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LOCAL REFLEXION SPACES

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Abstract. A reflexion space is generalization of a symmetric space introduced by O. Loos in [4]. We generalize locally symmetric spaces to local reflexion spaces in the similar way. We investigate, when local reflexion spaces are equivalently given by a locally flat Cartan connection of certain type.

There are several equivalent definitions of symmetric spaces and locally symmetric spaces. For example, an (affine) locally symmetric space is a connected smooth manifold M with a torsion-free linear connection with parallel curvature. Another definition is, that a (homogeneous) locally symmetric spaces is a locally flat Cartan geometry of type (G, H) on a connected manifold M if there is $h \in H$ such, that $h^2 = \mathrm{id}_G$ and H is open in the centralizer of h in G. The equivalence of these two definitions can be found for example in [6]. Details about Cartan connections can be also found in [2].

The reflexion spaces were introduced by O. Loos in [4]. He found, that reflexion spaces are equivalent to fibre bundles associated to homogeneous symmetric space $G \to G/H$. Precisely, if G/H is a homogeneous symmetric space and $H \times F \to F$ a left action of H on a smooth manifold F, then $G \times_H F$ is a reflexion space. If the Lie group G acts transitively on the reflexion space or equivalently H acts transitively on the fiber, then if we denote K stabilizer of one point of the reflexion space, the structure of reflexion space is equivalently given by a Maurer-Cartan form of $G \to G/K$ i.e. by a flat Cartan connection of type (G, K). We note, that there are further generalizations of reflexion spaces in [5].

Now, we introduce a local version of the reflexion spaces and investigate, under which conditions they are equivalently given by a locally flat Cartan connection of certain type.

Definition 1. Let M be a connected smooth manifold, N a neighborhood of the diagonal in $M \times M$ and $S\colon N \to M$ a smooth mapping. We denote

$$S(x, y) = S_x y = S^y x$$

and we say that S_x is a (local) reflexion at x. We call (M, S) a local reflexion space under the following three conditions:

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(A1) $S_x x = x$

(A2) If $U_x := \{y : (x,y) \in N\}$, then S_x is a diffeomorphism of U_x satisfying $S_x(S_x y) = y$ for all $y \in U_x$.

(A3) There is a neighborhood W of the diagonal in $M \times M \times M$ such, that

$$S_x S(y,z) = S(S_x y, S_x z)$$

holds for all $(x,y,z) \in W$.

Let (M,S) and (M',S') be two local reflexion spaces and $U \subset M$. Then $f\colon U \to M'$ is a local homomorphism of local reflexion spaces (we will say only homomorphism), if $f((U \times U) \cap N) \subset N'$ and

$$f(S_x y) = S'_{f(x)} f(y)$$

for $(x,y) \in (U \times U) \cap N$.

The meaning of conditions (A2) and (A3) is, that all (local) reflexions have to be involutive local automorphisms of local reflexion spaces.

There are the following examples of local reflexion spaces:

Example 2. Let $(p\colon \mathcal{G} \to M, \omega)$ be a locally flat Cartan geometry of type (G,K) and assume, that there is $h \in K$ satisfying $h^2 = \mathrm{id}_G$ and $hk = kh$ for any $k \in K$.

Since the Cartan geometry is locally flat, there is an atlas of M such, that the images of charts are open subsets of G/K and transition maps are restrictions of left actions of elements of G. In particular for all $x \in M$, there is a chart $V_x \subset G/K$ such that $x = eK$.

If it is a flat Cartan geometry of type (G,K), then $M = G/K$ and the chart V_{gK} is given just by the left multiplication by $g^{-1} \in G$. If we take h as the model of the reflexion at gK in the chart V_{gK}, then the reflexions are $S_{gK} fK = ghg^{-1} fK$ on G/K. It is easy to check, that $(G/K, S)$ is a (global) reflexion space. In the proof of Lemma 7 we will prove that it is the unique way how to define S.

Now we return to the locally flat situation. We denote $\bar{V}_x \subset \mathfrak{g}$ some neighborhood of 0 such, that of $p(\exp(\bar{V}_x)) \subset V_x$ and let $\bar{U}_x \subset \bar{V}_x$ be such, that

$$\exp(X) \exp\big(Ad(h)(-X)\big) \exp\big(Ad(h)Y\big)K \in V_x$$

for all $X, Y \in \bar{U}_x$. Then we define

$$N := \bigcup_{x \in M} \big(p\big(\exp(\bar{U}_x)\big), p\big(\exp(\bar{U}_x)\big)\big)\,,$$

so N is a neighborhood of diagonal in $M \times M$ and we define

$$S_{\exp(X)K} \exp(Y)K := \exp(X) \exp\big(Ad(h)(-X)\big) \exp\big(Ad(h)Y\big)K\,.$$

Since K commutes with h, the definition is correct.

We show, that (M,S) is a local reflexion space. Let tfK, $tgK \in p(\exp(\bar{U}_x))$ and fK, $gK \in p(\exp(\bar{U}_y))$ be two different coordinates of the same points of M, where the transition map between those coordinates is a left action of $t \in G$, then

$$S_{tfK} tgK = tfhf^{-1}t^{-1}tgK = tS_{fK} gK$$

i.e. the definition of S does not depend on the choice of coordinates.

Let $\bar{W}_x \subset \bar{U}_x$ be such, that

$$\exp(X)\exp\big(Ad(h)(-X)\big)\exp\big(Ad(h)Y\big)\exp(-Y)\exp(Z)K \in V_x$$

for all X, Y, $Z \in \bar{W}_x$. We define

$$W = \bigcup_{x\in M}\big(p(\exp(\bar{W}_x)), p(\exp(\bar{W}_x)), p(\exp(\bar{W}_x))\big)\,.$$

Checking that (A1), (A2) and (A3) holds, is then an easy computation.

For later use, we will notice that we can reconstruct the local Cartan geometry, under certain conditions. Consider the one parameter subgroup $f_t = \exp(tX)$. Then

$$\frac{d}{dt}|_{t=0} S_{f_tK} S_{eK} gK = R_X(gK) - R_{Ad(h)X}(gK)\,,$$

where R_X is the projection of right invariant vector field of $X \in \bar{W}_x$ on $p(\exp(\bar{W}_x))$. Since $h^2 = \mathrm{id}_G$, we denote $\mathfrak{g}^-$ the -1 eigenspace of $Ad(h)$. Then for $X \in \mathfrak{g}^-$ is $\frac{d}{dt}|_{t=0} S_{f_tK} S_{eK} gK = 2R_X(gK)$ i.e.

$$S_{\exp(X)K} S_{eK} gK = \exp(2X) gK\,.$$

Thus if $\mathfrak{g}^-$ generates the Lie algebra $\mathfrak{g}$ by the Lie bracket, we can reconstruct the right invariant vector fields from $S_x S_e$ action i.e. we can reconstruct locally flat Cartan geometry of type $(\mathfrak{g}, \mathfrak{k})$.

We choose the following representative for the equivalence class of the Cartan geometries obtained in the example:

Definition 3. We say that a local reflexion space (M, S) is locally homogeneous, if it is locally equivalent (as in previous example) to a locally flat Cartan geometry $(p\colon \mathcal{G} \to M, \omega)$ of type (G, K) such, that

- (H1) there is $h \in K$ such, that $h^2 = \mathrm{id}_G$, $hk = kh$ for any $k \in K$
- (H2) the -1 eigenspace of $Ad(h)$ in $\mathfrak{g}$ generates whole $\mathfrak{g}$ by the Lie bracket
- (H3) G/K is connected, simply connected and the maximal normal subgroup of G contained in K is trivial.

Let us discuss the assumptions of our definition on the following simple examples.

Example 4. Let M be $\mathbb{R}^n$ without the origin. Since we can view M as an open subset in Euclidean space $E^n = (e_1, \dots, e_n)$ i.e. homogeneous model of Cartan geometry of type $(E(n), O(n))$, where $E(n)$ is the group of Euclidean motions and $O(n)$ the group stabilizing the origin. Then the pullback to M is a flat Cartan geometry of type $(E(n), O(n))$.

Let us denote $|X, Y|$ the distance of $X, Y \in E^n$ and $|X|$ the distance of $X \in E^n$ from origin. We define $N = \{(X, Y) \in M \times M : \ |X, Y| < |X|\}$ and $W = \{(X, Y, Z) \in M \times M \times M : \ |X, Y| < \frac{1}{3}|X|,\ |X, Z| < \frac{1}{3}|X|\}$, and observe $|X + h(Y - X)| > 0$ and $|X + h(Y - X) + (Z - Y)| > 0$ for any $h \in O(n)$. If $h \in O(n)$ is such that $h^2 = id_{E(n)}$, then we set $S : N \to M$ as $S^h_X Y = X + h(Y - X)$. It is easy to check that (M, S^h) is a local reflexion space. Let us discuss particular choices of $h \in O(n)$ in comparison with the first example.

Let $h_1 \in O(n)$ be the reflexion with respect to the hyperplane orthogonal to first coordinate. Then we can not use the first example to reconstruct S^{h_1}, because h_1 does not commute with $O(n)$. However, there is also a flat Cartan geometry of type $(\mathbb{R}^n \rtimes \mathbb{Z}_2, \mathbb{Z}_2)$ on M, where $\mathbb{R}^n$ is generated by transitions and $\mathbb{Z}_2$ by the reflexion h_1. In this case, the procedure from the first example reconstructs the local reflexion space (M, S^{h_1}). The pseudogroup generated by symmetries are just translations in the first coordinate and we can not reconstruct neither of those two Cartan geometries form the local reflexion space (M, S^{h_1}). Clearly, the local reflexion space is not locally homogeneous. In particular, the first Cartan geometry of type $(E(n), O(n))$ does not satisfy conditions (H1) and (H2), the second Cartan geometry of type $(\mathbb{R}^n \rtimes \mathbb{Z}_2, \mathbb{Z}_2)$ does not satisfy condition (H2).

Let $h_c \in O(n)$ be the central symmetry. Now we use the first example to reconstruct S^{h_c} for both types of Cartan geometries on M. The pseudogroup generated by symmetries are all (local) translations in this case. The local reflexion space is locally homogeneous, because we can reconstruct the Cartan geometry of type $(\mathbb{R}^n \rtimes \mathbb{Z}_2, \mathbb{Z}_2)$ according to the first example. We can not reconstruct the Cartan geometry of type $(E(n), O(n))$, because it does not satisfy condition (H2).

Finally, let us restrict to $n = 3$ and h_{13} reflexion with respect to the second coordinate. Then we identify M with an open subset M' of $SO(3)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the element of $SO(3)$ with adjoint action h. The identification is using map $M \to \mathfrak{so}(3) = E^3 \wedge E^3 : \ e_1 \mapsto e_1 \wedge e_2, e_2 \mapsto e_1 \wedge e_3, e_3 \mapsto e_2 \wedge e_3$ composed with the exponential map. Since there is element of $SO(3)$ with adjoint action h_{13}, we can use the first example to obtain local reflexion space (M', S). The -1 eigenspaces of h_{13} are $e_1 \wedge e_2$ and $e_2 \wedge e_3$ so (H2) is satisfied. So locally, there is an action of the group $Spin(3) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by h_{13} and its action on $Spin(3)$. Since the symmetries are covered by the action of elements of $Spin(3) \rtimes \mathbb{Z}_2$, we obtain flat Cartan geometry of type $(Spin(3) \rtimes \mathbb{Z}_2, \mathbb{Z}_2)$ satisfying (H1), (H2) and (H3) on M'. We can not reconstruct the original flat Cartan geometry of type $(SO(3), \mathbb{Z}_2)$ on M' just by using the first example, because the Cartan geometry does not satisfy $(H3)$. One can reconstruct the original Cartan geometry after a careful investigation of the homotopy classes of M'.

We are interested, when are the local reflexion spaces locally homogeneous? The answer is the following:

Theorem 5. *Let (M, S) be a local reflexion space and let $\mathfrak{g}_x$ be a Lie subalgebra of Lie algebra of vector fields on some neighborhood of $x \in M$ generated by $\frac{d}{dt}|_{t=0} S_{x(t)} S_x$, where $x(t)$ is a smooth curve such, that $x(0) = x$. If for any $x \in M$ is $\mathfrak{g}_x(x) = T_x M$, then (M, S) is a locally homogeneous local reflexion space.*

Before we start the proof, we fix the following notation:

- choose W as in condition (A3) in definition, and denote W_x a neighborhood of x such that $W_x \times W_x \times W_x \subset W$
- $V_x := \{S_y z : y, z \in W_x\}$
- we denote by X, Y, … vector fields on $U \subset M$ and we assume, that we have chosen for any point x, $y \dots \in U$ a smooth curve $x(t), y(t), \dots$ in U satisfying $x(0) = x$, $y(0) = y, \dots$ and $x'(0) = X(x)$, $y'(0) = Y(y), \dots$

– we shall write $TS_xY := \frac{d}{dt}|_{t=0}S_xy(t)$, $TS^xY := \frac{d}{dt}|_{t=0}S_{y(t)}x$
– we denote XY the differential operator acting on $f\colon U\subset M\to\mathbb{R}$ as $X(Yf)$
– we denote $T^2S(X,Y)$ the differential operator acting on $f\colon U\subset M\to\mathbb{R}$ as

$$\begin{aligned}(T^2S(X,Y))f(S_xy) &= \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}f(S_{x(t)}y(s))\\ &= (TS_xY)(TS^yX)f(y) = (TS^yX)(TS_xY)f(x)\end{aligned}$$

– we denote $R_x(X)$ a vector field extension of $X\in T_xM$ given by

$$R_x(X)(y) := \frac{1}{2}TS^{S_xy}X.$$

We see, that the axioms (A1), (A2) and (A3) are defined for all points of W_x and further we shall restrict ourselves to W_x if not stated otherwise.

We call a map ϕ defined on an interval $(a,b)\subset\mathbb{R}$ containing zero with values in the pseudogroup of locally defined diffeomorphisms of M a local one parameter subgroup of local automorphisms on W_x, if ϕ satisfies:

$$\phi_0 = id_{V_x},\ \phi_{t+s} = \phi_t\circ\phi_s,\ \phi_t(S_pq) = S_{\phi_t(p)}\phi_t(q)$$

for all $p,q\in W_x$. Then we obtain an infinitesimal version of local automorphisms of local reflexion spaces by differentiation of ϕ:

Definition 6. Let (M,S) be a local reflexion space. We say that a vector field X defined on V_x is an infinitesimal automorphism if

$$X(S_pq) = TS_pX(q) + TS^qX(p)$$

for all $p,q\in W_x$.

The following lemma shows equivalence between the local one parameter subgroups of local automorphisms and the infinitesimal automorphisms. Moreover, we obtain condition, when they are generated by reflexions:

Lemma 7. *Let ϕ_t be a local one parameter subgroup of locally defined diffeomorphisms given as a flow of some vector field X on V_x. Then ϕ_t is a local one parameter subgroup of local automorphisms at W_x if and only if X is an infinitesimal automorphism. If X is an infinitesimal automorphism and $(S_x)^*X = -X$, then*

$$S_{\phi_t(x)}S_x = \phi_{2t}\,.$$

Proof. One of the implications is obvious, we prove the other one. Let

$$\gamma(t) := \phi_{-t}\big(S_{\phi_t(p)}\phi_t(q)\big)\,.$$

Then

$$\begin{aligned}\gamma'(t) &= -X(\gamma(t)) + T\phi_{-t}(TS_pX(q) + TS^qX(p))\\ &= -X(\gamma(t)) + T\phi_{-t}X(S_{\phi_t(p)}\phi_t(q))\\ &= -X(\gamma(t)) + T\phi_{-t}\circ X\circ\phi_t(\gamma(t)) = 0\,.\end{aligned}$$

Thus the curve γ is constant and

$$\phi_{-t}(S_{\phi_t(p)}\phi_t(q)) = \gamma(0) = S_pq.$$

Then for the flow Fl^X of X holds

$$S_{Fl^X_t(x)}S_x = Fl^X_t S_x Fl^X_{-t} S_x = Fl^X_t Fl^{(S_x)^*(X)}_{-t}\,.$$

If $(S_x)^*X = -X$, then

$$S_{Fl^X_t(x)}S_x = Fl^X_t Fl^{-X}_{-t} = Fl^X_{2t} = \phi_{2t}\,.$$

□

We see, that $R_x(X)(y)$ is a candidate for an infinitesimal automorphism. We show that this is indeed the case:

Lemma 8.

(1) *The set $\mathcal{D}_x$ of all infinitesimal automorphisms on V_x is a Lie subalgebra of the Lie algebra of vector fields on V_x.*

(2) *$(S_x)^*$ is an involutive automorphism of $\mathcal{D}_x$ and we denote $\mathfrak{g}^-_x$ the -1 eigenspace of $(S_x)^*$.*

(3) *Let $T^-_x M + T^-_x M$ be the decomposition of $T_x M$ with respect to the -1 and 1 eigenspaces of $(S_x)^*$. Then $TM = T^- M + T^+ M$ is a decomposition to subbundles, which is preserved by the local reflexions.*

(4) *R_x is an isomorphism of the vector spaces $T^-_x M$ and $\mathfrak{g}^-_x$ and for $X \in T^+_x M$ is $R_x(X) = 0$.*

(5) *$[[\mathfrak{g}^-_x, \mathfrak{g}^-_x], \mathfrak{g}^-_x] \subset \mathfrak{g}^-_x$ and, moreover, the Lie subalgebra $\mathfrak{g}_x \subset \mathcal{D}_x$ generated by $\mathfrak{g}^-_x$ is finite dimensional. In particular, $\mathfrak{g}_x = \mathfrak{g}^-_x + [\mathfrak{g}^-_x, \mathfrak{g}^-_x]$ and any ideal of $\mathfrak{g}_x$ contained in $[\mathfrak{g}^-_x, \mathfrak{g}^-_x]$ is contained in the center of $\mathfrak{g}_x$.*

(6) *Let ϕ be a local automorphism given by a composition of local reflexions such, that $\phi(x) = z$. Then $T\phi\colon \mathfrak{g}_x \to \mathfrak{g}_z$ is an isomorphism of Lie algebras.*

Proof. (1) For $Y \in \mathcal{D}_x$,

$$\begin{aligned}(TS^q P)(Y)(S_p q) &= (TS^q P)(TS_p Y)(q) + (TS^q P)(TS^q Y)(p)\\ &= T^2 S(P, Y)(S_p q) + TS^q(PY)(p)\end{aligned}$$

and in the same way obtain

$$(TS_p Q)Y(S_p q) = T^2 S(Y, Q)(S_p q) + TS_p(QY)(q)\,.$$

For $X, Y \in \mathcal{D}_x$,

$$\begin{aligned}[X, Y](S_p q) &= XY(S_p q) - YX(S_p q)\\ &= (TS_p X(q) + TS^q X(p))Y(S_p q) - (TS_p Y(q) + TS^q Y(p))X(S_p q)\\ &= TS^q(XY)(p) + T^2 S(X, Y)(S_p q) + TS_p(XY)(q) + T^2 S(Y, X)(S_p q)\\ &\quad - TS^q(YX)(p) - T^2 S(Y, X)(S_p q) - TS_p(YX)(q) - T^2 S(X, Y)(S_p q)\\ &= TS^q[X, Y](p) + TS_p[X, Y](q)\,,\end{aligned}$$

i.e. we have shown that $[X, Y] \in \mathcal{D}_x$.

(2) Differentiating $S_x S_y z(t) = S_{S_x y} S_x z(t)$ we obtain

$$TS_x TS_y Z = TS_{S_x y} TS_x Z$$

and differentiating $S_xS_{y(t)}z = S_{S_xy(t)}S_xz$ we obtain

$$TS_xTS^zY = TS^{S_xz}TS_xY\,.$$

Then for $X \in \mathcal{D}_x$

$$\begin{aligned}(TS_x \circ X(S_pq) \circ S_x) &= TS_xX(S_xS_pq) = TS_xX(S_{S_xp}S_xq)\\ &= TS_xTS^{S_xq}X(S_xp) + TS_xTS_{S_xp}X(S_xq)\\ &= TS^qTS_xX(S_xp) + TS_pTS_xX(S_xq)\\ &= TS_q(TS_x \circ X \circ S_x)(p) + TS_p(TS_x \circ X \circ S_x)(q)\,,\end{aligned}$$

i.e. we have shown that $(S_x)^*$ is an automorphism of $\mathcal{D}_x$. Differentiating $S_xS_xy(t) = y(t)$ we obtain

$$(TS_x)^2Y = Y.$$

Thus $(S_x)^*|_{T_xM} = TS_x$ has only eigenvalues ± 1 and, since $S_x^2 = \mathrm{id}$, $((S_x)^*)^2 = \mathrm{id}$.
(3) Differentiating $S_{x(t)}x(t) = x(t)$ we obtain

$$TS_xX + TS^xX = X(x)\,.$$

Thus TS^x is a projection from $T_xM \to T_x^-M$ with kernel T_x^+M and T^-M+T^+M is a decomposition of TM to subbundles. Further, we have shown that $TS_y(TS_xX) = TS_{S_yx}(TS_yX)$, so we see, that the reflexions preserve the decomposition $TM = T^-M + T^+M$.

(4) Differentiating $S_{S_{x(t)}y}z = S_{x(t)}S_yS_{x(t)}z$ we obtain

$$TS^zTS^yX = TS^{S_yS_xz}X + TS_xTS_yTS^zX$$

and differentiating $S_{x(t)}S_{x(t)}y = y$ we obtain

$$TS^{S_xy}X + TS_xTS^yX = 0\,.$$

So

$$\begin{aligned}R_x(TS^x(X))(y) &= \frac{1}{4}TS^{S_xy}TS^xX\\ &= \frac{1}{4}TS^{S_xS_xS_xy}X + \frac{1}{4}TS_xTS_xTS^yX = R_x(X)(y)\,.\end{aligned}$$

Thus for $X \in T^+M$ we obtain $R_x(X) = 0$.

Next we show $R_x(X) \in \mathfrak{g}_x^-$. Differentiating $S_{x(t)}S_yz = S_{S_{x(t)}y}S_{x(t)}z$ we obtain

$$TS^{S_yz}X = TS^{S_xz}TS^yX + TS_{S_xy}TS^zX\,.$$

Thus

$$\begin{aligned}2R_x(X)(S_yz) &= TS^{S_xS_yz}X = TS^{S_{S_xy}S_xz}X\\ &= TS^{S_xS_xz}TS^{S_xy}X + TS_{S_xS_xy}TS^{S_xz}X\\ &= 2(TS^zR_x(X)(y) + TS_yR_x(X)(z))\,,\end{aligned}$$

and

$$2(TS_x \circ R_x(X) \circ S_x)(y) = TS_xTS^{S_xS_xy}X = -TS^{S_xy}X = -2R_x(X)(y)\,.$$

Since $R_x(X)(x) = TS^x(X)$, the map R_x is injective.

Differentiating $S_xS_{y(t)}z = S_{S_xy(t)}S_xz$ we obtain

$$TS_xTS^zY = TS^{S_xz}TS_xY\,.$$

Then for $X \in \mathfrak{g}_x^-$, we may conclude:

$$\begin{aligned}-X(y) &= TS_x \circ X(y) \circ S_x = TS_xX(S_xy)\\ &= TS_xTS^yX(x) + TS_xTS_xX(y)\\ &= TS^{S_xy}TS_xX(x) + X(y) = -TS^{S_xy}X(x) + X(y).\end{aligned}$$

Thus $X(y) = R_x(X(x))(y)$ and R_x is surjective.

(5) Since

$$\begin{aligned}(S_x)^*([[R_x(X), R_x(Y)], R_x(Z)]) &= [[(S_x)^*(R_x(X)), (S_x)^*(R_x(Y))], (S_x)^*(R_x(Z))]\\ &= -[[R_x(X), R_x(Y)], R_x(Z)],\end{aligned}$$

we get $[[\mathfrak{g}_x^-, \mathfrak{g}_x^-], \mathfrak{g}_x^-] \in \mathfrak{g}_x^-$. Further, $[[R_x(X), R_x(Y)], R_x(Z)]$ is linear in all entries, thus the Lie algebra $\mathfrak{g}_x$ generated by $\mathfrak{g}_x^-$ is finite dimensional and $\mathfrak{g}_x = \mathfrak{g}_x^- + [\mathfrak{g}_x^-, \mathfrak{g}_x^-]$. From the isomorphism $\mathfrak{g}_x^- = T_x^-M$ we get $[\mathfrak{g}_x^-, \mathfrak{g}_x^-] \subset \mathrm{End}(T_x^-M)$ and any ideal of $\mathfrak{g}_x$ contained in $[\mathfrak{g}_x^-, \mathfrak{g}_x^-]$ is contained in center of $\mathfrak{g}_x$.

(6) $T\phi$ induces a vector space isomorphism between T_x^-M and T_z^-M. Since

$$\begin{aligned}T\phi X(S_pq) &= T\phi(TS^qX(p) + TS_pX(q))\\ &= TS^{\phi(q)}T\phi(X)(\phi(p)) + TS_{\phi(p)}T\phi(X)(\phi(q))\,,\end{aligned}$$

it maps $\mathfrak{g}_x^-$ to $\mathfrak{g}_z^-$. Further,

$$\begin{aligned}2(TS_w \circ R_x(X) \circ S_w)(y) &= TS_wTS^{S_xS_wy}X = TS^{S_{S_wx}y}T_xS_w(X)\\ &= 2R_{S_wx}(T_xS_w(X))(y)\,.\end{aligned}$$

Thus, if ϕ is a composition of local reflexions, then it is compatible with the Lie bracket of vector fields and the claim follows. □

Thus if $\mathfrak{g}_x(x) = T_xM$ for any $x \in M$, the previous two lemmas show, that there are infinitesimal automorphisms in all directions. Now we need a version of the Lie second fundamental theorem for this situation:

Lemma 9. *Let $\mathfrak{g}$ be a finite dimensional Lie subalgebra of Lie algebra $\mathcal{D}_x$. Then there is a connected, simply connected Lie group G with the Lie algebra $\mathfrak{g}$, an open subset $U \in G$ and a local left action $l\colon U \times W_x \to M$.*

Proof. The lemma is a local version of [1, Lemma 2.3]. For the convenience of the reader we include the proof below. The details on the parallel transport can be found in [3, Chapter 9].

Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. There is the integrable distribution (L_X, X) on $G \times V_x$, where L_X is a left invariant vector field corresponding to $X \in \mathfrak{g}$. We will denote $L(y)$ the leaf through (e, y). The $pr_1\colon G \times V_x \to G$ is a trivial fibre bundle with a flat connection (for a horizontal distribution given by (L_X, X)). Further, $pr_1|_{L(y)}$ is a local diffeomorphism onto an open neighborhood $Q(y)$ of e in G.

We will use the parallel transport $Pt(c,(g,y),t)$ with respect to the flat connection. For a curve $c\colon (a,b) \to G$, $c(0) = g$, $Pt(c,(g,y),t)$ is defined on some neighborhood V of $g \times V_x \times 0$ in $g \times V_x \times \mathbb{R}$.

Let $c\colon [0,1] \to Q(y)$ be a piecewise smooth curve with $c(0) = e$. Since $e \times y \times [0,1] \subset V$, then $Pt(c,(e,\cdot),0)$ is defined for points in a open subset $U(y)$ containing y and is a diffeomorphism of $c(0) \times U(y)$ onto its image $c(1) \times U'$. We choose $U(y)$ maximal with this property. Since the connection is flat, the parallel transport depends on the homotopy classes of the curve c (with fixed end points). Thus, $Pt(c,(e,\cdot),0)$ defines the map $\gamma_y(c) := pr_2 \circ Pt(c,(e,\cdot),0)\colon U(y) \to U'$.

Now let $\bar{V}$ be a neighborhood of 0 in $\mathfrak{g}$ such, that $Fl^X_1(y)$ is defined for all $y \in W_x$ for $X \in \bar{V}$. Then there is $\bar{U} \subset \bar{V}$ such, that $U = \exp(\bar{U}) \subset Q(y)$ for all $y \in W_x$. Thus $Pt(c,(c(0),\cdot),0)$ is defined for all $y \in W_x$ and for all $c\colon [0,1] \to U$.

We define the local left action $l\colon U \times W_x \to M$ as $l(g,y) = \gamma_x(c)(y)$, where c is a piecewise smooth curve with $c(0) = e$ and $c(1) = g$. Obviously, the definition is correct and it is a left action. Indeed $l(\exp(tX),y) = Fl^X_t(y)$ is the local one parameter group of local automorphisms generated by $X \in \mathfrak{g}$. □

As a corollary of the Lemmas 7, 8 and 9 we get the following:

Corollary 10. *If there is $x \in M$ such that $\mathfrak{g}_x(x) = T_xM$, then the pseudogroup of locally defined diffeomorphisms generated by pairs of local reflexions acts transitively on M and locally is generated by $\mathfrak{g}_x$.*

Now we can prove the main theorem:

Proof. Lemma 7 and Corollary 10 imply, that $\mathfrak{g}_x$ are isomorphic Lie algebras for all $x \in M$, and there are local actions of G from Lemma 9 around all points. We denote K the connected component of identity of stabilizer of some point $x \in M$. We have shown that maximal normal subgroup of G contained in K is contained in center of G and we factor out this part to satisfy condition (H3). The local actions of G provide an atlas of M such that the images of charts are open subsets of G/K and transition functions are elements of G. If we glue the pullbacks of restrictions of the images of the charts in $G \to G/K$ using the same transition functions, we get principal K-bundle over M. The pullbacks of the Maurer Cartan form restricted to those pieces can be glued together to a Cartan connection on this K-bundle. Thus we get a locally flat Cartan geometry of type (G,K).

Now G/K is a connected, simply connected and S_x acts as an automorphism on G. If it is not an inner automorphism, we can extend G and K by $h := S_x$ and (G,K) still satisfies (H3). Clearly h satisfies (H1) and (H2). It is obvious that the local reflexions are equivalent to those defined in the first example. □

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