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ON SPECIAL TYPES OF NONHOLONOMIC 3-JETS

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Abstract. We deduce a classification of all special types of nonholonomic 3-jets. In the introductory part, we summarize the basic properties of nonholonomic r-jets.

Generally speaking, a very attractive phenomenon of the problem of classifying the special types of nonholonomic 3-jets is that its solution is heavily based on the Weil algebra technique, even though no algebras appear in the formulation of the problem. We start with summarizing some properties of classical r-jets from the viewpoint used in the present paper and then we mention the basic properties of nonholonomic r-jets, [2]. The second part of Section 1 is devoted to the categorial approach to the concept of special type of nonholonomic r-jets from [7]. In Section 2 we describe the nonholonomic 3-jets in detail. Section 3 contains our previous classification results concerning nonholonomic 2-jets, [5], semiholonomic 3-jets, [3], and two lemmas on the invariant homomorphisms of the related Weil algebras. Section 4 is devoted to the fundamental algebraic properties of the Weil algebra $\widetilde{\mathbb{D}}^3_m$ corresponding to the nonholonomic 3-jets. In Section 5 we classify those nonholonomic 3-jets that are not one-semiholonomic. The classification list is completed in the last section.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [8].

1. Nonholonomic r-jets. The classical, or holonomic, r-jets $X = j^r_x\varphi$ of smooth maps $\varphi\colon M \to N$ form a fibered manifold $J^r(M,N) \to M \times N$ with respect to the source and target projections $\alpha X = x \in M$ and $\beta X = \varphi(x) \in N$. All r-jets form a category J^r over pointed manifolds (M,x): if $X \in J^r_x(M,N)_y$ and $Z = j^r_y\psi \in J^r_y(N,Q)_z$, then $Z \circ X = j^r_x(\psi\circ\varphi) \in J^r_x(M,Q)_z$. We write $L^r_{m,n} = J^r_0(\mathbb{R}^m,\mathbb{R}^n)_0$. Then

$$L^r = \bigcup_{m,n\in\mathbb{N}} L^r_{m,n}$$

is a category over integers called the skeleton of J^r. Clearly, J^r can be reconstructed from L^r: if $\dim M = m$ and $\dim N = n$, then $J^r(M,N)$ coincides with the

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associated bundle

$$J^r(M,N) = (P^rM \times P^rN)[L^r_{m,n}]\,,$$

where $P^rM(M,G^r_m)$ or $P^rN(N,G^r_n)$ is the r-th order frame bundle of M or N, [8]. Further, J^r is a bundle functor of the product category $\mathcal{M}f_m \times \mathcal{M}f$, where $\mathcal{M}f$ denoted the category of all manifolds and all smooth maps and $\mathcal{M}f_m$ is the category of m-dimensional manifolds and local diffeomorphisms. If $f\colon M \to M'$ is a local diffeomorphism and $g\colon N \to N'$ is a smooth map, then

$$J^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1} \in J^r(M',N')\,, \tag{1}$$

[8]. Clearly, J^r preserves products in the second factor, i.e.

$$J^r(M,N_1 \times N_2) = J^r(M,N_1) \times_M J^r(M,N_2)\,,$$

where the fiber product is constructed with respect to the source projection. Moreover, $\varphi\colon M \to N$ defines a map $j^r\varphi\colon M \to J^r(M,N)$, $(j^r\varphi)(x) = j^r_x\varphi$. Then $j^r_x\varphi$ can be identified with $j^1_x(j^{r-1}\varphi)$.

The bundle $\widetilde{J}^2(M,N)$ of nonholonomic 2-jets of M into N is the space of 1-jets $X = j^1_x f$ of the α-sections of $J^1(M,N)$, i.e. the maps $f\colon M \to J^1(M,N)$ satisfying $\alpha f(u) = u$, $u \in M$. This is a bundle over $M \times N$ with respect to the source projection $\alpha(j^1_x f) = x$ and the target projection $\beta(j^1_x f) = \beta(f(x)) \in N$, where β on the right hand side is the target projection of $J^1(M,N)$. Local coordinates x^i on M and y^p on N induce the additional coordinates y^p_i on $J^1(M,N)$. It will be useful to write $y^p = y^p_0$. So the coordinate expression of an α-section $f(u)$ is $f^p_0(u)$, $f^p_i(u)$. Then x^i and $y^p_{h_1h_2}$, h_1, $h_2 = 0,1\dots,m$,

$$y^p_{00} = f^p_0(x)\,, \quad y^p_{i0} = f^p_i(x)\,, \quad y^p_{0i} = \frac{\partial f^p_0(x)}{\partial u^i}\,, \quad y^p_{ij} = \frac{\partial f^p_i(x)}{\partial u^j} \tag{2}$$

are the induced coordinates on $\widetilde{J}^2(M,N)$. The subset $J^2(M,N) \subset \widetilde{J}^2(M,N)$ is characterized by

$$y^p_{i0} = y^p_{0i} \quad \text{and} \quad y^p_{ij} = y^p_{ji}\,. \tag{3}$$

We have two projections ϱ_1, $\varrho_2\colon \widetilde{J}^2(M,N) \to J^1(M,N)$, $\varrho_1(j^1_x f) = f(x)$, $\varrho_2(j^1_x f) = j^1_x(\beta \circ f)$. In coordinates, $\varrho_1(X) = (x^i, y^p_{00}, y^p_{i0})$, $\varrho_2(X) = (x^i, y^p_{00}, y^p_{0i})$. We say that $X \in \widetilde{J}^2(M,N)$ is semiholonomic, if $\varrho_1(X) = \varrho_2(X)$, i.e. $y^p_{i0} = y^p_{0i}$. We write $\bar{J}^2(M,N)$ for the bundle of all semiholonomic 2-jets of M into N, By (3), $J^2(M,N) \subset \bar{J}^2(M,N)$.

Even the nonholonomic 2-jets form a category $\widetilde{J}^2$ over pointed manifolds. For $X = j^1_x f(u) \in \widetilde{J}^2_x(M,N)_y$ and $Z = j^1_y g(v) \in \widetilde{J}^2_y(N,Q)_z$ one defines

$$Z \circ X = j^1_x\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}^2_x(M,Q)_z \tag{4}$$

with the composition of 1-jets on the right hand side. If X and Z are holonomic, then (4) coincides with the classical composition of 2-jets. Indeed, if $X = j^1_x(j^1_u\varphi)$ and $Z = j^1_y(j^1_v\psi)$, then

$$j^1_x\big(j^1_u(\psi \circ \varphi)\big) = j^1_x\big(j^1_{\varphi(u)}\psi \circ j^1_u\varphi\big) \tag{5}$$

and the right hand sides of (4) and (5) are the same.

Assume by induction that we have constructed the bundle $\widetilde{J}^{r-1}(M,N) \to M \times N$ with the source projection $\alpha\colon \widetilde{J}^{r-1}(M,N) \to M$ and the target projection $\beta\colon \widetilde{J}^{r-1}(M,N) \to N$ and $r-1$ canonical projections $\varrho_1, \dots, \varrho_{r-1}\colon \widetilde{J}^{r-1}(M,N) \to \widetilde{J}^{r-2}(M,N)$. Then we define the bundle of nonholonomic r-jets of M into N to be the space $\widetilde{J}^r(M,N)$ of 1-jets of α-sections $f\colon M \to \widetilde{J}^{r-1}(M,N)$. This is a bundle over $M \times N$ with respect to the source and target projections $\alpha(j^1_x f) = x$ and $\beta(j^1_x f) = \beta(f(x)) \in N$ where β on the right hand side means the target projection of $\widetilde{J}^{r-1}(M,N)$.

Assume by induction that $(x^i, y^p_{h_1\dots h_{r-1}})$, $h_1, \dots, h_{r-1} = 0, 1, \dots, m$, are the local coordinates on $\widetilde{J}^{r-1}(M,N)$ induced by x^i on M and y^p on N and the coordinate form of ϱ_s is

$$\varrho_s(x^i, y^p_{h_1\dots h_{r-1}}) = \left(x^i, y^p_{h_1\dots h_{r-s-1} 0 h_{r-s+1}\dots h_{r-1}}\right), \quad s = 1, \dots, r-1\,.$$

The coordinate expression of f is $f_{h_1\dots h_{r-1}}(u)$. This induces the coordinates $(x^i, y^p_{h_1\dots h_r})$, $h_1, \dots, h_r = 0, 1, \dots, m$ on $\widetilde{J}^r(M,N)$ by

$$y^p_{h_1\dots h_{r-1}0} = f^p_{h_1\dots h_{r-1}}(x)\,, \quad y^p_{h_1\dots h_{r-1}j} = \frac{\partial f_{h_1\dots h_{r-1}}(x)}{\partial u^j}\,. \tag{6}$$

We construct r projections $\beta_s\colon \widetilde{J}^r(M,N) \to \widetilde{J}^{r-1}(M,N)$ by

$$\beta_1(j^1_x f) = f(x)\,, \quad \beta_{s+1}(j^1_x f) = j^1_x(\varrho_s \circ f)\,, \qquad s = 1, \dots, r-1\,. \tag{7}$$

Hence the coordinate form of β_s is

$$\beta_s\left(x^i, y^p_{h_1\dots h_r}\right) = \left(x^i, y^p_{h_1\dots h_{r-s} 0 h_{r-s+2}\dots h_r}\right). \tag{8}$$

An element $X \in \widetilde{J}^r(M,N)$ is called semiholonomic, if $\beta_1 X = \dots = \beta_r X$. The bundle of all semiholonomic r-jets is denoted by $\bar{J}^r(M,N)$. Write $\langle h_1 \dots h_r\rangle = (i_1 \dots i_s)$ for the subsequence of all nonzero indices and $|h_1 \dots h_r|$ for the set $\{i_1, \dots, i_s\}$. Then the elements of $\bar{J}^r(M,N)$ are characterized by

$$y^p_{h_1\dots h_r} = y^p_{l_1\dots l_r} \quad \text{whenever} \quad \langle h_1 \dots h_r\rangle = \langle l_1 \dots l_r\rangle\,.$$

The inclusion $J^r(M,N) \hookrightarrow \widetilde{J}^r(M,N)$ is defined by

$$j^r_x\varphi \mapsto j^1_x(j^{r-1}\varphi) \tag{9}$$

Hence the elements of $J^r(M,N) \subset \bar{J}^r(M,N)$ are characterized by

$$y^p_{h_1\dots h_r} = y^p_{l_1\dots l_r} \quad \text{whenever} \quad |h_1 \dots h_r| = |l_1 \dots l_r|\,.$$

Remark 1. G. Virsik clarified, [12], that the composition of various canonical projections $\widetilde{J}^r(M,N) \to \dots \to \widetilde{J}^k(M,N)$ define $\binom{r}{k}$ canonical projections $\widetilde{J}^r(M,N) \to \widetilde{J}^k(M,N)$. They are in bijection with the fixations of $r-k$ zeros in the sequence $h_1, \dots, h_r$. Given a fixation $\varkappa$ of $r-k$ elements from $1, \dots, r$, the coordinate form of the corresponding projection is $(x^i, y^p_{h_1\dots h_r}) \mapsto (x^i y^p_{\varkappa(h_1\dots h_r)})$, where $\varkappa(h_1 \dots h_r)$ means that we replace the indices at the prescribe places by zeros. Then we can define an element of $\widetilde{J}^r(M,N)$ to be k-semiholonomic, if all its canonical projections into $\widetilde{J}^k(M,N)$ coincide. We write $\widetilde{J}^{r,k}(M,N)$ for the bundle of all k-semiholonomic r-jets. Clearly, $(r-1)$-semiholonomic means semiholonomic in the classical sense. In [4] we deduced some geometric properties of one-semiholonomic r-jets.

The composition of $X = j^1_x f \in \widetilde{J}^r(M,N)_y$ and $Z = j^1_y g \in \widetilde{J}^r(N,Q)_z$ is defined by

$$Z \circ X = j^1_x\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}^r_x(M,Q)_z \tag{10}$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. The associativity of (10) is proved e.g. in [7]. So $\widetilde{J}^r$ is a category over pointed manifolds. Modifying (5), we deduce that for two holonomic r-jets, (10) coincides with the classical composition.

Lemma 1. *Every projection β_s, $s = 1, \dots, r$, preserves the composition of nonholonomic jets.*

Proof. For β_1 this follows directly from(10). Assume by induction that every ϱ_s, $s = 1, \dots, r-1$, preserves the composition of nonholonomic jets. Then we have

$$\begin{aligned}\beta_s\big(j^1_x(g(\beta f(u)) \circ f(u))\big) &= j^1_x\big(\varrho_{s-1}(g(\beta f(u)) \circ f(u))\big)\\ &= j^1_x\big(\varrho_{s-1}(g(\beta f(u))) \circ \varrho_{s-1}(f(u))\big) = \beta_s(j^1_y g) \circ \beta_s(j^1_x f)\,.\end{aligned}$$

In particular, Lemma 1 implies that the composition of two k-semiholonomic r-jets is a k-semiholonomic r-jet.

Even $\widetilde{J}^r$ can be interpreted as a functor on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$\widetilde{J}^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1} \tag{11}$$

with the composition of nonholonomic r-jets. Using induction, one verifies directly

$$\widetilde{J}^r(M, N_1 \times N_2) = \widetilde{J}^r(M,N_1) \times_M \widetilde{J}^r(M,N_2)\,.$$

The same holds in the k-semiholonomic case. Write

$$\widetilde{L}^r_{m,n} = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^n)_0\,, \quad \widetilde{L}^r = \bigcup_{m,n\in\mathbb{N}} \widetilde{L}^r_{m,n}\,.$$

Then $\widetilde{L}^r$ is a category over integers, called the skeleton of $\widetilde{J}^r$. Analogously to the holonomic case,

$$\widetilde{J}^r(M,N) = (P^r M \times P^r N)[\widetilde{L}^r_{m,n}]\,.$$

We recall that $X \in \widetilde{J}^r_x(M,N)_y$ is said to be regular, if there exists $Z \in \widetilde{J}^r_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$. In [7], we introduced the following concept. □

Definition 1. A nonholonomic r-jet category C is a rule transforming every pair (M,N) of manifolds into a fibered submanifold $C(M,N) \subset \widetilde{J}^r(M,N)$ such that

(i) $J^r(M,N) \subset C(M,N)$ is fibered submanifold,

(ii) if $X \in C_x(M,N)_y$ and $Z \in C_y(N,Q)_z$, then $Z \circ X \in C_x(M,Q)_z$,

(iii) if X is regular in $\widetilde{J}^r$, then there exists $Z \in C_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$,

(iv) $C(M, N_1 \times N_2) = C(M,N_1) \times_M C(M,N_2)$.

Write

$$L^C_{m,n} = C_0(\mathbb{R}^m,\mathbb{R}^n)_0 \subset \widetilde{L}^r_{m,n} \quad \text{and} \quad L^C = \bigcup_{m,n\in\mathbb{N}} L^C_{m,n}\,.$$

This skeleton is a subcategory of $\widetilde{L}^r$ and we have

$$C(M,N) = (P^rM \times P^rN)[L^C_{m,n}]\,.$$

According to [7] and [3], the special types of nonholonomic r-jets are identified with the nonholonomic r-jet categories.

2. Nonholonomic 3-jets. We summarize some facts and formulae concerning nonholonomic 2-jets and 3-jets, that will be applied to our classification problem.

We have three projections $\beta_1,\beta_2,\beta_3\colon \widetilde{J}^3(M,N)\to\widetilde{J}^2(M,N)$ and two projections $\varrho_1,\ \varrho_2\colon \widetilde{J}^2(M,N)\to J^1(M,N)$. According to Remark 1, this gives rise to three projections $\gamma_1,\gamma_2,\gamma_3\colon \widetilde{J}^3(M,N)\to J^1(M,N)$, $\gamma_1=\varrho_1\circ\beta_1=\varrho_1\circ\beta_2$, $\gamma_2=\varrho_2\circ\beta_1=\varrho_1\circ\beta_3$, $\gamma_3=\varrho_2\circ\beta_2=\varrho_2\circ\beta_3$.

It is sufficient to write the related coordinate expressions on $\widetilde{L}^2_{m,n}$ and $\widetilde{L}^3_{m,n}$. The canonical coordinates of $X\in\widetilde{L}^2_{m,n}$ are $(y^p_{i0},y^p_{0i},y^p_{ij})$ and $\varrho_1(X)=(y^p_{i0})$, $\varrho_2(X)=(y^p_{0i})$. If $Z\in\widetilde{L}^2_{n,q}$, $Z=(z^a_{p0},z^a_{0p},z^a_{pq})$, then the coordinates $(v^a_{i0},v^a_{0i},v^a_{ij})$ of $V=Z\circ X\in\widetilde{L}^2_{m,q}$ are

$$v^a_{i0}=z^a_{p0}y^p_{i0}\,,\quad v^a_{0i}=z^a_{0p}y^p_{0i}\,,\quad v^a_{i0}=z^a_{pq}y^p_{i0}y^q_{0j}+z^a_{p0}y^p_{ij}\,. \tag{12}$$

In the third order, the coordinate expression of $X\in\widetilde{L}^3_{m,n}$ is $(y^p_{i00},y^p_{0i0},y^p_{00i},y^p_{ij0},y^p_{i0j},y^p_{0ij},y^p_{ijk})$. Then

$$\begin{aligned}&\beta_1X=(y^p_{i00},y^p_{0i0},y^p_{ij0})\,,\qquad \beta_2X=(y^p_{i00},y^p_{00i},y^p_{i0j})\,,\\&\beta_3X=(y^p_{0i0},y^p_{00i},y^p_{0ij})\,,\\&\gamma_1X=(y^p_{i00})\,,\quad \gamma_2X=(y^p_{0i0})\,,\quad \gamma_3X=(y^p_{00i})\,.\end{aligned} \tag{13}$$

If we express $Z\in\widetilde{L}^3_{n,q}$ and $V\in\widetilde{L}^3_{n,q}$ analogously as above, then the coordinate expression of $V=Z\circ X$ is

$$v^a_{i00}=z^a_{p00}y^p_{i00}\,,\quad v^a_{0i0}=z^a_{0p0}y^p_{0i0}\,,\quad v^a_{00i}=z^a_{00p}y^p_{00i}\,, \tag{14}$$

$$\begin{aligned}&v^a_{ij0}=z^a_{pq0}y^p_{i00}y^q_{0j0}+z^a_{p00}y^p_{ij0}\,,\quad v^a_{i0j}=z^a_{p0q}y^p_{i00}y^q_{00j}+z^a_{p00}y^p_{i0j}\\&v^a_{0ij}=z^a_{0pq}y^p_{0i0}y^q_{00j}+z^a_{0p0}y^p_{0ij}\end{aligned} \tag{15}$$

$$\begin{aligned}v^a_{ijk}=\ &z^a_{pqr}y^p_{i00}y^q_{0j0}y^r_{00k}+z^a_{pq0}(y^p_{i0k}y^q_{0j0}+y^p_{i00}y^q_{0jk})\\&+z^a_{p0q}y^p_{ij0}y^q_{00k}+z^a_{p00}y^p_{ijk}\,.\end{aligned} \tag{16}$$

Now we describe the related Weil algebras. In general, a Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A=\mathbb{R}\times N$, where N is the ideal of all nilpotent elements, [6]. There exists an integer r such that $N^{r+1} = 0$, the smallest r is called the order of A. The multiplication in A is determined by the multiplication in N, for

$$(x_1 + n_1)(x_2 + n_2) = x_1x_2 + x_1n_2 + x_2n_1 + n_1n_2\,, \quad x_1, x_2 \in \mathbb{R},\ n_1, n_2 \in N\,.$$

According to the general theory, $\widetilde{D}^r_m = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times \widetilde{N}^r_m$ is a Weil algebra, [6]. Clearly, $\widetilde{N}^r_m = \widetilde{L}^r_{m,1}$. So $Z \in \widetilde{N}^3_m$ is of the form

$$Z = (z_{i00}, z_{0i0}, z_{00i}, z_{ij0}, z_{i0j}, z_{0ij}, z_{ijk})\,. \tag{17}$$

The algebra multiplication in $\widetilde{\mathbb{D}}^r_m$ is induced by the multiplication of reals, [6]. This implies that the product Z of X, $Y \in \widetilde{N}^3_m$ is of the form

$$\begin{aligned} &z_{i00} = 0\,, \quad z_{0i0} = 0\,, \quad z_{00i} = 0\,, \quad z_{ij0} = x_{i00}y_{0j0} + y_{i00}x_{0j0} \\ &z_{i0j} = x_{i00}y_{00j} + y_{i00}x_{00j}\,, \quad z_{0ij} = x_{0i0}y_{00j} + y_{0i0}x_{00j}\,, \end{aligned} \tag{18}$$

$$z_{ijk} = x_{0jk}y_{i00} + x_{0j0}y_{i0k} + y_{0jk}x_{i00} + y_{0j0}x_{i0k} + x_{00k}y_{ij0} + y_{00k}x_{ij0}\,. \tag{19}$$

The composition of 3-jets defines a right action of the group $G^3_m = \mathrm{inv}\ J^3_0(\mathbb{R}^m, \mathbb{R}^m)_0$ on $\widetilde{\mathbb{D}}^3_m$. Its coordinate expression can be obtained by specifying (14)–(16).

Clearly, $\mathbb{D}^3_m = J^3_0(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times N^3_m$ and $\bar{\mathbb{D}}^3_m = \bar{J}^3_0(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times \bar{N}^3_m$ are G^3_m-invariant subalgebras of $\widetilde{\mathbb{D}}^3_m$. Consider the canonical injection $i_{m,n}\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$, $x \mapsto (x, 0)$ and write $I^3_{m,n} = j^3_0 i_{m,n}$. The rule $Z \to Z \circ I^3_{m,n}$ defines an algebra epimorphism $\widetilde{\mathbb{D}}^3_{m+n} \to \widetilde{\mathbb{D}}^3_m$, whose coordinate expression is

$$\bar{z}_{h_1h_2h_3} = z_{h_1h_2h_3} \tag{20}$$

with no appearence of $z_{q_1q_2q_3}$ with at least one q_s greater then m, $q_s = 0, 1, \dots, m+n$, $s = 1, 2, 3$ on the right hand side.

3. Some previous classification results. In [5] we deduced that the only nontrivial G^2_m-invariant subalgebra of $\widetilde{\mathbb{D}}^2_m$ containing $\mathbb{D}^2_m$ is $\bar{\mathbb{D}}^2_m$. This implies directly that the only nonholonomic 2-jet categories are $\widetilde{J}^2$, $\bar{J}^2$ and J^2, [3].

In [3] we determined all semiholonomic 3-jet categories C, i.e. $C(M,N) \subset \bar{J}^3(M,N)$. We write $\bar{J}^{3,2}(M,N) \subset \bar{J}^3(M,N)$ for the semiholonomic 3-jets holonomic in the second order. For every sixtuple $d = (d_1, d_2, d_3, d_4, d_5, d_6)$ of reals satisfying $d_1 + \dots + d_6 = 1$, we introduce $\bar{L}^d_{m,n} = \{(y^p_i, y^p_{ij} = y^p_{ji}, d_1y^p_{ijk} + d_2y^p_{ikj} + d_3y^p_{jik} + d_4y^p_{jki} + d_5y^p_{kij} + d_6y^p_{kji})\} \subset \bar{L}^{3,2}_{m,n} = \bar{J}^{3,2}_0(\mathbb{R}^m, \mathbb{R}^n)_0$. Then

$$\bar{L}^d = \bigcup_{m,n\in\mathbb{N}} \bar{L}^d_{m,n} \quad \text{is a subcategory of} \quad \bar{L}^{3,2} = \bigcup_{m,n\in\mathbb{N}} \bar{L}^{3,2}_{m,n}\,,$$

which defines $\bar{J}^d(M,N) \subset \bar{J}^{3,2}(M,N)$. Our result from [3] is: All semiholonomic 3-jet categories are $\bar{J}^3$, $\bar{J}^{3,2}$, J^3 and $\bar{J}^d$ for all d.

In [8], Section 32, we deduced certain properties of second order jet functors, that we now reformulate on the algebra level.

Lemma 2. *The only invariant algebra epimorphism $\mathbb{D}^2_m \to \mathbb{D}^2_m$ is the identity. The only invariant algebra epimorphism $\widetilde{\mathbb{D}}^2_m \to \widetilde{\mathbb{D}}^2_m$ transforming $\mathbb{D}^2_m$ into $\mathbb{D}^2_m$ is* *the identity. All invariant algebra epimorphisms* $\bar{\mathbb{D}}^2_m \to \bar{\mathbb{D}}^2_m$ *form the one parameter family* φ_t

$$\bar{z}_i = z_i\,, \quad \bar{z}_{ij} = tz_{ij} + (1-t)z_{ji}\,, \quad t \in \mathbb{R}\,,\ z_i = z_{i0} = z_{0i}\,. \tag{21}$$

Further, the geometric results from [10] can be reformulated as follows.

Lemma 3. *There is no invariant algebra epimorphism* $\widetilde{\mathbb{D}}^2_m \to \bar{\mathbb{D}}^2_m$ *or* $\widetilde{\mathbb{D}}^2_m \to \mathbb{D}^2_m$. *The only invariant algebra epimorphism* $\sigma\colon \bar{\mathbb{D}}^2_m \to \mathbb{D}^2_m$ *is the symmetrization* $(z_i, z_{ij}) \mapsto (z_i, \frac{1}{2}(z_{ij} + z_{ji}))$.

At the bundle level, σ maps $\bar{J}^2(M,N)$ into $J^2(M,N)$. We remark that σ preserves the jet composition: if $X \in \bar{J}^2_x(M,N)_y$, $Z \in \bar{J}^2_y(N,Q)_z$, then $\sigma(X) \in J^2_x(M,N)_y$, $\sigma(Z) \in J^2_y(N,Q)_z$ and $\sigma(Z \circ X) = \sigma(Z) \circ \sigma(X)$.

Consider a nonholonomic 3-jet category C. Then C defines a bundle functor C_m on $\mathcal{M}f_m \times \mathcal{M}f$. According to the general theory, $\mathbb{D}^C_m = C_0(\mathbb{R}^m, \mathbb{R})$ is a G^3_m-invariant Weil algebra satisfying $\mathbb{D}^3_m \subset \mathbb{D}^C_m \subset \widetilde{\mathbb{D}}^3_m$, [7], [3]. We can reconstruct C from the sequence $\mathbb{D}^C_m = \mathbb{R} \times N^C_m$ by setting $L^C_{m,n} = (N^C_m)^n$.

Thus the problem of finding all nonholonomic 3-jet categories can be divided into 3 steps.

I. We determine all invariant subalgebras $\mathbb{D}^3_m \subset A \subset \widetilde{\mathbb{D}}^3_m$.

II. We consider sequences of them $\mathbb{D}^S_m = \mathbb{R} \times N^S_m$ satisfying (20).

III. We define $L^S_{m,n} = (N^S_m)^n$ and we discuss whether $L^S = \bigcup\limits_{m,n\in\mathbb{N}} L^S_{m,n}$ is a subcategory of $\widetilde{L}^3$.

We point out that in all concrete cases mentioned in this sections, condition III is automatically satisfied. However, in Section 5 below we meet a case in which this is not true. This clarifies that III is independent of I and II.

4. The Weil algebra $\widetilde{\mathbb{D}}^3_m$**.** In general, every Weil algebra $A = \mathbb{R} \times N$ of order r defines the underlying Weil algebra $A_k = A/N^{k+1}$ in every order $k \le r$. Every algebra homomorphism $\mu\colon A \to B$ induces the underlying algebra homomorphism $\mu_k\colon A_k \to B_k$, [6]. We have

$$\begin{aligned}(\widetilde{\mathbb{D}}^3_m)_2 =: \mathbb{B}^{3,2}_m = \{&(X_1, X_2, X_3) \in \widetilde{\mathbb{D}}^2_m \times \widetilde{\mathbb{D}}^2_m \times \widetilde{\mathbb{D}}^2_m, \varrho_1 X_1 = \varrho_1 X_2,\\ &\varrho_2 X_1 = \varrho_1 X_3, \varrho_2 X_2 = \varrho_2 X_3\}\,.\end{aligned} \tag{22}$$

The injection $\mathbb{D}^2_m \hookrightarrow \mathbb{B}^{3,2}_m$ is $X \mapsto (X,X,X)$. We write $\mathbb{B}^{3,2}_m = \mathbb{R} \times N^{3,2}_m$. Hence the coordinates on $N^{3,2}_m$ are

$$(z_{i00}, z_{0i0}, z_{00i}, z_{ij0}, z_{i0j}, z_{0ij})\,. \tag{23}$$

We have $\widetilde{\mathbb{D}}^3_m = \mathbb{B}^{3,2}_m \times \overset{3}{\bigotimes}\mathbb{R}^{m*}$. The right action of G^3_m on $\widetilde{\mathbb{D}}^3_m$ induces a right action of G^2_m on $\mathbb{B}^{3,2}_m$. We define $\widetilde{\mathbb{D}}^{3,1}_m = \widetilde{J}^{3,1}_0(\mathbb{R}^m, \mathbb{R})$. This is an invariant subalgebra of $\widetilde{\mathbb{D}}^3_m$, that is characterized by $z_{i00} = z_{0i0} = z_{00i}$.

A subalgebra $\mathbb{D}^3_m \subset A \subset \widetilde{\mathbb{D}}^3_m$ or $\mathbb{D}^2_m \subset B \subset \mathbb{B}^{3,2}_m$ will be called admissible, if it is G^3_m-invariant or G^2_m-invariant, respectively. Write $\delta\colon \widetilde{\mathbb{D}}^3_m \to \mathbb{B}^{3,2}_m$ for the factor projection. We say that $\delta(A)$ is the boundary of A. Consider the canonical injection $GL(m,\mathbb{R}) = G^1_m \hookrightarrow G^3_m$, $a^i_j \mapsto (a^i_j, 0, 0)$. Using (10), we find immediately

Lemma 4. *If $A \subset \widetilde{\mathbb{D}}^3_m$ is admissible, then the kernel K_A of $\delta \mid A$ is a G^1_m-invariant subspace of $\bigotimes^3 \mathbb{R}^{m*}$ containing $S^3\mathbb{R}^{m*}$.*

Consider an admissible subalgebra $B \subset \mathbb{B}^{3,2}_m$. By (14)–(19) we obtain directly

Lemma 5. $B \times \bigotimes^3 \mathbb{R}^{m*} \subset \mathbb{B}^{3,2}_m \times \bigotimes^3 \mathbb{R}^{m*}$ *is an admissible subalgebra of $\widetilde{\mathbb{D}}^3_m$.*

We write $B \times \bigotimes^3 \mathbb{R}^{m*} = [B]$ and we say that $[B]$ is the inverse of B. So every admissible subalgebra $B \subset \mathbb{B}^{3,2}_m$ is the boundary of an admissible subalgebra of $\widetilde{\mathbb{D}}^3_m$. Thus, we will be interested in the admissible subalgebras of $\mathbb{B}^{3,2}_m$.

Remark 2. The general problem of the underlying lower order bundle functors was studied by M. Doupovec, [1]. In general, every Weil algebra of order r and a group homomorphism $H\colon G^r_m \to \operatorname{Aut} A$ defines a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ by $F(M,N) = P^rM[T^AN, H_N]$, where H_N is the action of G^r_m on the Weil bundle T^AN determined by the natural transformations corresponding to the algebra automorphisms $H(v)\colon A \to A$, $v \in G^r_m$, and $F(f,g) = P^rf[T^Ag]\colon P^rM[T^AN] \to P^rM'[T^AN']$, $f\colon M \to M'$, $g\colon N \to N'$. Then the bundle functor $(\delta\widetilde{J}^3)$ corresponding to $\mathbb{B}^{3,2}_m$ is determined by the generalized fiber product

$$\begin{aligned}(\delta\widetilde{J}^3)(M,N) = &\{X_1, X_2, X_3 \in \widetilde{J}^2(M,N), \varrho_1 X_1 = \varrho_1 X_2,\\ &\varrho_2 X_1 = \varrho_1 X_3, \varrho_2 X_2 = \varrho_2 X_3\}\end{aligned}$$

and $\delta\colon \widetilde{\mathbb{D}}^3_m \to \mathbb{B}^{3,2}_m$ induces a projection

$$\delta_{M,N}\colon \widetilde{J}^3(M,N) \to (\delta\widetilde{J}^3)(M,N)\,, \quad \delta_{M,N}(X) = (\beta_1 X, \beta_2 X, \beta_3 X)\,,$$

[1]. Every admisssible subalgebra $B \subset \mathbb{B}^{3,2}_m$ defines a subbundle $P^2M[T^BN] \subset (\delta\widetilde{J}^3)(M,N)$. The subbundle of $\widetilde{J}^3(M,N)$ determined by $[B]$ is the inverse image of $P^2M[T^BN] \subset (\delta\widetilde{J}^3)(M,N)$ with respect to $\delta_{M,N}$.

Let $B = \mathbb{R} \times N_B$ be an admissible subalgebra of $\mathbb{B}^{3,2}_m$. Write π_s, $s = 1,2,3$, for the projections of B into the individual components of (22). By invariancy and Section 3, $\pi_s(B)$ is $\widetilde{\mathbb{D}}^2_m$ or $\bar{\mathbb{D}}^2_m$ or $\mathbb{D}^2_m$, $s = 1,2,3$. We write D_s for any of them, if suitable. We say that the pair (π_r, π_s), $r \neq s$, is projectable, if there exists an induced map $\psi\colon D_r \to D_s$ satisfying $\psi \circ \pi_r = \pi_s$. By (14)–(16), one deduced easily

Lemma 6. *In the projectable case, ψ is an G^2_m-invariant algebra epimorphism transforming $\mathbb{D}^2_m$ into $\mathbb{D}^2_m$.*

Let $A \subset \widetilde{\mathbb{D}}^3_m$ be an admissible subalgebra with $\delta A = B$. Consider $B_1 = B/N^2_B = \mathbb{R} \times N_1$. If $\xi \in N^2_B$ and $\eta \in N_1$, then

$$\xi = x + K_A\,,\quad \eta = y + N_B^2\,,\quad x \in (\widetilde{N}_m^3)^2\,,\quad y \in \widetilde{N}_m^3 \quad \text{and}$$
$$\xi\eta = (x + K_A)(y + N_B^2) = xy \in \bigotimes^3 \mathbb{R}^{m*}\,.$$

Definition 2. The linear span of all these elements will be denoted by Q_B.

Hence Q_B is a linear subspace of $\overset{3}{\bigotimes}\mathbb{R}^{m*}$ that is determined by B only. Clearly, $Q_B \subset K_A$. According to the definition, the coordinate expression of Q_B is

$$z_{ijk} = x_{0jk}y_{i00} + x_{0j0}y_{i0k} + y_{0jk}x_{i00} + y_{0j0}x_{i0k} + x_{00k}y_{ij0} + y_{00k}x_{ij0}\,, \tag{24}$$

x, $y \in N_B$. In particular, $S^3\mathbb{R}^{m*} \subset Q_B$.

Using (14)–(19), we verify easily

Lemma 7. *$B \times Q_B$ is an admissible subalgebra of $\widetilde{\mathbb{D}}_m^3$.*

Example 1. For $B = \mathbb{B}_m^{3,2}$, we have $Q_B = \overset{3}{\bigotimes}\mathbb{R}^{m*}$. Indeed, the x's and y's are arbitrary. For $x_{ij0} = 1$, $y_{00k} = 1$ and all zeros otherwise, we obtain that every $z_{ijk} = 1$, all zeros otherwise, belongs to Q_B. Since Q_B is a linear subspace, we have $Q_B = \overset{3}{\bigotimes}\mathbb{R}^{m*}$.

Example 2. In the case of $B = (\mathbb{D}_m^2, \widetilde{\mathbb{D}}_m^2, \widetilde{\mathbb{D}}_m^2)$ with $D_2 = D_3$, we have $Q_B = S^2\mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$. Indeed, our conditions are $z_{i00} = z_{0i0}$, $z_{ij0} = z_{ji0}$, $z_{i0j} = z_{0ij}$. This implies that (24) is symmetric in i and j.

So, if we start from an admissible subalgebra $B \subset \mathbb{B}_m^{3,2}$, every admissible subalgebra $A \subset \widetilde{\mathbb{D}}_m^3$ with $\delta A = B$ must satisfy

$$S^3\mathbb{R}^{m*} \subset Q_B \subset K_A \subset \bigotimes^3 \mathbb{R}^{m*}\,,$$

where K_A is a G_m^1-invariant linear subspace. In all cases that we shall meet in the sequel, this implies $K_A = Q_B$ or $K_A = \overset{3}{\bigotimes}\mathbb{R}^{m*}$.

5. Strongly nonholonomic 3-jets. A nonholonomic 3-jet will be called strongly nonholonomic, if it is not one-semiholonomic.

In [5], we deduced: If $A \subset \widetilde{\mathbb{D}}_m^3$ is an admissible subalgebra such that $\gamma_s \mid A$, $s = 1, 2, 3$ do not lie on the same straight line in $\operatorname{Hom}(A, \mathbb{D}_m^1)$, then $A = \widetilde{\mathbb{D}}_m^3$. Take $a = (a_1, a_2, a_3) \neq 0$ with $a_1 + a_2 + a_3 = 0$ and define $\widetilde{\mathbb{D}}_m^a = \{Z \in \widetilde{\mathbb{D}}_m^3, a_1 z_{i00} + a_2 z_{0i0} + a_3 z_{00i} = 0\}$. Using (14)–(16) and (18), one verifies directly that this is an admissible subalgebra. (Hence $\widetilde{\mathbb{D}}_m^a = \mathbb{R} \times \widetilde{N}_m^a$ defines a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, [5].) Write

$$\widetilde{L}_{m,n}^a = (\widetilde{N}_m^a)^n \quad \text{and} \quad \widetilde{L}^a = \bigcup_{m,n\in\mathbb{N}} \widetilde{L}_{m,n}^a\,.$$

Lemma 8. *$\widetilde{L}^a$ is a subcategory of $\widetilde{L}^3$ if and only if $z_{i00} = z_{0i0}$ or $z_{i00} = z_{00i}$ or $z_{0i0} = z_{00i}$.*

Proof. We start with the case $a_1 \neq 0$, so that

$$z_{i00} = bz_{0i0} + cz_{00i} \qquad b + c = 1\,.$$

Then the elements of $\widetilde{L}^a_{m,n}$ satisfy $y^p_{i00} = by^p_{0i0} + cy^p_{00i}$. In the notation of Section 2, we also have $z^a_{p00} = bz^a_{0p0} + cz^a_{00p}$ and $v^a_{i00} = bv^a_{0i0} + cv^a_{00i}$. Then (14) implies $v^a_{i00} = (bz^a_{0p0} + cz^a_{00p})\,(by^p_{0i0} + cy^p_{00i}) = b^2v^a_{0i0} + bcz^a_{0p0}y^p_{00i} + bcz^a_{00p}y^p_{0i0} + c^2v^a_{00i}$. This yields $bc = 0$, $b^2 = b$, $c^2 = c$, $b + c = 1$. Hence $b = 1$, $c = 0$ or $b = 0$, $c = 1$. If we start with $a_2 \neq 0$ or $a_3 \neq 0$, we obtain the remaining possibility $z_{0i0} = z_{00i}$. □

In general, we write $D_{12} = \{(X_1, X_2) \in D_1 \times D_2, \varrho_1 X_1 = \varrho_1 X_2\}$. We define $\pi_{12} \colon B \to D_{12}$, $X \mapsto (\pi_1 X, \pi_2 X)$. Since the algebras $\pi_s(B)$ can be $\widetilde{\mathbb{D}}^2_m$, $\bar{\mathbb{D}}^2_m$ and $\mathbb{D}^2_m$ only, we shall indicate them by writing simply n, s and h in the expression for B of the form (22).

Lemma 9. *Let $B \subset (n, n, s)$ be an admissible subalgebra that is not projectable over π_1, $\pi_2 \colon B \to \widetilde{\mathbb{D}}^2_m$. Then $\pi_{12} \colon B \to D_{12}$ is surjective. The same holds in the cases of (n, n, h), (s, n, n) and (h, n, n).*

Proof. Consider $(x_{i00}, x_{0i0} = x_{00i}, x_{ij0}, x_{i0j}, x_{0ij}) \in N_B$. In the nonprojectable case, there exist x, $y \in B$ such that $\pi_1(x) = \pi_1(y)$ and $\pi_2(x) \neq \pi_2(y)$. Hence y is of the form

$$(x_{i00}, x_{0i0} = x_{00i}, x_{ij0}, x_{i0j} + w_{i0j}, x_{0ij} + w_{0ij}) \quad \text{with} \quad (w_{i0j}) \neq o\,.$$

By continuity, this holds on a neighbourhood of y. Hence π_{12} is surjective on an open subset. But π_{12} is a linear map, so that π_{12} is surjective everywhere. The cases (n, n, h), (s, n, n) and (h, n, n) are discussed in the same way. □

In what follows, the inverses satisfying some additional conditions are indicated by adding them into the square bracket. Further, in the case of $Q_B \neq \overset{3}{\bigotimes}\mathbb{R}^{m*}$, we write all data determining A into a round bracket.

Proposition 1. *All strongly nonholonomic* 3*-jet categories are determined by the inverses*

(i)	$[n, n, n]$,	*which corresponds to*	$\widetilde{J}^3$,
(ii)	$[s, n, n]$,	— " —	$\beta_1^{-1}(\bar{J}^2)$,
(iii)	$[n, s, n]$,	— " —	$\beta_2^{-1}(\bar{J}^2)$,
(iv)	$[n, n, s]$,	— " —	$\beta_3^{-1}(\bar{J}^2)$,
(v)	$[s, n, n, \pi_2 = \pi_3]$,	— " —	$J^1\bar{J}^2$,
(vi)	$[n, n, s, \pi_1 = \pi_2]$,	— " —	$\bar{J}^2J^1$,
(vii)	$[h, n, n]$,	— " —	$\beta_1^{-1}(J^2)$,
(viii)	$[n, h, n]$,	— " —	$\beta_2^{-1}(J^2)$,
(ix)	$[n, n, h]$,	— " —	$\beta_3^{-1}(J^2)$,
(x)	$[h, n, n, \pi_2 = \pi_3]$,	— " —	$\beta_1^{-1}(J^2) \cap J^1\bar{J}^2$,
(xi)	$[n, n, h, \pi_1 = \pi_2]$,	— " —	$\beta_3^{-1}(J^2) \cap \bar{J}^2J^1$

and there are two more cases with nontrivial Q_B

(xii) $(h, n, n, \pi_2 = \pi_3, z_{ijk} = z_{jik})$ *corresponding to* J^1J^2,

(xiii) $(n, n, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$ *corresponding to* J^2J^1.

Proof. By the above mentioned result from [5] and Lemma 8, the only nonholonomic 3-jet category corresponding to an algebra A with $\beta_1(A) = \widetilde{\mathbb{D}}^2_m$, $\beta_2(A) = \widetilde{\mathbb{D}}^2_m$ and $\beta_3(A) = \widetilde{\mathbb{D}}^2_m$ is $\widetilde{J}^3$. By Lemmas 2 and 9, the cases (ii)–(xi) correspond to all possible inverses in this situation. We point out that the case $[n, s, n, \pi_1 = \pi_3]$ is characterized by $(z_{i00}, z_{0i0}, z_{ij0}) = (z_{0i0}, z_{00i}, z_{0ij})$, so that it is one-semiholonomic. In the cases (ii)–(ix), we deduce, analogously to Example 1, that $Q_B = \overset{3}{\otimes}\mathbb{R}^{m*}$ is the only possibility. The case (x) was discussed in Example 2. The conditions characterizing (xi) are $(z_{i00}, z_{0i0}, z_{ij0}) = (z_{i00}, z_{00i}, z_{i0j})$ and $z_{0ij} = z_{0ji}$. If we put these data into Example 2, we obtain $z_{ijk} = z_{ikj}$. One verifies directly that all algebras in question determine subcategories of $\widetilde{L}^3$. □

6. Properly one-semiholonomic 3-jets. It remains to discuss the 3-jets that are properly one-semiholonomic, i.e. not semiholonomic. We write $\mathbb{B}^{3,1}_m \subset \mathbb{B}^{3,2}_m$ for the subset $z_{i00} = z_{0i0} = z_{00i}$.

Lemma 10. *Let $B \subset \mathbb{B}^{3,1}_m$ be an admissible subalgebra that is not projectable over π_1, $\pi_2\colon B \to \bar{\mathbb{D}}^2_m$. Then $\pi_{12}\colon B \to D_{12}$ is surjective. The same holds in the case π_1, $\pi_2\colon B \to \mathbb{D}^2_m$.*

The proof is a replica of the proof of Lemma 9.

The description of the inverses in Propositions 2–5 are quite analogous to Proposition 1, so that we do not express them explicitely.

Proposition 2. *In the case of three s's, we have the categories determined by the inverses*
- (i) $[s, s, s]$,
- (ii) $[s, s, s, \pi_2 = \varphi_t \circ \pi_3]$,
- (iii) $[s, s, s, \pi_1 = \varphi_t \circ \pi_3]$,
- (iv) $[s, s, s, \pi_1 = \varphi_t \circ \pi_2]$,
- (v) $[s, s, s, \pi_2 = \varphi_t \circ \pi_1, \pi_3 = \varphi_\tau \circ \pi_1]$, $t \neq 1$ *or* $\tau \neq 1$.

Proof. This is a direct consequence of Lemma 2 and the evaluations similar to Example 1. In the case $\pi_1 = \pi_2$ and $\pi_2 = \pi_3$ we obtain $\bar{J}^3$. To verify that all algebras in question determine subcategories of $\widetilde{L}^3$, one uses the following geometric fact. At the bundle level, φ_t induces a map $\varphi_t\colon \bar{J}^2(M,N) \to \bar{J}^2(M,N)$ for every t. These maps preserve the jet composition: if $X \in \bar{J}^2_x(M,N)_y$ and $Z \in \bar{J}^2_y(N,Q)_z$, then $\varphi_t(Z \circ X) = \varphi_t(Z) \circ \varphi_t(X)$, $t \in \mathbb{R}$. □

The proofs of Propositions 3–5 are quite analogous to Propositions 1 and 2.

Proposition 3. *In the case of two s's, we have the categories determined by the inverses*
- (i) $[h, s, s]$,
- (ii) $[h, s, s, \pi_1 = \sigma \circ \pi_2]$,
- (iii) $[h, s, s, \pi_1 = \sigma \circ \pi_3]$,
- (iv) $[h, s, s, \pi_2 = \varphi_t \circ \pi_3]$,
- (v) $[s, s, h]$,
- (vi) $[s, s, h, \pi_3 = \sigma \circ \pi_1]$,
- (vii) $[s, s, h, \pi_3 = \sigma \circ \pi_2]$,
- (viii) $[s, s, h, \pi_1 = \varphi_t \circ \pi_2]$,
- (ix) $[s, h, s]$,
- (x) $[s, h, s, \pi_2 = \sigma \circ \pi_1]$,
- (xi) $[s, h, s, \pi_2 = \sigma \circ \pi_3]$,
- (xii) $[s, h, s, \pi_1 = \varphi_t \circ \pi_3]$

and there are two more cases with nontrivial Q_B

(xiii) $(h, s, s, \pi_2 = \pi_3, z_{ijk} = z_{jik})$ *corresponding to* $J^1J^2 \cap \widetilde{J}^{3,1}$,

(xiv) $(s, s, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$ *corresponding to* $J^2J^1 \cap \widetilde{J}^{3,1}$.

Proposition 4. *In the case of two h's, we have the categories determined by the inverses*

(i) $[s, h, h]$,
(ii) $[s, h, h, \pi_2 = \pi_3]$,
(iii) $[s, h, h, \pi_2 = \sigma \circ \pi_1]$,
(iv) $[s, h, h, \pi_3 = \sigma \circ \pi_1]$,
(v) $[h, h, s]$,
(vi) $[h, h, s, \pi_1 = \pi_2]$,
(vii) $[h, h, s, \pi_1 = \sigma \circ \pi_3]$,
(viii) $[h, h, s, \pi_2 = \sigma \circ \pi_3]$,
(ix) $[h, s, h]$,
(x) $[h, s, h, \pi_1 = \pi_3]$,
(xi) $[h, s, h, \pi_1 = \sigma \circ \pi_2]$,
(xii) $[h, s, h, \pi_3 = \sigma \circ \pi_2]$.

In the following assertion, $(\beta_2 = \beta_3)^{-1}(J^2)$ or $(\beta_1 = \beta_2)^{-1}(J^2)$ means all $X \in \widetilde{J}^{3,1}(M, N)$ satisfying $\beta_2 X = \beta_3 X \in J^2(M, N)$ or $\beta_1 X = \beta_2 X \in J^2(M, N)$, respectively.

Proposition 5. *In the case of three h's, we have the categories determined by the inverses*

(i) $[h, h, h]$,
(ii) $[h, h, h, \pi_2 = \pi_3]$,
(iii) $[h, h, h, \pi_1 = \pi_2]$,
(iv) $[h, h, h, \pi_1 = \pi_3]$

and there are two more cases with nontrivial Q_B

(v) $(h, h, h, \pi_2 = \pi_3, z_{ijk} = z_{jik})$ *corresponding to* $J^1J^2 \cap (\beta_2 = \beta_3)^{-1}(J^2)$,

(vi) $(h, h, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$ *corresponding to* $J^2J^1 \cap (\beta_1 = \beta_2)^{-1}(J^2)$.

Clearly, the case $[h, h, h, \pi_1 = \pi_2 = \pi_3]$ corresponds to $\bar{J}^{3,2}$.

Since the special types of semiholonomic 3-jets are determined in [3], we have classified all special types of nonholonomic 3-jets.

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