with* $\gamma_i\in\mathcal{C}^{k+1}(\Omega,\mathbb{C})$ *for* $i=1,2,3,\ldots,r$ *and* $a_i(x)\neq a_j(x)$ *for arbitrary* $x\in\Omega$ $(i,j=1,2,\ldots,r,\ i\neq j,\ n_1+n_2+\cdots+n_r=k,\ n_r,r,k\in\mathbb{N})$*, then there exist unique functions* $w_j\in\mathcal{C}^{n_j}(\Omega,\mathbb{C}_n)$ *satisfying* $(\underline{\mathcal{D}}-a_j(x))^{n_j}w_{n_j}(x)=0$ $(j=1,2,\ldots,r)$ *such that*

$$w(x)=w_{n_1}(x)+w_{n_2}(x)+\cdots+w_{n_r}(x)\,, \tag{20}$$

where

$$w_{n_j}(x)=\sum_{i=1}^{n_j}\frac{1}{(n_j-i)!}\Big[\frac{d^{n_j-i}}{d\lambda^{n_j-i}}\frac{(\lambda-a_j(x))^{n_j}}{l(\lambda)}\Big]\Big|_{\lambda=a_j(x)}l_j(\underline{\mathcal{D}})w(x)\,,$$

$l(\lambda)=\prod_{j=1}^{r}(\lambda-a_j(x))^{n_j}$ *and* $l_j(\lambda)=\frac{l(\lambda)}{(\lambda-a_j(x))^{n_j}}$*. That is,*

$$\ker\,p(\underline{\mathcal{D}})=\ker\big(\underline{\mathcal{D}}-a_1(x)\big)^{n_1}\oplus\ker\big(\underline{\mathcal{D}}-a_2(x)\big)^{n_2}\oplus\cdots\oplus\ker\big(\underline{\mathcal{D}}-a_r(x)\big)^{n_r} \tag{21}$$

Moreover,

$$w(x)=\sum_{j=1}^{r}\sum_{i=0}^{n_j-1}x_0^i e^{\gamma_j(x)}w_{i,j}(x)\,, \tag{22}$$

where the functions $w_{i,j}(x)$ *satisfying* $\underline{\mathcal{D}}w_{i,j}(x)=0$ $(j=1,2,\ldots,r,\ i=0,1,\ldots,n_j-1)$ *are given similarly to* (2)*. That is,*

$$\ker\,p(\underline{\mathcal{D}})=e^{\gamma_1(x)}\ker\underline{\mathcal{D}}^{n_1}\oplus e^{\gamma_2(x)}\ker\underline{\mathcal{D}}^{n_2}\oplus\cdots\oplus e^{\gamma_r(x)}\ker\underline{\mathcal{D}}^{n_r}\,, \tag{23}$$

Remark 4. Theorem 3.2 and Theorem 3.3 give the Fischer–type decomposition theorems for null solutions to a class of polynomially generalized Vekua–type operators with variable cofficients. As special cases, Corollary 3.2 and Corollary 3.3 correspond to the Fischer–type decomposition theorems for null solutions to a class of polynomially generalized Bers–Vekua operators with variable cofficients, which imply the corresponding results for null solutions to the iterated Bers–Vekua operator in [10] and to the polynomially generalized Cauchy–Riemann operator in [16, 9, 13].

4. Solutions to polynomial generalized Bers–Vekua equation $p(\underline{D})w = 0$

In this section, under the assumption of the existence of a Bauer–type differential operator for the solutions to the generalized Bers–Vekua equation $\underline{D}w(x) = 0$, we will obtain the solutions to the polynomially generalized Bers–Vekua equation (i.e., $p(\underline{D})w = 0$) with variable coefficients in the domain Ω of $\mathbb{R}^{n+1}$.

Lemma 4.1 ([1, 2]). *Under the assumption of the existence of a Bauer–type differential operator for the solutions of generalized Bers–Vekua equation $\underline{D}w(x) = 0$, the solutions to the generalized Bers–Vekua equation $\underline{D}w(x) = 0$ are given by*

$$w(x) = \sum_{i=0}^{l} f_i(x_0)(u(x)\overline{\partial}^i) + \sum_{i=0}^{l-1} g_i(x_0)b(\partial^i\overline{u}(x)) \quad (l \in \mathbb{N}), \tag{24}$$

where $\partial u(x) = 0$ and $f_i(x_0)$ $(i = 0, 1, 2, \ldots, l)$, $g_i(x_0)$ $(i = 0, 1, 2, \ldots, l-1)$ are complex–valued functions of the variable x_0.

Remark 5. In Lemma 4.1, under the condition of the existence of the a Bauer–type differential operator, the coefficients of $f_i(x_0)$ $(i = 0, 1, 2, \ldots, l)$ and $g_i(x_0)$ $(i = 0, 1, 2, \ldots, l-1)$ of (24) can be explicitly given (see [2]) similarly to those of (11) in [1]. If it exists, the Bauer–type differential operator is defined similarly to (7) and (8) of Section 3 in [1], which is implied in [2, 10], when considering the generalized Bers–Vekua equation. A sufficient condition for the existence of a Bauer–type differential operator was provided in [2]; however, designing other sufficient conditions for the existence of a Bauer–type differential operator is still work in progress.

In the sequel we make use of the operator δ given by

$$\delta u(x) \triangleq u(x)\overline{\partial} - \partial\overline{u}(x), \delta^k u(x) \triangleq \delta(\delta^{k-1}u(x)), \quad \delta^0 u(x) = u(x).$$

Lemma 4.2. *Under the assumption of the existence of a Bauer–type differential operator for the solutions to the generalized Bers–Vekua equation $\underline{D}w(x) = 0$, if $u(x)$ is a solution to the equation $(\partial - a(x))^k w(x) = 0$ where $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\underline{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$, then for arbitrary $i \in \mathbb{N}$,*

$$\delta^i u(x) = \sum_{j=0}^{k-1} x_0^j e^{\gamma(x)}(u_j(x)\overline{\partial}^i), \tag{25}$$

where $u_j(x)$ $(j = 0, 1, 2, \ldots, k-1)$ is a solution to the equation $\partial u(x) = 0$ satisfying $u(x) = \sum\limits_{j=0}^{k-1} x_0^j e^{\gamma(x)} u_j(x)$.

Proof. As $u(x)$ is a solution to the equation $(\partial - a(x))^k w(x) = 0$, by Remark 3, there exist unique functions $u_j(x)$ $(j = 0, 1, 2, \ldots, k-1)$ which are solutions to the equation $\partial u(x) = 0$, satisfying

$$u(x) = \sum_{j=0}^{k-1} x_0^j e^{\gamma(x)} u_j(x)\,.$$

When $i = 1$, we get

$$\delta u(x) = \sum_{j=0}^{k-1} \big(x_0^j e^{\gamma(x)} u_j(x)\big)\overline{\partial} - \partial\big(x_0^j e^{\gamma(x)} u_j(x)\big) = \sum_{j=0}^{k-1} x_0^j e^{\gamma(x)} \big(u_j(x)\overline{\partial}\big)\,. \tag{26}$$

Letting act the operator δ $(i-1)$ consecutive times on both sides of (26), the result follows. □

Theorem 4.1. *Under the assumption of the existence of a Bauer–type differential operator for the solutions of generalized Bers–Vekua equation $\underline{D}w(x) = 0$, if $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{D} - a(x))^k w(x) = 0$ where $a \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function such that $\underline{D}\gamma(x) = a(x)$ with $\gamma \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$, then the solution $w(x)$ is expressed by*

$$w(x) = \sum_{i=0}^{l} f_i(x_0)\big(\delta^i u(x)\big) + \sum_{i=0}^{l-1} g_i(x_0)\overline{\delta^i u(x)}\,, \tag{27}$$

where $u(x)$ is a solution to the equation $\partial^k u(x) = 0$, called a polymonogenic function.

Proof. The result follows from Corollary 3.1, Lemma 4.1 and Lemma 4.2. □

Theorem 4.2. *Under the assumption of the existence of a Bauer–type differential operator for the solutions of generalized Bers–Vekua equation $\underline{D}w(x) = 0$, if $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{D} - a_1(x))(\underline{D} - a_2(x)) \dots (\underline{D} - a_k(x))w(x) = 0$ where $a_i \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\underline{D}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for $i = 1, 2, 3, \dots, k$ and $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i, j = 1, 2, \dots, k,\ i \neq j,\ k \in \mathbb{N})$, then the solution $w(x)$ is expressed by*

$$w(x) = \sum_{j=1}^{k}\sum_{i=0}^{l} f_i(x_0) e^{\gamma_j(x)}\big(u_j(x)\partial^i\big) + \sum_{j=1}^{k}\sum_{i=0}^{l-1} g_i(x_0) e^{\gamma_j(x)}\big(\overline{\partial}^i u_j(x)\big)\,, \tag{28}$$

where $u_j(x)$ is a solution to the equation $\partial u_j(x) = 0$ $(j = 1, 2, \dots, k)$.

Proof. The result follows from Corollary 3.2 and Lemma 4.1. □

Theorem 4.3. *Under the assumption of the existence of a Bauer–type differential operator for the solutions of generalized Bers–Vekua equation $\underline{D}w(x) = 0$, if $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{D} - a_1(x))^{n_1}(\underline{D} - a_2(x))^{n_2} \dots (\underline{D} - a_r(x))^{n_r} w(x) = 0$ where $a_i \in \mathcal{C}^k(\Omega, \mathbb{C}^{n+1})$ is the chosen function satisfying $\underline{D}\gamma_i(x) = a_i(x)$ with $\gamma_i \in \mathcal{C}^{k+1}(\Omega, \mathbb{C})$ for $i = 1, 2, 3, \dots, r$ and $a_i(x) \neq a_j(x)$ for arbitrary $x \in \Omega$ $(i, j = 1, 2, \dots, r,\ i \neq j,\ n_1 + n_2 + \cdots + n_r = k,\ n_r, r, k \in \mathbb{N})$, the solution $w(x)$ is expressed by*

$$w(x) = \sum_{j=1}^{r}\sum_{t=0}^{l} a_t(x_0)\big(\delta_{n_j}^t u_j(x)\big) + \sum_{j=1}^{r}\sum_{t=0}^{l-1} b_t(x_0)\overline{\delta_{n_j}^t u_j(x)}\,, \tag{29}$$

where $u_j(x)$ is a solution to the equation $\partial^{n_j} u(x) = 0$ $(j = 1, 2, \ldots, r)$ and

$$\delta^t_{n_j} u_j(x) = \sum_{i=0}^{n_j-1} x_0^i e^{\gamma_j(x)} \big(u_{i,j}(x)\overline{\partial}^t\big)$$

where $u_{i,j}(x)$ is a solution to the equation $\partial u(x) = 0$ $(j = 1, 2, \ldots, r;$ $i = 0, 1, 2, \ldots, n_j - 1)$.

Proof. By Corollary 3.3, we get

$$w(x) = \sum_{j=1}^{r} \sum_{i=0}^{n_j-1} x_0^i e^{\gamma_j(x)} w_{i,j}(x)\,,$$

where $\underline{\mathcal{D}} w_{i,j}(x) = 0$ $(j = 1, 2, \ldots, r;\ i = 0, 1, \ldots, n_j - 1)$.
By means of Lemma 4.1, we have

$$w(x) = \sum_{j=1}^{r} \sum_{i=0}^{n_j-1} x_0^i e^{\gamma_j(x)} \Big(\sum_{t=0}^{l} f_t(x_0)\big(u_{i,j}(x)\overline{\partial}^t\big) + \sum_{t=0}^{l-1} g_t(x_0)\big(\partial^t \overline{u}_{i,j}(x)\big)\Big) \tag{30}$$

where $u_{i,j}(x)$ is a solution to the equation $\partial u(x) = 0$ $(j = 1, 2, \ldots, r,\ i = 0, 1, 2, \ldots, n_j - 1)$.
Similarly to Lemma 4.2,

$$\delta^t_{n_j} u_j(x) = \sum_{i=0}^{n_j-1} x_0^i e^{\gamma_j(x)} \big(u_{i,j}(x)\overline{\partial}^t\big)\,, \tag{31}$$

where $u_j(x)$ is a solution to the equation $\partial^{n_j} u(x) = 0$ $(j = 1, 2, \ldots, r)$ and $u_{i,j}(x)$ is a solution to the equation $\partial u(x) = 0$ $(j = 1, 2, \ldots, r,\ i = 0, 1, 2, \ldots, n_j - 1)$.
Substituting (31) into (30) completes the proof. □

Remark 6. From Theorems 4.1, 4.2, 4.3, it follows that the solutions to the class of polynomially generalized Bers–Vekua operators with variable coefficients, can be given in terms of polymonogenic functions and monogenic functions. As a Corollary, Theorems 2 in [2] and 4.2, 4.3 in [10] are derived. Moreover, by Theorems 4.1, 4.2, 4.3 and [5, 16], the solutions to the class of polynomially generalized Bers–Vekua equations with variable coefficients may be obtained in virtue of the integral representation and of the Taylor series, respectively.

5. Solutions to the inhomogeneous equation $p(\underline{\mathcal{D}})w = v$

Following the ideas of the previous sections, we will discuss the inhomogeneous polynomially generalized Bers–Vekua equation with variable coefficients (i.e., $p(\underline{\mathcal{D}})w = v$) in a domain Ω of $\mathbb{R}^{n+1}$. In particular we will establish the structure of its solutions.

In this section $p(\mathcal{D}) = \prod_{j=1}^{k} (\mathcal{D} - a_j(x))$ or $p(\mathcal{D}) = \prod_{j=1}^{r} (\mathcal{D} - a_j(x))^{n_j}$ with a_j $(j = 1, 2, \ldots, k)$ as in Section 4, and $p(\underline{\mathcal{D}}) = \prod_{j=1}^{k} (\underline{\mathcal{D}} - a_j(x))$ or $p(\underline{\mathcal{D}}) = \prod_{j=1}^{r} (\underline{\mathcal{D}} -$

$a_j(x))^{n_j}$ with a_j $(j = 1, 2, \dots, r)$ as in Section 4, which are appropriately chosen according to the context.

Theorem 5.1. *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $p(\mathcal{D})w(x) = v(x)$, then each solution $w(x)$ is expressed as*

$$w(x) = W_0(x) + W_1(x)\,, \tag{32}$$

where $W_1 \in \ker p(\mathcal{D})$ and $W_0(x)$ is a special solution to the equation $p(\mathcal{D})w = v$.

Proof. $w(x)$ is a solution to the equation $p(\mathcal{D})w(x) = v(x)$ and $W_0(x)$ is a special solution to the equation $p(\mathcal{D})w(x) = v(x)$, then $w - W_0 \in \ker p(\mathcal{D})$.

Conversely, if $W_1 \in \ker p(\mathcal{D})$ and $W_0(x)$ is a special solution to $p(\mathcal{D})w = v$, then the function $w(x) = W_0(x) + W_1(x)$ is a solution to the equation $p(\mathcal{D})w = v$. The result follows. □

Corollary 5.1. *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{\mathcal{D}} - a_1(x))(\underline{\mathcal{D}} - a_2(x)) \dots (\underline{\mathcal{D}} - a_k(x))w(x) = v(x)$ with a_j $(j = 1, 2, \dots, k)$ as in Section 4, then each solution $w(x)$ is expressed as follows*

$$w(x) = W_0(x) + \sum_{j=1}^{k} e^{\gamma_j(x)} \ker \underline{\mathcal{D}}\,, \tag{33}$$

where $W_0(x)$ is a special solution to the equation $p(\underline{\mathcal{D}})w = v$.

Corollary 5.2. *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{\mathcal{D}} - a_1(x))^{n_1}(\underline{\mathcal{D}} - a_2(x))^{n_2} \dots (\underline{\mathcal{D}} - a_r(x))^{n_r} w(x) = v(x)$ with a_j $(n_1 + \dots + n_r = k,\ n_j \in \mathbb{N},\ j = 1, 2, \dots, r)$ as in Section 4, then each solution $w(x)$ is expressed as*

$$w(x) = W_0(x) + \sum_{j=1}^{r} e^{\gamma_j(x)} \ker(\underline{\mathcal{D}} - a_j(x))^{n_j}\,, \tag{34}$$

where $W_0(x)$ is a special solution to the equation $p(\underline{\mathcal{D}})w = v$.

Remark 7. Theorem 5.1 and Corollaries 5.1, 5.2 provide the structure of the solutions of the inhomogeneous polynomially generalized Vekua–type equation $p(\mathcal{D})w = v$ with variable coefficients, and the inhomogeneous polynomially generalized Bers–Vekua equation $p(\underline{\mathcal{D}})w = v$ with variable coefficients, respectively. In general there is no way to obtain the special solution to the equations $p(\mathcal{D})w = v$ and $p(\underline{\mathcal{D}})w = v$ for an arbitrary $\mathbb{C}_n$–valued function $v(x)$.

Combining Theorems 4.1, 4.2, 4.3 with Corollaries 5.1, 5.2, we obtain

Theorem 5.2. *Assume the existence of a Bauer–type differential operator for the solutions of generalized Bers–Vekua equation $\underline{\mathcal{D}}w(x) = 0$, and let $W_0(x)$ be a special solution to $p(\underline{\mathcal{D}})w = v$ and $l \in \mathbb{N}$, $f_i(x_0)$ $(i = 0, 1, 2, \dots, l)$, $g_j(x_0)$ $(j = 0, 1, 2, \dots, l-1)$, δ^i $(i = 0, 1, \dots, l)$ as Section 4.*
(i) *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{\mathcal{D}} - a(x))^k w(x) = 0$, then the solution $w(x)$ is expressed by*

$$w(x) = W_0(x) + \sum_{i=0}^{l} a_i(x_0)\big(\delta^i u(x)\big) + \sum_{i=0}^{l-1} b_i(x_0)\overline{\delta^i u(x)}\,, \tag{35}$$

where $u(x)$ is a solution to the equation $\partial^k u(x) = 0$.
(ii) *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation $(\underline{D} - a_1(x))(\underline{D} - a_2(x)) \dots (\underline{D} - a_k(x))w(x) = 0$ with a_j $(j = 1, 2, \dots, k)$ as in Section 4, then the solution $w(x)$ is expressed by*

$$\begin{aligned}(36)\quad w(x) &= W_0(x)\\ &+ \sum_{j=1}^{k}\sum_{i=0}^{l} f_i(x_0) e^{\gamma_j(x)} \big(u_j(x)\partial^i\big) + \sum_{j=1}^{k}\sum_{i=0}^{l-1} g_i(x_0) e^{\gamma_j(x)} \big(\overline{\partial}^i u_j(x)\big)\,,\end{aligned}$$

where $u_j(x)$ is a solution to the equation $\partial u(x) = 0$ $(j = 1, 2, \dots, k)$.
(iii) *If $w \in \mathcal{C}^k(\Omega, \mathbb{C}_n)$ is a solution to the equation*

$$(\underline{D} - a_1(x))^{n_1}(\underline{D} - a_2(x))^{n_2} \dots (\underline{D} - a_r(x))^{n_r} w(x) = 0$$

with a_j $(n_1 + \dots + n_r = k,\ n_j \in \mathbb{N},\ j = 1, 2, \dots, r)$ as in Section 4, then the solution $w(x)$ is expressed by

$$(37)\qquad w(x) = W_0(x) + \sum_{j=1}^{r}\sum_{t=0}^{l} a_t(x_0)\big(\delta^t_{n_j} u_j(x)\big) + \sum_{j=1}^{r}\sum_{t=0}^{l-1} b_t(x_0)\overline{\delta^t_{n_j} u_j(x)}\,,$$

where $u_j(x)$ is a solution to the equation $\partial^{n_j} u(x) = 0$ $(i = 1, 2, \dots, r)$ and

$$\delta^t_{n_j} u_j(x) = \sum_{i=0}^{n_j-1} x_0^i e^{\gamma_j(x)} \big(u_{i,j}(x)\overline{\partial}^t\big)\,,$$

where $u_{i,j}(x)$ is a solution to the equation $\partial u(x) = 0$ $(j = 1, 2, \dots, r;\ i = 0, 1, 2, \dots, n_j - 1)$.

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