ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 353–370

TREE ALGEBRAS: AN ALGEBRAIC AXIOMATIZATION OF INTERTWINING VERTEX OPERATORS

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Abstract. We describe a completely algebraic axiom system for intertwining operators of vertex algebra modules, using algebraic flat connections, thus formulating the concept of a *tree algebra*. Using the Riemann-Hilbert correspondence, we further prove that a vertex tensor category in the sense of Huang and Lepowsky gives rise to a tree algebra over $\mathbb{C}$. We also show that the chiral WZW model of a simply connected simple compact Lie group gives rise to a tree algebra over $\mathbb{Q}$.

1. Introduction

The aim of the present paper is to describe a set of purely algebraic axioms designed to capture the structure of genus 0 (or "tree level") amplitudes in conformal field theory. To explain this very briefly, in mathematical physics, a chiral quantum conformal field theory consists of a system of state spaces indexed by a set of *labels* Λ. A Riemann surface with parametrized boundary components whose boundary components are labelled by elements of Λ should specify, roughly, a linear operator from the tensor product of the state spaces corresponding to the inbound boundary components to the tensor product of the state spaces corresponding to the outbound boundary components. In fact, the operator is not specified uniquely. Rather, there should be a finite vector space of such operators, and these vector spaces should form a holomorphic bundle on the moduli space of Riemann surfaces, called a *modular functor*. Certain axioms should be satisfied, in particular, when cutting a Riemann surface Σ along an analytically parametrized simple curve, the space of operators corresponding to Σ should be isomorphic, via a prescribed isomorphism, to the direct sum of the vector spaces corresponding to all possible labels on the cutting curve (the spaces should be 0 for all but finitely many choices of labels on the cutting curve). The isomorphisms should be subject to certain canonical coherence diagrams. The operators on state spaces corresponding to the uncut and cut Riemann surfaces should be related by trace. There should also be a

2010 *Mathematics Subject Classification*: primary 17B69; secondary 81T40, 35Q15.
Key words and phrases: vertex algebra, Riemann-Hilbert correspondence, D-module, KZ-equations, WZW-model.
The first author was supported in part by NSA grant H98230-09-1-0045, NSF grant DMS 1102614, and the James D. Wolfensohn Fund at the Institute for Advanced Study. The second author was supported by a Fellowship from the University of Michigan.
DOI: 10.5817/AM2012-5-353

special label called the *zero label* such that the operator corresponding to annuli approaching the unit circle "converge" to the identity.

The main issue with such an approach is inherited from the usual issues of quantum mechanics: in interesting examples, the state spaces are infinite-dimensional, and to make the approach mathematically rigorous, we must select which category of vector spaces we will be working in. Quantum mechanics suggests Hilbert spaces (cf. [17]), in which case we are firmly in the realm of analysis. Involving analysis certainly seems necessary if we want to make rigorous the entire structure outlined in the last paragraph.

In the 80's, however, it has been noticed by Borcherds [2] and Frenkel, Lepowsky and Meurman [5] that the structure present on the state space corresponding to the 0 label and genus 0 Riemann surfaces can be axiomatized purely algebraically. In this approach, the state space is not a Hilbert space, but simply a graded vector space over a field of characteristic 0. Operators corresponding to round domains (the unit disk with a finite set of disks removed, all boundary components parametrized linearly) are encoded in a structure called *(graded) vertex algebra*. If one wishes to look beyond round domains, one does not consider arbitrary Riemann surfaces of genus 0, but rather infinitesimal variations of boundary parametrization, the effect of which is described by what is known as the *conformal element* or *energy-momentum tensor*.

The goal of the present paper is to propose an extension of the vertex algebra axiomatization to *intertwining operators*, i.e. operators on state spaces involving the entire set of labels Λ of a chiral conformal field theory, purely in the language of algebra. In particular, a test of the success of such an approach is that it should be valid over any field of characteristic 0. (Note: in this paper, we only deal with round domains, we do not consider conformal elements). This advances the program of making conformal field theory a purely algebraic object, which is a well known desideratum. It is related, for example, to the Grothendieck-Teichmüller program [8, 10], and, viewed from a different perspective, the geometric Langlands program (cf. [1]).

There is, however, a major difficulty. It is well known that in interesting examples (e.g. parafermions or the chiral WZW models), the intertwining operators corresponding to non-zero labels (even for round domains) fail to be algebraic. Typically, rather, one gets hypergeometric functions (which are transcendental) and their generalizations (cf. [7, 15]). Considering this, our task may seem impossible. Because of this, axiomatic systems have been developed which combine algebra and analysis, in particular the concept of *vertex tensor category* by Huang and Lepowsky [14].

Yet, the same examples point toward a possible solution: although the correlation functions for non-zero labels are not algebraic, they satisfy *differential equations* which are algebraic. An example is given by the Knizhnik-Zamolodchikov equations in the case of the WZW model (cf. [15]). In fact, similar equations have been formulated by Y.Z. Huang for a large range of examples in [13].

What we do in the present paper is formulate purely algebraic axioms the Huang-Knizhnik-Zamolodchikov equations should satisfy in an algebraic model of chiral conformal field theory in genus 0. The main obstacle to doing so is the sheer complexity of the structure involved. Even the algebraic axiomatization of a vertex algebra [2, 5] (i.e. the case of the 0 label) involves the remarkably complicated Jacobi identity. To make things worse, a straightforward generalization of the Jacobi identity to intertwining operators is false. Although Huang [12] has obtained a more complicated generalization of the Jacobi identity for intertwining operators, devicing a workable system of axioms from this approach seems daunting. To get a handle on the structure in the present paper, we make substantial use a simple interpretation of the vertex algebra axioms obtained by Hortsch, Kriz and Pultr [9]: a graded vertex algebra is essentially the same thing as an algebra over a certain co-operad (the "correlation function co-operad"), which can be easily described explicitly.

The present paper is organized as follows: We write down the general axioms in Section 2 below; we call the resulting algebraic structure a *tree algebra*. In Section 4, we discuss a version of the Riemann-Hilbert correspondence (cf. [4]) in the present setting, and prove that a vertex tensor category in the sense of Huang and Lepowsky [14] can be realized as a tree algebra over $\mathbb{C}$ "with regular singularities", which is a completely algebraic object. At the end of Section 4, we also give an example showing that the chiral WZW model for a simply connected simple compact Lie group can be realized as a tree algebra over $\mathbb{Q}$.

2. The basic definitions

Let F be any field of characteristic 0. We recall from [9] the ordered configuration space $C(n) = C(z_1, \dots, z_n)$ of n distinct points $z_1, \dots, z_n$ in $\mathbb{A}^1$. Denote the coefficient ring of the affine variety $C(n)$ by $\mathcal{C}(n)$. It is advantageous to consider $\mathcal{C}(n) = \mathcal{C}(z_1, \dots, z_n)$ a $\mathbb{Z}$-graded ring with grading by total degree of a homogeneous rational function. In [9], it is shown that the system $(\mathcal{C}(n))$ is a graded co-operad: the co-multiplication

$$\begin{aligned}(1)\quad & \mathcal{C}(z_{11}, \dots, z_{1n_1}, \dots, z_{k1}, \dots, z_{kn_k})_\ell \\ & \qquad \to \mathcal{C}(z_1, \dots, z_k)_{\ell_0} \otimes \mathcal{C}(t_{11}, \dots, t_{1n_1})_{\ell_1} \otimes \cdots \otimes \mathcal{C}(t_{k1}, \dots, t_{kn_k})_{\ell_k}\end{aligned}$$

with $\ell_0 + \cdots + \ell_k = \ell$ is given by setting

$$z_{ij} = t_{ij} + z_i$$

and expanding

$$(t_{ij} + z_i - t_{i'j'} - z_{i'})^{-1}, \quad i \neq i' \tag{2}$$

by rewriting (2) as

$$(t_{ij} + (z_i - z_{i'}) - t_{i'j'})^{-1},$$

and expanding in increasing powers of t_{ij} and $t_{i'j'}$. Moreover, it is shown in [9] that vertex algebras in a suitable sense can be characterized as graded algebras over the graded co-operad $\mathcal{C}$.

Define, for $\mathbb{Z}$-graded F-vector spaces A, B, a $\mathbb{Z}$-graded vector space

$$(A\widehat{\otimes}B)_n = \underset{k_0}{colim} \prod_{k\geq k_0} A_k \otimes B_{n-k}$$

If A, B are (commutative) rings, so is (in a natural way) $A\widehat{\otimes}B$. One checks that the structure map (1) takes the form

$$\mathcal{C}(m_1 + \cdots + m_n) \to (\mathcal{C}(m_1) \otimes \cdots \otimes \mathcal{C}(m_n))\widehat{\otimes}\mathcal{C}(n)\,, \tag{3}$$

and is a homomorphism of graded rings. (Note: In the definition of a *graded* co-operad, separate operations appear in each multi-degree on the right which adds up to the degree on the left. Because of this, when encoding the structure map by a single symbol, infinite sums on the right can occur. This is the reason for the appearance of the $\widehat{\otimes}$ sign; see [9].)

Note that $\mathcal{C}(0) = F$, so selecting i among n coordinates gives us a "co-augmentation"

$$\mathcal{C}(n-i) \to \mathcal{C}(n)\,. \tag{4}$$

The purpose of this paper is to extend this definition to a fully algebraic treatment of tree-level amplitudes of chiral conformal field theories. This makes it possible to define, at least on the level of chiral tree-level amplitudes, "rational conformal field theory" as "conformal field theory over $\mathbb{Q}$". This is possible despite of the fact that these amplitudes are usually transcendental: typical examples are hypergeometric functions [7]. The reason that an algebraic treatment is possible is because we can describe *algebraic flat connections with regular singularities* whose solutions are the desired amplitudes, and characterize the precise algebraic structure these connections are required to obey. To this end, we must first develop the theory of such connections: the reason this is non-trivial is that we need a suitable treatment of the grading, which in particular should capture a concept of "total degree" of the solution, which is to be an element of the field F.

We consider the ring of Kähler differentials $\Omega^1_{\mathcal{C}(n)/F}$, and equip it with a $\mathbb{Z}$-grading so that the universal differentiation

$$d\colon \mathcal{C}(n) \to \Omega^1_{\mathcal{C}(n)/F}$$

is a graded homomorphism of F-modules (of degree 0). Then $\Omega^1_{\mathcal{C}(n)/F}$ is a free $\mathcal{C}(n)$-module on the basis $dz_1, \ldots, dz_n$. Let M be a projective graded $\mathcal{C}(n)$-module. Then a *homogeneous connection* is a map of graded F-modules

$$\nabla\colon M \to M \otimes_{\mathcal{C}(n)} \Omega^1_{\mathcal{C}(n)/F}$$

which satisfies, for $f \in \mathcal{C}(n)$, $m \in M$,

$$\nabla(fm) = mdf + f\nabla(m)\,.$$

We say that the connection ∇ is *flat* when it satisfies the Maurer-Cartan equation

$$(1 \otimes d)\nabla = -(\nabla \otimes 1)\nabla \tag{5}$$

where both sides are considered as maps into $M \otimes_{\mathcal{C}(n)} \Omega^2_{\mathcal{C}(n)/F}$. (Note: This is equivalent to equipping M with a structure of an algebraic D-module. A general D-module, however, does not have to be projective, although finitely generated $\mathcal{C}(n)$-modules are (see [3] VI, Proposition 1.7.)

Next, our aim is to define the *degree* of a flat homogeneous connection. We first define the degree of the *difference* of two flat homogeneous connections. In effect, such difference

$$E = \nabla_1 - \nabla_2 \,,$$

written in matrix form as

$$E = E_1 dz_1 + \cdots + E_n dz_n \,, \quad E_i \in \mathrm{Hom}_{\mathcal{C}(n)}(M, M) \tag{6}$$

is said to *have a degree* when

$$E_1 z_1 + \cdots + E_n z_n = F(z_1, \ldots, z_n) \, \mathrm{Id}_M \tag{7}$$

for some function

$$F(z_1, \ldots, z_n) \in \mathcal{C}(n) \,.$$

Clearly, this property is invariant under linear change of the variables $z_1, \ldots, z_n$.

Lemma 1. *When a difference E of two flat connections has a degree, then* $\deg(E) = F(z_1, \ldots, z_n)$ *is a constant function of $z_1, \ldots, z_n$.*

Proof. In coordinates, (5) reads

$$\frac{\partial E_i}{\partial z_j} - \frac{\partial E_j}{\partial z_i} = -\frac{1}{2}[E_i, E_j] \,. \tag{8}$$

Now compute

$$\begin{aligned} \frac{\partial (E_1 z_1 + \cdots + E_n z_n)}{\partial z_1} &= E_1 + z_1 \frac{\partial E_1}{\partial z_1} + \cdots + z_n \frac{\partial E_n}{\partial z_1} \\ &= E_1 + z_1 \frac{\partial E_1}{\partial z_1} + \cdots + z_n \frac{\partial E_1}{\partial z_n} - \frac{1}{2}[E_1, z_1 E_1 + \cdots + z_n E_n] \,. \end{aligned} \tag{9}$$

The Lie bracket (9) vanishes because of our assumption (7). The first term vanishes by the following Lemma and the fact that E is a homogeneous connection. Thus, (9) vanishes. An analogous argument holds with z_1 replaced by any z_i, which proves the statement of the Lemma. $\square$

Lemma 2. *A function f of n variables $z_1, \ldots, z_n$ is homogeneous of degree k if and only if*

$$z_1 \frac{\partial f}{z_1} + \cdots + z_n \frac{\partial f}{\partial z_n} = kf. \tag{10}$$

Proof. Suppose f is homogeneous of degree k. By the chain rule,

$$ka^{k-1}f(z_1,\dots,z_n)=\frac{da^k}{da}f(z_1,\dots,z_n)=\frac{df(az_1,\dots,az_n)}{da} \tag{11}$$

$$=\frac{\partial f(t_1,\dots,t_n)}{\partial t_1}\Big|_{(az_1,\dots,az_n)}\cdot z_1+\dots+\frac{\partial f(t_1,\dots,t_n)}{\partial t_n}\Big|_{(az_1,\dots,az_n)}\cdot z_n \tag{12}$$

$$=a^{k-1}\Big(z_1\frac{\partial f}{\partial z_1}+\dots+z_n\frac{\partial f}{\partial z_n}\Big). \tag{13}$$

Conversely, if (10) holds,

$$\frac{df(\lambda z_1,\dots,\lambda z_n)}{d\lambda}=\sum z_i\frac{\partial f(t_1,\dots,t_n)}{\partial t_i}\Big|_{t=\lambda z_i}=\frac{kf(\lambda z_1,\dots,\lambda z_n)}{\lambda}.$$

Studying the solutions of the ODE $y'(\lambda)=\frac{ky}{\lambda}$ gives the result. □

Lemma 3. *Suppose M is a graded $\mathcal{C}(n)$-module and ∇ is a flat homogeneous connection. Suppose $g\colon M\to M$ be an isomorphism of $\mathcal{C}(n)$-modules which is homogeneous of degree ℓ. Then*

$$\nabla+g^{-1}dg \tag{14}$$

is a flat homogeneous connection, and the difference $g^{-1}dg$ has degree ℓ.

Proof. A direct consequence of Lemma 2. □

The connections ∇, (14) of Lemma 3 will be said to *have the same monodromy.*

Now we define the degree of a homogeneous flat connection ∇ on M. Let us first assume that M is a free graded $\mathcal{C}(n)$-module. Let $\phi=(\phi_1,\dots,\phi_k)$ be a $\mathcal{C}(n)$-basis of M consisting of elements of degree 0. Then we say that the flat homogeneous connection

$$\nabla_\phi\colon a_1\phi_1+\dots+a_k\phi_k\mapsto(da_1)\phi_1+\dots+(da_k)\phi_k$$

has degree 0. An arbitrary flat homogeneous connection ∇ is said to have degree 0 when the difference $\nabla-\nabla_\phi$ has degree 0. This definition is consistent since by Lemma 3, for two degree 0 bases ϕ, ψ, $\nabla_\phi-\nabla_\psi$ has degree 0.

When M is a projective $\mathcal{C}(n)$-module, consider a direct summand embedding $M\subseteq N$ where N is a free $\mathcal{C}(n)$-module. We say that a flat connection ∇ on M has degree 0 when there exists a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "\iota"', "\subseteq"] & M\otimes_{\mathcal{C}(n)}\Omega^1_{\mathcal{C}(n)/k} \arrow[d, "\subseteq"', "\iota\otimes\mathrm{Id}"] \\ N \arrow[r, "\nabla'"] & N\otimes_{\mathcal{C}(n)}\Omega^1_{\mathcal{C}(n)/k} \end{tikzcd} \tag{15}$$

where N is a free $\mathcal{C}(n)$-module and ∇' is a flat homogeneous connection of degree 0, and ι is an inclusion of a graded direct summand.

Lemma 4. *This notion of degree 0 flat homogeneous connection does not depend on the choice of the graded direct summand inclusion ι.*

Proof. Consider two such direct summand embeddings $\iota\colon M \to N$, $\iota'\colon M \to N'$. Clearly, we may assume

$$N = N'\,. \tag{16}$$

Let the direct complements of ι, ι' be K, K', respectively. We may assume

$$K \cong K' \tag{17}$$

(by replacing, if necessary, N with $N \oplus M \oplus K$). Now assuming (16), (17), we can produce a diagram of $\mathcal{C}(n)$-modules

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "="'] & N \arrow[d, "f", "\cong"'] \\ M \arrow[r, "\iota'"'] & N \end{tikzcd}$$

for some graded isomorphism $f\colon N \to N$ (of degree 0) of graded $\mathcal{C}(n)$-modules. The statement then follows from Lemma 3. $\square$

Comment: We do not know whether every projective $\mathcal{C}(n)$-module M, or even one endowed with a flat homogeneous connection of degree 0, is free.

Let M_i be modules over $\mathbb{Z}$-graded commutative rings R_i, and suppose we have homogeneous connections E_i on M_i. Then there is a natural homogeneous connection E on the $\bigotimes R_i$-module $\bigotimes M_i$ given by

$$E(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes E_i(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n\,. \tag{18}$$

The same formula also defines a natural homogeneous connection on $M_1 \widehat{\otimes} M_2$.

If $R \to S$ is a map of $\mathbb{Z}$-graded commutative rings, M is a graded R-module and we have a homogeneous flat connection E on M, then there is a natural "pushforward" homogeneous flat connection $E \otimes_R S$ on $M \otimes_R S$, since we have a natural map

$$\Omega^1_{R/F} \otimes_R S \to \Omega^1_{S/F}\,.$$

3. Tree functors and tree algebras

Let Λ be a set (called *set of labels*). A *pre-tree functor* (Λ, M) is a system of graded $\mathcal{C}(n)$-modules $M_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$ and graded module isomorphisms

(19)
$$\begin{array}{c} M_{\lambda_{11},\dots,\lambda_{nm_n},\lambda_\infty} \otimes_{\mathcal{C}(m_1+\dots+m_n)} (\mathcal{C}(m_1)\otimes\dots\otimes\mathcal{C}(m_n)\widehat{\otimes}\mathcal{C}(n)) \\ \Big\downarrow \cong \\ \bigoplus_{\lambda_1,\dots,\lambda_n} \Big(\bigotimes_{i=1}^{n} M_{\lambda_{i1},\dots,\lambda_{im_i},\lambda_i}\Big)\widehat{\otimes} M_{\lambda_1,\dots,\lambda_n,\lambda_\infty} \end{array}$$

which must make $M_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$ a Λ-sorted $\mathbb{Z}$-graded $\mathcal{C}$-module co-operad. We will also assume that there be a distinguished element $1\in\Lambda$ such that

(20)
$$\begin{array}{ll} M_{\lambda_\infty} = 0 & \text{if} \quad \lambda_\infty \neq 1 \\ F = \mathcal{C}(0) & \text{if} \quad \lambda_\infty = 1\,. \end{array}$$

It follows that the co-augmentation (4) induce an isomorphism

(21)
$$M_{\lambda_{i+1},\dots,\lambda_n,\lambda_\infty} \otimes_{\mathcal{C}(n-i)} \mathcal{C}(n) \xrightarrow{\cong} M_{1,\dots,1,\lambda_{i+1},\dots,\lambda_n,\lambda_\infty}\,.$$

A pre-tree functor is called *finite* if all the projective $\mathcal{C}(n)$-modules $M_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$ are finite rank. A *pre-tree algebra* $(\Lambda, M, V, \alpha, \phi)$ consists of a pre-tree functor M, a system of $\mathbb{Z}+\alpha_\lambda$-graded vector spaces V_λ such that $\alpha_1 = 0$, and homomorphisms of $\mathcal{C}(n)$-modules

(22)
$$\phi_{\lambda_1,\dots,\lambda_n,\lambda_\infty} : M_{\lambda_1,\dots,\lambda_n,\lambda_\infty} \to \mathrm{Hom}((V_{\lambda_1})\otimes\dots(\otimes V_{\lambda_n}), (V_{\lambda_\infty})\otimes_F \mathcal{C}(n))$$

homogeneous of degree

(23)
$$\alpha_\infty - \alpha_1 - \dots - \alpha_n\,,$$

such that if we choose labels $\lambda_{11},\dots,\lambda_{nm_n},\lambda_\infty$, and denote, for a graded $\mathcal{C}(m_1+\dots+m_n)$-module X, put

$$X' := X\otimes_{\mathcal{C}(m_1+\dots+m_n)} (\mathcal{C}(m_1)\otimes\dots\otimes\mathcal{C}(m_n)\widehat{\otimes}\mathcal{C}(n))\,,$$

and put

$$\mathcal{M} = \bigoplus_{\lambda_i} \Big(\bigotimes_{i=1}^{n} M_{\lambda_{i1},\dots,\lambda_{in_i},\lambda_i}\Big)\widehat{\otimes} M_{\lambda_1,\dots,\lambda_n,\lambda_\infty}\,,$$

$$\mathcal{V} = \bigotimes_{\lambda_i} \Big(\bigotimes_{i=1}^{n} \mathrm{Hom}\Big(\bigotimes_j V_{\lambda_{ij}}, V_{\lambda_i}\otimes\mathcal{C}(m_i)\Big)\Big)\widehat{\otimes}\,\mathrm{Hom}\Big(\bigotimes_i V_{\lambda_i}, V_{\lambda_\infty}\otimes\mathcal{C}(n)\Big)\,,$$

we have a commutative diagram

$$
\begin{tikzcd}
(M_{\lambda_{11},\dots,\lambda_{nm_n},\lambda_\infty})' \arrow[r] & \left(\mathrm{Hom}(V_{\lambda_{11}}\otimes\dots\otimes V_{\lambda nm_n}, \mathcal{C}(\sum\limits_{i=1}^{n} m_i))\right)' \\
\mathcal{M} \arrow[u, "\cong"] \arrow[r, "\phi_*\hat{\otimes}\phi_*"'] & \mathcal{V}, \arrow[u]
\end{tikzcd}
\tag{24}
$$

where the top and bottom row are induced by the appropriate cases of (22), the left column is induced by an appropriate case of (19), and the right hand column is induced by composition.

There is also the obvious equivariance axiom and a unitality axiom which asserts that $1 \in M_{\lambda,\lambda}$ maps in

$$\mathrm{Hom}(V_\lambda, V_\lambda \otimes \mathcal{C}(1))$$

to $\mathrm{Id}\otimes 1$. A pre-tree algebra is called *finite* if its pre-tree functor is finite.

A *tree functor* (Λ, M, E) consists of a pre-tree functor (Λ, M), and a system of homogeneous flat connections

$$E_{\lambda_1,\dots,\lambda_n,\lambda_\infty} \tag{25}$$

of degree (23) on $M_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$ such that

(1) The connections (25) are equivariant with respect to the obvious action of Σ_n

(2) The connection (25) for $\lambda_1 = \dots = \lambda_i = 1$ has the same monodromy as the pushforward of the connection $E_{\lambda_{i+1},\dots,\lambda_n,\lambda_\infty}$ along the natural map $\mathcal{C}(n-i) \to \mathcal{C}(n)$ induced by co-inserting $\mathcal{C}(0)$ to the first i coordinates.

(3) The pushforward of the connection

$$E_{\lambda_{11},\dots,\lambda_{nm_n}}$$

via the structure map

$$\mathcal{C}(m_1 + \dots + m_n) \to \mathcal{C}(m_1)\otimes \dots \mathcal{C}(m_n)\widehat{\otimes}\mathcal{C}(n) \tag{26}$$

is equal to

$$\bigoplus_{\lambda_i,\dots,\lambda_n} \left(\bigotimes_{i=1}^{n} E_{\lambda_{i1},\dots,\lambda im_i,\lambda_i}\right)\widehat{\otimes} E_{\lambda_1,dots,\lambda_n}.$$

(4) The connection $E_{\lambda,\lambda}$ is the pushforward of the connection dz via the map $\mathcal{C}(1) \to M_{\lambda,\lambda}$.

A *tree algebra* $(\Lambda, M, V, \alpha, \phi, E)$ consists of a pre-tree algebra $(\Lambda, M, V, \alpha, \phi)$, and a structure (Λ, M, E) of a tree functor on its pre-tree functor (Λ, M).

This data are somewhat redundant. There is no natural choice of the numbers α_λ, they are only determined modulo 1. Because of that, it is important to define isomorphism of tree algebras. To this end, there is an obvious notion of isomorphism of tree functors, and an obvious notion of isomorphism involving graded isomorphisms of the spaces V_λ, so all we need to discuss is isomorphism of tree algebra on the same tree functor and the same data V, ϕ. An isomorphism of tree algebras $(\Lambda, M, V, \alpha, \phi, E)$, $(\Lambda, M, V, \beta, \phi, F)$ consists of homogeneous isomorphisms

$$g_{\lambda_1,\dots,\lambda_n,\lambda_\infty} : M_{\lambda_1,\dots,\lambda_n,\lambda_\infty} \xrightarrow{\cong} M_{\lambda_1,\dots,\lambda_n,\lambda_\infty} \tag{27}$$

homogeneous of degree

$$(\beta_{\lambda_\infty} - \alpha_{\lambda_\infty}) - (\beta_{\lambda_1} - \alpha_{\lambda_1}) - \dots - (\beta_{\lambda_n} - \alpha_{\lambda_n})$$

such that

$$F_{\lambda_1,\dots,\lambda_n,\lambda_\infty} - E_{\lambda_1,\dots,\lambda_n,\lambda_\infty} = g^{-1}_{\lambda_1,\dots,\lambda_n,\lambda_\infty} dg_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$$

and the pushforward of $g_{\lambda_{11},\dots,\lambda_{nm_n},\lambda_\infty}$ via (19) is equal to

$$\bigoplus_{\lambda_1,\dots,\lambda_n} \bigotimes g_{\lambda_{i1},\dots,\lambda_{im_i},\lambda_i} \widehat{\otimes} g_{\lambda_1,\dots,\lambda_n,\lambda_\infty} .$$

4. Regularity

Our treatment of regular connections follows Deligne [4]. Let C be a smooth algebraic curve over F, and let $\overline{C}$ be a smooth projective curve containing C. Let M be a (finite-dimensional) algebraic vector bundle on C, and let

$$E \colon M \to M \otimes_{\mathcal{O}_C} \Omega^1_{C/\operatorname{Spec} F} \tag{28}$$

be an algebraic connection. We say that E has *regular singularities* if for every smooth projective curve $\overline{C}$ and every embedding $C \subset \overline{C}$, and every point $x \in \overline{C} - C$,

$$M \otimes_{\mathcal{O}_C} \mathcal{O}_x$$

has a basis in which the pushforward of the connection E has simple poles. A connection E on an n-dimensional smooth separated algebraic variety X is said to have *regular singularities* if for every smooth algebraic curve C in X, the restriction of E to C has regular singularities. We will call a tree functor (Λ, M, E) *regular* when all the connections $E_{\lambda_1,\dots,\lambda_n,\lambda_\infty}$ have regular singularities. A tree algebra is called regular when its tree functor is regular. This notion is clearly invariant under isomorphism of tree algebras.

Our main goal is to establish a version of the Riemann-Hilbert correspondence for tree functors and tree algebras over $\mathbb{C}$. To this end, we must define the analytic versions of our concepts. In effect, we may define $\mathcal{C}(n)_{an}$ to be the $\mathbb{Z}$-graded ring of all holomorphic functions f on the configuration space $C(n)$ of n ordered distinct points in $\mathbb{C}$, homogeneous in the sense that

$$f(\lambda z_1, \dots, \lambda z_n) = \lambda^k f(z_1, \dots, z_n) .$$

In our grading, the degree of f is $-k$. We would like to have a $\mathbb{Z}$-graded co-operad structure

$$(29)\qquad \mathcal{C}(m_1+\cdots+m_n)_{an}\to\big(\mathcal{C}(m_1)_{an}\otimes\cdots\otimes\mathcal{C}(m_n)_{an}\big)\widehat{\otimes}\mathcal{C}(n)_{an}\,,$$

but the difficulty is that expanding an analytic function by a Laurent series may present elements of arbitrarily low degrees. Replacing $\widehat{\otimes}$ by the product of its bigraded summands of the same total degree, we do get a $\mathbb{Z}$-graded co-operad, but the resulting objects $\mathcal{P}(m_1,\dots,m_n,n)$ aren't rings, so we cannot study vector bundles in the sense of finite rank projective modules. Our solution is to replace (29) by

$$(30)\qquad \mathcal{C}(m_1+\cdots+m_n)_{an}\to\chi(m_1,\dots,m_n,n)_{an}\,,$$

where the right hand side denotes the ring of all partially defined holomorphic functions f on

$$(31)\qquad C(m_1)\times\cdots\times C(m_n)\times C(n)$$

where, if we denote the coordinates of (31) by

$$(32)\qquad t_{11},\dots,t_{1m_1},\dots,t_{n1},\dots,t_{nm_n},z_1,\dots,z_n\,,$$

then for each choice of $z_1,\dots,z_n$ there exists a locally uniform $\epsilon>0$ such that f is defined on (32) when

$$(33)\qquad \|t_{ij}\|<\epsilon\,.$$

Using (30) and the natural inclusion

$$(34)\qquad \mathcal{C}(m_1)_{an}\otimes\cdots\otimes\mathcal{C}(m_n)_{an}\otimes\mathcal{C}(n)_{an}\subset\chi(m_1,\dots,m_n,n)_{an}\,,$$

we may define analytic (pre)-tree functors and (pre)-tree algebras in precise analogy with the algebraic definitions. We start with $\mathbb{C}^\times$-equivariant holomorphic vector bundles Ξ on $C(n)$. By this we mean a holomorphic bundle with a holomorphic action of $\mathbb{C}^\times$ on the total space which is compatible with the $\mathbb{C}^\times$-action on $C(n)$ by

$$\lambda(z_1,\dots,z_n)=(\lambda z_1,\dots,\lambda z_n)\,.$$

Then the space M of global sections of Ξ is then naturally a graded $\mathcal{C}(n)_{an}$-module.

We will called a projective module *of finite rank* if it is a direct summand of a free module on a finite set of generators.

Lemma 5. *Let X be a Stein manifold. The global sections functor defines an equivalence of categories from the category of finite-dimensional holomorphic vector bundles over X (and holomorphic maps over Id_X) and the category of finite rank projective modules over the ring $\mathrm{Hol}(X)$ of holomorphic functions on X.*

Proof. First we will prove that a holomorphic vector bundle ξ of finite dimension n on X is always a holomorphic direct summand of a finite dimensional trivial vector bundle. First, note that this is true topologically: X is of the homotopy type of a $\dim(X)$-dimensional CW complex, so by Whitehead's theorem, the topological classifying map ϕ of ξ factors through a map ϕ' into the $\dim(X)$-skeleton of $BU(n)$:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, dotted, "\phi'"'] & BU(n) \\ & BU(n)_{\dim(X)} \arrow[u, "\subseteq"'] \end{tikzcd}$$

But $BU(n)_{dim(X)}$ is compact, so the restriction γ_n' of the universal n-bundle γ_n on $BU(n)$ to $BU(n)_{\dim(X)}$ is a direct summand of a finite-dimensional trivial vector bundle. Hence, the same is true for ξ topologically, which we indicate by the subscript $(?)_{\mathrm{top}}$:

$$\xi_{\mathrm{top}} \oplus \eta_{top} \cong N_{\mathrm{top}}\,. \tag{35}$$

But now the data (35) may be represented topologically by a $GL_n(\mathbb{C}) \times GL_{N-n}(\mathbb{C})$--principal bundle, so by Grauert's principle [6], Satz 2, the data (35) can be represented in the holomorphic category, which we indicate by the subscript $(?)_{an}$:

$$\xi_{an} \oplus \eta_{an} \cong N_{an}\,. \tag{36}$$

Applying Grauert's principle again for the groups $GL_n(\mathbb{C})$, $GL_N(\mathbb{C})$, however, we see that

$$\xi_{an} \cong \xi\,, \quad N_{an} \cong N\,.$$

This shows that the global section functor in the statement of the Lemma lands in the category indicated. To show that the functor is onto on isomorphism classes of objects, recall that a finite rank projective $\mathrm{Hol}(X)$-module can be constructed from a free module by applying an idempotent matrix; since a free module always arises from a trivial bundle, applying the same matrix on the bundle gives the bundle corresponding to the finite rank projective module.

We now need to prove that the global section functor is fiathfully full. Since, however, every object in the source is a holomorphic direct summand of a trivial finite-dimensional vector bundle, it suffices to prove that the functor is faithfully full on the subcategory of finite-dimensional trivial holomorphic vector bundles, which in turn reduces to showing that the functor induces bijection of the set of holomorphic self-maps of the 1-dimensional trivial vector bundle to the set of holomorphic self-maps of the 1-dimensional free $\mathrm{Hol}(X)$-modules. Obviously, however, both sets are (compatibly) bijective to $\mathrm{Hol}(X)$. □

Corollary 6. *The category of finite-dimensional $\mathbb{C}^\times$-equivariant holomorphic vector bundles over $C(n)$ and holomorphic $\mathbb{C}^\times$-equivariant homomorphisms (over the identity on $C(n)$) is equivalent, via the global sections functor, to the category of finite rank projective graded $\mathcal{C}(n)_{an}$-modules and (degree 0) homomorphisms of graded modules.*

Proof. In the case $n = 1$, the isomorphism class of a $\mathbb{C}^\times$-equivariant bundle is determined by the representation on the 0 fiber, which shows that the functor is a bijection on isomorphism classes of objects. Additionally, non-equivariantly, the bundles are trivial by Grauert's theorem [6], so in the rank 1 case, maps are simply holomorphic functions on $\mathbb{C}$, and therefore homogeneous functions are simply z^k, $k \geq 0$. This is a graded morphism if and only if the $\mathbb{C}^\times$-action on 0-fiber of the target is z^{-k} tensored with the action of the 0-fiber of the source. This gives the required statement.

In the case $n > 1$, the action of $\mathbb{C}^\times$ on $C(n)$ is free. The quotient $C(n)_0 = C(n)/\mathbb{C}^\times$ is an affine variety whose ring of coefficients is $\mathcal{C}(n)_0$, the homogeneous submodule of $\mathcal{C}(n)$ of elements of degree 0. (In fact, $C(n)_0$ is canonically isomorphic to the configuration space of n distinct points $z_1, \dots, z_n \in \mathbb{A}^1$ with the condition $z_2 = z_1 + 1$.) The category of $\mathbb{C}^\times$-equivariant holomorphic bundles over $C(n)$ and $\mathbb{C}^\times$-equivariant holomorphic maps over Id is equivalent to the category of holomorphic vector bundles on $C(n)_0$. Since $C(n)_0$ is a Stein manifold, this is in turn equivalent to the category of finite rank projective $\mathcal{C}(n)_0$-modules, which is in turn equivalent to the category of graded finite type projective $\mathcal{C}(n)$-modules. $\square$

Also, a holomorphic connection on Ξ gives rise to a connection in the $\mathcal{C}(n)_{an}$-module sense

$$E\colon M \to M\{dz_1, \dots, dz_n\}\,; \tag{37}$$

note that we have a canonical differentiation

$$\mathcal{C}(n)_{an} \to \mathcal{C}(n)_{an}\{dz_1, \dots, dz_n\}.$$

We may then define flat, homogeneous connections and connections with a given degree in terms of the algebraic connection (37), and we can mimic the definitions of the previous section using the space $\chi(m_1, \dots, m_n, n)_{an}$. One complication to the compatibility of our notions however is that $(\mathcal{C}(m_1)\otimes\cdots\otimes\mathcal{C}(m_n))\widehat{\otimes}\mathcal{C}(n)$ is not a subspace of $\chi(m_1, \dots, m_n, n)_{an}$. Nevertheless, if we denote by $\chi(m_1, \dots, m_n, n)$ the intersection of $(\mathcal{C}(m_1) \otimes \cdots \otimes \mathcal{C}(m_n))\widehat{\otimes}\mathcal{C}(n)$ and $\chi(m_1, \dots, m_n, n)_{an}$ in $P(m_1, \dots, m_n, n)$, then the right hand side of (26) can be replaced by $\chi(m_1, \dots, m_n, n)$, which allows a comparison.

We will need to define a regular flat homogeneous connection

$$E\colon M' \to M'\{dt_{ij}, dz_i\}$$

on a finitely generated projective $\chi(m_1, \dots, m_n, n)$-module M where

$$M' = M \otimes_{\chi(m_1,\dots,m_n,n)} \chi(m_1, \dots, m_n, n)_{an}\,.$$

We will say that a flat connection E on M' is regular if each of its solutions has moderate growth. We say that a solution g has moderate growth if for every locally closed smooth algebraic curve C in (31), every holomorphic embedding $D \subset \overline{C}$ with non-zero derivative at $0 \in D$ (where C is a smooth compactification of C) and every continuous function ϵ on D such that for every $x = (t_{ij}, z_i) \in D - \{0\}$, g is defined for $x' = (\delta t_{ij}, z_i)$ whenever $\delta_i \leq \epsilon(x)$, there exists an $N > 0$ such that

$$g(x') < \Big(\prod \delta_i\Big)^{-N} \|u\|^{-N} \tag{38}$$

where u denotes the standard holomorphic coordinate on D and in (38), g denotes a branch of g on a sector S of $D - \{0\}$ ([4]); we may define a branch of a solution of E on S as a holomorphic function on S which satisfies $\nabla g = 0$ where ∇ is the pullback of E to S. The multi-valued section g will also be referred to as *regular.*

Our first version of Riemann-Hilbert correspondence is the following

Lemma 7. *There is an equivalence of categories (canonical up to canonical isomorphism) between the following categories:*

(39) *Finite-dimensional $\mathbb{C}^\times$-equivariant holomorphic vector bundles on $C(n)$ with a homogeneous flat connection of degree k*

(40) *Finite rank projective $\mathbb{Z}$-graded $\mathcal{C}(n)$-modules with a homogeneous regular flat connection of degree k.*

Proof. When $n = 1$, then every local system on $C(1) \cong \mathbb{A}^1_{\mathbb{C}}$ is trivial. Treating the local system as a free $\mathcal{C}(1)_{an}$-module M with a flat connection E, a priori the free generators of M may not be graded. However, since all homogeneous functions on $C(1)$ have non-negative integral degrees, we have a decreasing filtration F^kM consisting of all elements of degree $\geq k$. Then, we can consider the associated graded object

$$F^kM/F^{k+1}M\,. \tag{41}$$

Looking at the lowest k for which (41) is non-trivial, the fact that M clearly implies that (41) is generated by homogeneous elements of degree k. Further, the connection must be 0 on these generators by homogeneity. Consider the free graded submodule generated by these elements by M_0. Then M/M_0 is also a free module (the category of finite dimensional free $\mathcal{C}(1)_{an}$-modules with flat connection is equivalent to the category of finite dimensional $\mathbb{C}$-vector spaces), so we may repeat the argument with M replaced by M/M_0 to show by induction that M is free as a graded $\mathcal{C}(1)_{an}$-module with generators annihilated by the connection. Since clearly an analogous argument applies to the algebraic category, the statement follows.

Assume now $n > 1$. Then any z_i defines an isomorphism of vector spaces from $\mathcal{C}(n)_k$ to $\mathcal{C}(n)_{k-1}$. The category of graded $\mathcal{C}(n)$-modules and morphisms of degree 0 is therefore equivalent to the category of $\mathcal{C}(n)_0$-modules. Similar statements are also valid for the corresponding analytic categories. But now $\mathcal{C}(n)_0$ is in fact the coefficient ring of a smooth affine variety $C(n)_0$, which comes with an embedding into $\mathbb{P}^{n-1}$ with homogeneous coordinates $z_1, \dots, z_n$ (in fact, the embedding factors through the copy of $\mathbb{A}^{n-1}$ which is the complement of the locus of $z_1 - z_2$). The statement for connections is that flat connections (algebraic or analytic) on $C(n)_0$ correspond precisely to homogeneous connections on $\mathcal{C}(n)$ resp. $\mathcal{C}(n)_{an}$ of degree 0: This is because

$$\Omega^1_{\mathcal{C}(n)_0/F} = \{a_1\frac{dz_1}{z_1} + dots + a_n\frac{dz_n}{z_n} \in (\Omega^1_{\mathcal{C}(n)/F})_0 | a_1 + \cdots + a_n = 0\}\,.$$

Therefore, for homogeneous flat connections of degree 0, the result follows from Theorem 5.9 of [4], applied to the variety $C(n)_0 = \mathrm{Spec}(\mathcal{C}(n)_0)$. The case of general degree k can be reduced to the case of degree 0 by subtracting an algebraic connection of degree k. □

Our main interest is in the following statement:

Theorem 8. *There is an equivalence of categories (canonical up to canonical isomorphism) between the category of finite analytical tree functors (resp. algebras) and finite regular tree functors (resp. algebras).*

Proof. Given Lemma 7, and the fact that the two connections whose isomorphism we are seeking are obviously regular, the statement amounts to asserting that an isomorphism in

$$\chi(m_1, \dots, m_n, n)_{an} \tag{42}$$

between two graded regular connections on

$$\chi(m_1, \dots, m_n, n) \tag{43}$$

of the same degree is algebraic. However, this amounts to saying that every regular function in (42) is in (43). This can be shown as follows: consider a regular function in (42). Then considering g as a function of the t_{ij}'s for fixed z_i, expand in the total degree of all the t_{ij}'s for a fixed i. Then the coefficients of fixed total degree d_i in the t_{ij}'s for each i are obviously elements of

$$\mathcal{C}(m_1)_{d_1} \otimes \cdots \otimes \mathcal{C}(m_n)_{d_n}$$

and further each of the numbers d_i where the coefficient is non-zero is bounded below by a bound $N(z_1, \dots, z_n)$. Since this bound, however, is locally constant, it must be in effect constant because the function involved are analytic.

Now we claim that for each given assortment of degrees $d_1, \dots, d_n$, the corresponding component

$$g_{d_1, \dots, d_n} \tag{44}$$

of g is in effect an element of

$$\mathcal{C}(m_1) \otimes \cdots \otimes \mathcal{C}(m_n) \otimes \mathcal{C}(n)\,.$$

In fact, select the lowest $(d_1, \dots, d_n)$ (say, lexicographically) such that (44) is non-trivial. Then by (38), all summands of higher degree can be neglected, and (44) must be regular. Subtracting this component, we may show by induction that all of the components are regular. □

Definition 9. *Recall the definition of vertex tensor category of Huang and Lepowsky* [14]*, Definition 4.1. We call a vertex tensor category* semisimple *if there are finitely many objects (called* irreducible objects*) whose endomorphism groups are $\mathbb{C}$ and such that there are no nonzero morphisms between non-isomorphic irreducible objects, and every object is isomorphic to a direct sum of irreducible objects, and the unit object is irreducible.*

Theorem 10. *There is a 'realization' functor (canonical up to natural equivalence) from the category of vertex tensor categories and isomorphisms, to the category of analytic tree algebras and isomorphisms, and consequently, by Theorem 8, to the category of tree algebras and isomorphisms.*

Comment: By the work of Lepowsky and Huang, vertex algebras which satisfy certain 'rationality' conditions supply examples of vertex tensor categories in the sense of [14]. A good survey is Huang [16]. It should also be noted that the present result overlaps with Huang's construction [13] of genus 0 correlation functions for modules of vertex algebras. The major point of interest of our result is that it extends to the algebraic category of tree algebras over $\mathbb{C}$. It should be noted that we model only a part of the structure of vertex tensor category in our axioms (e.g. we do not treat the conformal element = energy-momentum tensor), which is one of the reasons why we do not get an equivalence of categories in our statement.

Proof of Theorem 10. We will study the definition of vertex tensor category [14], Definition 4.1. First of all, because we do not treat conformal element data, we restrict attention to the subspaces $K(n)_0$ of the moduli spaces $K(n)$ (see Huang [11], p.65) where the tube functions are the identity, and the scaling constant is 1. In this setting, there is a canonical splitting of the determinant bundle, so we get a canonical map

$$\psi : K(n)_0 \to \widetilde{K}^c(n)\,. \tag{45}$$

Next, for a semisimple vertex tensor category $\mathcal{V}$, the set of labels Λ is the set of representatives of isomorphism classes of irreducible objects (we shall also write $V_\lambda = \lambda$), and 1 corresponds to the unit object [14], Definition 4.1. (3). Now for and $Q \in K(2)_0$, and irreducible objects V_λ, V_μ, we consider their tensor product [14], Definition 4.1 (1)

$$W := V_\lambda \boxtimes_{\psi(Q)} V_\mu. \tag{46}$$

Then by our definition of semisimple vertex tensor category, we have a unique decomposition up to isomorphism

$$W = \bigoplus_{\nu\in\Lambda} W_\nu \tag{47}$$

where, non-canonically,

$$W_\nu \cong N \otimes V_\nu \tag{48}$$

for a finite-dimensional complex vector space N. Further, by our assumptions, non-canonically, we have

$$\mathrm{Aut}(W_\nu) \cong GL(N)\,. \tag{49}$$

Property (6), together with axioms (1) and (2) of Definition 4.1 of [14] imply that (49) define a principal smooth $GL(N)$-bundle with flat connection on the space $C(2)$. We let $M_{\lambda,\mu,\nu}$ be the dual of the associated vector bundle (which the flat connection automatically makes analytic). The correlation function $\phi_{\lambda,\mu,\nu}$ (the analytic version of (22)) then follows from the universal intertwining operator from $V_\lambda \otimes V_\mu$ to (46).

General correlations functions with an arbitrary number of arguments are then produced in an analogous way by iterating the tensor product (46). Co-operad associativity resp. unitality resp. equivariance follow from property (4) resp. (7) resp. (5) of [14], Definition 4.1. Vertex algebra unitality follows from property (8). $\square$

Example. *The chiral WZW model. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. In* [15]*, Huang and Lepowsky construct a vertex tensor category of finite sums of irreducible lowest weight-modules $L(k,\lambda)$ over the quotient $L(k,0)$ of the affine vertex algebra $M(k,0)$ by its maximal ideal (=maximal proper graded submodule) for $k = 0, 1, 2, \ldots$. By Theorem 10, this gives rise to a tree algebra T over $\mathbb{C}$. In effect, we have the following refinement:*

Theorem 11. *Let $\mathfrak{g}$ be defined over $\mathbb{C} \supseteq k \supseteq \mathbb{Q}$. Then the tree algebra corresponding to the Huang-Lepowsky construction can be defined over k.*

Proof. The key point is to study the so called Knizhnik-Zamolodchikov equations [15], (2.14), which define the desired homogeneous flat connection on the tree functor. The connection is defined on the trivial $\mathcal{C}(n)$-module

$$N_{\lambda_1,\ldots,\lambda_n,\lambda_\infty} = L(\lambda_1) \otimes \cdots \otimes L(\lambda_n) \tag{50}$$

where $L(\lambda)$ is the summand of lowest degree of $L(k,\lambda)$. The flat connection defined by the KZ-equations maps

$$f\colon L(\lambda_1) \otimes \cdots \otimes L(\lambda_n)$$

to

$$df - \frac{1}{k+h^\vee} \sum_{p\neq\ell} \frac{1}{z_\ell - z_p} \sum_i f(Id \otimes \cdots \otimes g^i \otimes \cdots \otimes g_i \otimes \cdots \otimes Id) dz_\phi \tag{51}$$

where on the right hand side, (g^i) and (g_i) are dual bases of $\mathfrak{g}$ with respect to the Killing form, and are inserted at the ℓ'th and p'th coordinate, respectively.

Manifestly, the connection (51) is defined over $\mathbb{Q}$. The tree functor is actually a direct summand of (50). It corresponds to $\mathfrak{g}$-equivariant maps from the lowest weight summand of

$$(L(k,\lambda_1) \boxtimes \cdots \boxtimes L(k,\lambda_n))_Q$$

to $L(k,\lambda_\infty)$ where Q encodes the moduli data [14], which, in our case, is just the n-tuple of points $(z_1,\ldots,z_n)$. By definition, all this is defined over k. $\square$

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