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CAUCHY PROBLEMS FOR DISCRETE AFFINE MINIMAL SURFACES

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Abstract. In this paper we discuss planar quadrilateral (PQ) nets as discrete models for convex affine surfaces. As a main result, we prove a necessary and sufficient condition for a PQ net to admit a Lelieuvre co-normal vector field. Particular attention is given to the class of surfaces with discrete harmonic co-normals, which we call discrete affine minimal surfaces, and the subclass of surfaces with co-planar discrete harmonic co-normals, which we call discrete improper affine spheres. Within this classes, we show how to solve discrete Cauchy problems analogous to the Cauchy problems for smooth analytic improper affine spheres and smooth analytic affine minimal surfaces.

1. Introduction

Discrete differential geometry has attracted much attention recently, mainly due to the growth of computer graphics. One of the main issues in discrete differential geometry is to define suitable discrete analogous of the concepts of smooth differential geometry ([4]).

Some work have been done in discrete affine differential geometry of surfaces in $\mathbb{R}^3$: In [3] a definition of discrete affine spheres is proposed, and the case of improper affine spheres were considered in [9] and [11]. In [8], we gave a constructive definition of discrete affine minimal surfaces with indefinite Berwald-Blaschke metric. In [3], [9] and [11], surfaces with definite metric were modelled by planar quadrilateral (PQ) nets, which are also called discrete conjugate nets. In [10], discrete affine minimal surfaces are discussed, both in the definite and in the indefinite case.

The co-normal vector field associated with a PQ net is basic in this work. It is defined by the discrete Lelieuvre's equations and, when it exists, it is unique up to black-white re-scaling. As a consequence, each co-normal vector is orthogonal to the corresponding planar face. Also, similarly to the smooth case, the discrete laplacian of the co-normal vector field is parallel to it.

But not every PQ net admit a Lelieuvre's co-normal vector field. We prove here a necessary and sufficient condition for this to occur in terms of certain volumes of tetrahedra associated with the net. So we shall only consider in this paper PQ nets

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satisfying this condition and also an orientation condition insuring that the surface is locally convex at the vertices.

Smooth affine minimal surfaces with definite metric are critical points of the affine area functional. It was shown in [6, 7] that such surfaces in fact maximize the affine area, and because of that they are sometimes called *maximal* surfaces. In this paper we introduce a class of discrete surfaces that corresponds to these surfaces. We shall call *discrete affine minimal surface* any member of this class.

The smooth affine minimal surfaces are also characterized by the fact that the components of the co-normal vector fields are harmonic. We shall use the corresponding discrete property as a definition of discrete minimal surfaces. A nice consequence of this definition is that the discrete affine minimal surfaces admit a discrete Weierstrass representation formula. In order to obtain explicit examples of affine discrete minimal surfaces, it is better to start from the harmonic co-normal vector fields.

We consider two Cauchy problems in this paper: one for improper affine spheres and the other for affine minimal surfaces. The corresponding smooth analytic problems were considered in [1] and [2]. We show here that the discrete problems can be solved in a very simple way. The algorithms for solving these problems are straightforward, although their implementations require some care with numerical instabilities.

The paper is organized as follows: in Section 2 we review the basic equations of smooth surfaces in affine geometry, with special attention to definite affine minimal surfaces with isothermal parameters. In Section 3, we relate Lelieuvre's co-normal vector field with oriented PQ nets. In section 4, we discuss the definition of discrete affine minimal surfaces and its consequences. In section 5, we consider the discrete Cauchy problem for improper affine spheres, while in section 6 we consider the discrete Cauchy problem for minimal surfaces.

Notation. Given two vectors $V_1, V_2 \in \mathbb{R}^3$, we denote by $V_1 \times V_2$ the cross product and by $V_1 \cdot V_2$ the dot product between them. Given three vectors $V_1, V_2, V_3 \in \mathbb{R}^3$, we denote by $[V_1, V_2, V_3] = (V_1 \times V_2) \cdot V_3$ their determinant. For a discrete real or vector function f defined on a domain $D \subset \mathbb{Z}^2$, we denote the discrete partial derivatives with respect to u or v by

$$\begin{aligned} f_1(u+\tfrac{1}{2},v) &= f(u+1,v) - f(u,v) \\ f_2(u,v+\tfrac{1}{2}) &= f(u,v+1) - f(u,v)\,. \end{aligned}$$

The second order partial derivatives are defined by

$$\begin{aligned} f_{11}(u,v) &= f(u+1,v) - 2f(u,v) + f(u-1,v) \\ f_{22}(u,v) &= f(u,v+1) - 2f(u,v) + f(u,v-1) \\ f_{12}\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big) &= f(u+1,v+1) + f(u,v) - f(u+1,v) - f(u,v+1)\,. \end{aligned}$$

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2. Review of affine concepts for smooth surfaces

In this section we review some affine concepts and equations of smooth surfaces. Although we shall not use them explicitly, they are important for comparing with the corresponding concepts and equations for discrete surfaces defined along this paper.

An affine transformation of $\mathbb{R}^3$ is determined by an invertible linear transformation and a translation. An affine transformation is called equi-affine if the determinant of its linear part is one. The affine concepts considered in this section are all invariant under equi-affine transformations.

2.1. **Affine concepts in isothermal coordinates.** Consider a parameterized smooth surface $q\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where U is an open subset of the plane and denote

$$\begin{aligned} L(u,v) &= [q_u, q_v, q_{uu}] \ , \\ M(u,v) &= [q_u, q_v, q_{uv}] \ , \\ N(u,v) &= [q_u, q_v, q_{vv}] \ . \end{aligned}$$

The surface is non-degenerate if $LN - M^2 \neq 0$, and, in this case, the Berwald-Blaschke metric is defined by

$$ds^2 = \frac{1}{|LN - M^2|^{1/4}}(Ldu^2 + 2Mdudv + Ndv^2)\,.$$

If $LN - M^2 > 0$, the metric is *definite* while if $LN - M^2 < 0$, the metric is *indefinite*. The Berwald-Blaschke metric is conformal to the second fundamental form. In the definite case, the surface is locally convex, while in the indefinite case, the surface is locally hyperbolic, i.e., the tangent plane crosses the surface.

Assume that the affine surface has definite metric. We can make a change of coordinates such that $L - N = M = 0$. Such coordinates are called *isothermal*. Moreover, we may assume that $L = N > 0$, and we define Ω by $\Omega^2 = L = N$. In this case, the metric takes the form $ds^2 = \Omega(du^2 + dv^2)$.

The vector field $\nu = \frac{q_u \times q_v}{\Omega}$ is called the co-normal vector field and satisfies Lelieuvre's equations

$$\begin{aligned} q_u &= -\nu \times \nu_v \ , \\ q_v &= \nu \times \nu_u \ . \end{aligned}$$

It also satisfies the equation $\nu_{uu} + \nu_{vv} = H\Omega\nu$, where H is a scalar function called the affine mean curvature. The normal vector field is defined as $\xi = \frac{q_{uu}+q_{vv}}{2\Omega}$.

Affine minimal surfaces. A surface is called affine minimal if $H = 0$, or equivalently, if $\nu_{uu} + \nu_{vv} = 0$. Since ν is harmonic, we can consider an holomorphic function $\Psi(u,v)$ with imaginary part ν. The representation of q in terms of ν is called the affine Weierstrass representation formula (see [12]).

Improper affine spheres. An affine minimal surface is called an improper affine sphere if the normal vector field ξ is parallel to a fixed direction, or equivalently, if all co-normal vectors are co-planar. An improper affine sphere is locally the graph of a function f satisfying the Monge-Ampère equation $det(D^2(f)) = 1$. Considering $q(u, v) = (p(u, v), f(u, v))$, with $p(u, v) \in \mathbb{R}^2$, in isothermal coordinates (u, v), the Monge-Ampère equation can be re-written as

$$f_{uu} + f_{vv} = 2[p_u, p_v]\,. \tag{1}$$

2.2. **The analytic Cauchy Problem for improper affine spheres.** We can pose the Cauchy problem for improper affine spheres as follows: Given an analytic curve $q(s)$ in $\mathbb{R}^3$ and an analytic co-normal vector field $\nu(s) = (\phi(s), 1)$, with $\phi(s) \in \mathbb{R}^2$, satisfying the compatibility condition $\nu \cdot q_s = 0$ and the non-degeneracy condition $\nu \cdot q_{ss} > 0$, find an improper affine sphere that contains $q(s)$ with co-normal $\nu(s)$ along it.

In [1] it is proved that this problem admits a unique solution. Moreover, they describe a Weierstrass representation for the solution: Let $z = s + it$ be a conformal parameter. Writing $q = (p, f)$, we extend $p^1(s) + i\phi^2(s)$ and $p^2(s) - i\phi^1(s)$ to holomorphic functions $p^1 + i\phi^2(z)$ and $p^2 - i\phi^1(z)$. The component f is determined by the condition $\nabla(f) = (-\phi^1, -\phi^2)$.

Example 1. Let $q(s) = (s, s^3 - 3s, 2s)$, $s > 0$, and $\nu(s) = (1 - 3s^2, 1, 1)$. Then $p^1(s) + i\phi^2(s)$ is a restriction of the holomorphic function $z = s + it$ to $t = 1$, and $p^2(s) - i\phi^1(s)$ is a restriction of the holomorphic function $z^3 = (s + it)^3$ to the same line. Thus we have $(p^1, p^2)(s, t) = (s, s^3 - 3st^2)$ and $(\phi^1, \phi^2)(s, t) = (t^3 - 3s^2t, t)$. The third coordinate $f(s, t)$ is obtained from

$$\begin{aligned} f_s &= -\phi^1 p^1_s - \phi^2 p^2_s = 2t^3 \\ f_t &= -\phi^1 p^1_t - \phi^2 p^2_t = 6st^2\,, \end{aligned}$$

and so $f(s, t) = 2st^3$. Note that $\Omega(s, t) = 6st > 0$.

Surfaces with singular sets. Sometimes we can relax the non-degeneracy condition $\nu \cdot q_{ss} > 0$ and even so obtain an improper affine sphere, but in this case, with singularities at the original curve. At the singular curve, the metric degenerates, i.e., $\Omega = 0$ (for details, see [1]).

Example 2. Let $q(s) = (p^1(s), p^2(s), 0)$ be a convex plane curve and $\nu(s) = (0, 0, 1)$. Following [1], we can obtain an improper affine sphere

$$q(s, t) = (p^1(s, t), p^2(s, t), f(s, t) \quad \text{with} \quad t = 0$$

as its singular set. It is also proved in [9] that $f(s, t)$, $t > 0$, is the area of a plane region bounded by some isothermal lines starting at $(p^1(s, t), p^2(s, t))$, tangent to the curve, and an arc of the curve.

As a particular example, consider $p(s) = (\cos(s), \sin(s), 0)$. Since $\cos(t + is) = \cos(s)\cosh(t) + i\sin(s)\sinh(t)$ is a holomorphic function extending $p^1(s, 0)$ and $-i\sin(t + is) = \sin(s)\cosh(t) - i\cos(s)\sinh(t)$ is a holomorphic function extending $p^2(s, 0)$, we conclude that $p^1(s, t) = \cos(s)\cosh(t)$, $p^2(s, t) = \sin(s)\cosh(t)$, $\phi^1(s, t) = \cos(s)\sinh(t)$ and $\phi^2(s, t) = \sin(s)\sinh(t)$. The third component f is obtained from $f_s = 0$ and $f_t = -\sinh(t)^2$. Thus $f(s,t) = \frac{1}{2}(t - \frac{\sinh(2t)}{2})$. Observe that $\Omega(s,t) = \frac{\sinh(t)\cosh(t)}{2}$ vanishes at $t = 0$.

2.3. **The analytic affine Cauchy Problem.** The Cauchy problem concerns finding affine maximal surfaces containing a prescribed strip. It is also called *affine Björling problem.*

We shall call *Problem I* the following analytic affine Cauchy problem for minimal surfaces: Given a curve $q(s)$ together with a co-normal vector field $\nu(s)$ and a normal vector field $\xi(s)$ satisfying the compatibility equations $q_s \cdot \nu = 0$, $\xi \cdot \nu = 1$, $\xi_s \cdot \nu = 0$ and the non-degeneracy condition $q_{ss} \cdot \nu > 0$, find an affine maximal surface containing $q(s)$ with co-normal $\nu(s)$ and normal $\xi(s)$ along the curve. In [2], Problem I is shown to have a unique solution.

We can also consider *Problem II*, which is equivalent to Problem I: Given a curve $q(s)$ together with co-normal vector field $\nu(s)$ and a transversal derivative vector field $w(s) = \nu_t(s)$ satisfying $q_s = w(s) \times \nu$ and $\rho(s) = [\nu, \nu_s, w(s)] > 0$, find an affine maximal surface containing $q(s)$ with co-normal vector field $\nu(s)$ and transversal derivative $w(s)$ along the curve.

To show that Problem II also admits a unique solution, define

$$\xi(s) = \frac{1}{\rho(s)} \nu_s \times w(s)\,.$$

It is easy to see that the triple (q, ν, ξ) satisfies the conditions of Problem I and thus there exists a unique surface $q(s,t)$ with co-normal $\nu(s)$ and normal $\xi(s)$ along the initial curve. And this surface has transversal derivative $w(s)$.

Let us describe the solution of Problem II: Given analytic functions ν and w, one can obtain $\eta(s)$ analytic satisfying $\eta_s = w$ along the curve. Then extend $\eta + i\nu(s)$ to a holomorphic function $\eta + i\nu(s,t)$. We remark that $\nu(s,t)$ is in fact the unique harmonic extension of $\nu(s)$ with transversal derivative $w(s)$. Finally use Lelieuvre's formulas to calculate $q(s,t)$.

Example 3. Let $\nu = (-1, -s, 2s)$, $w(s) = (-1, 0, 2s)$. Then $\eta(s) = (-s, 1, s^2 + 1)$ and so $\Phi(z) = (-z, -iz, z^2)$. Thus $\nu(s,t) = (-t, -s, 2st)$. Thus we can calculate q_s and q_t from Lelieuvre's formulas obtaining

$$\begin{aligned} q_s &= (2s^2, 0, s) \\ q_t &= (0, 2t^2, t)\,. \end{aligned}$$

One concludes that

$$q(s,t) = \Big(\frac{2s^3}{3}, \frac{2t^3}{3}, \frac{s^2 + t^2}{2} \Big)\,.$$

Surfaces with singular sets. As in the case of improper affine spheres, we can relax the non-degeneracy condition and even so obtain a solution to the Cauchy problem. For example, consider a planar curve $q(s) = (q^1(s), q^2(s), 0)$, $\nu(s) = (0, 0, 1)$ and $w(s) = \nu_t(s) = (-q_s^2, q_s^1, h)$. Applying the same algorithm as above, we obtain an affine minimal surface with the original curve as its singular set ([2]).

Example 4. Let $q(s)=(s,\frac{s^2}{2},0)$ and $h(s)=3s^2$. Then $\nu_t(s)=(-s,1,3s^2)$. We can extend ν harmonically to $\nu(s,t)=(-st,t,3s^2t-t^3+1)$, satisfying the initial conditions at $t=0$. Lelieuvre's equations imply that $q_s=(2t^3+1,2st^3+s,0)$ and $q_t=(6st^2,3s^2t^2+t^4-t,t^2)$. Thus $q(s,t)=(2t^3s+s,s^2t^3+s^2+t^5/5-t^2/2,t^3/3)$. Observe that $\Omega(s,t)=t+2t^4$, thus vanishing at $t=0$.

3. Discrete co-normal vector fields and oriented PQ nets

A *PQ net* is defined to be a $\mathbb{R}^3$-valued function defined on a subset D of $\mathbb{Z}^2$, such that faces are planar, i.e., $q(u,v)$, $q(u+1,v)$, $q(u,v+1)$ and $q(u+1,v+1)$ are co-planar (see [5, Definition 2.1]).

We say that the discrete conjugate net is *definite* if the sign of the following four quantities is the same and does not depend on (u,v):

$$\begin{aligned}
\Omega_1(u,v) &:= \left[q_1\left(u+\tfrac{1}{2},v\right),q_1\left(u-\tfrac{1}{2},v\right),q_2\left(u,v+\tfrac{1}{2}\right)\right]\ ,\\
\Omega_2(u,v) &:= \left[q_1\left(u-\tfrac{1}{2},v\right),q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right)\right]\ ,\\
\Omega_3(u,v) &:= \left[q_1\left(u+\tfrac{1}{2},v=\right),q_1\left(u-\tfrac{1}{2},v\right),q_2\left(u,v-\tfrac{1}{2}\right)\right]\ ,\\
\Omega_4(u,v) &:= \left[q_1\left(u+\tfrac{1}{2},v\right),q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right)\right]\ .
\end{aligned}$$

All PQ nets considered in this paper will have $\Omega_i(u,v)>0$, $\forall(u,v)\in D$, $1\leq i\leq 4$.

Lemma 5. *Assume that $\Omega_i(u,v)>0$ for two consecutive values of i and any $(u,v)\in D$. Then the PQ net is definite. Any definite PQ net is convex.*

Proof. Assume, without loss of generality, that $\Omega_1(u,v)>0$ and $\Omega_2(u,v)>0$. Then the points $q(u+1,v)$ and $q(u,v-1)$ must be at same side of the plane passing through $q(u,v)$, $q(u-1,v)$ and $q(u,v+1)$. Thus $\Omega_3(u,v)>0$ and $\Omega_4(u,v)>0$. We also conclude that the definiteness assumption guarantees the *convexity* of the discrete surface. □

We take Lelieuvre's formulas as a definition of the co-normal vector field: A *co-normal vector field* ν with respect to a PQ net q is a vector-valued map defined at any face $(u+\frac{1}{2},v+\frac{1}{2})$ of the net satisfying the discrete Lelieuvre's equations

$$q_1\left(u+\tfrac{1}{2},v\right)=\nu\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)\times\nu\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right) \tag{2}$$

$$q_2\left(u,v+\tfrac{1}{2}\right)=\nu\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\times\nu\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right). \tag{3}$$

It is easy to see that, when it exists, the co-normal vector field ν is unique up to black-white re-scaling, i.e., up to multiplication by a non-zero constant ρ, if $u+v$ is even, and by ρ^{-1}, if $u+v$ is odd.

Given a net $\nu:(\mathbb{Z}^2)^*\to\mathbb{R}^3$, define the discrete laplacian of ν by

$$\Delta\nu\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=\nu_{11}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)+\nu_{22}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right).$$

We shall consider below nets ν whose laplacian is parallel to ν at every point. In [3], such nets are called discrete affine harmonic, but we shall not use this term in order to avoid confusion with the discrete harmonic co-normal field defined in Section 4.

Proposition 6. *A vector field $\nu\colon (\mathbb{Z}^2)^* \to \mathbb{R}^3$ is the co-normal vector field of a PQ net if and only if its discrete laplacian $\Delta\nu$ is parallel to ν.*

Proof. Observe that

$$q_{12}\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big) = \nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)\times\nu\big(u+\tfrac{1}{2},v+\tfrac{3}{2}\big)-\nu\big(u+\tfrac{1}{2},v-\tfrac{1}{2}\big)\times\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)$$

and

$$q_{21}\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big) = -\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)\times\nu\big(u+\tfrac{3}{2},v+\tfrac{1}{2}\big)+\nu\big(u-\tfrac{1}{2},v+\tfrac{1}{2}\big)\times\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big).$$

So

$$q_{12}-q_{21} = \nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)\times\big(\nu_{11}(u+\tfrac{1}{2},v+\tfrac{1}{2})+\nu_{22}\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)\big).$$

Assume that ν is a co-normal vector field of a PQ net. Then the first member of the above equation is 0, and so $\Delta\nu$ is parallel to ν. Reciprocally, given the vector field ν, one define the immersion $q(u,v)$ by Lelieuvre's equations, and the condition of $\Delta\nu$ being parallel to ν guarantees that $q_{12}=q_{21}$. This proves that ν is a co-normal vector field of the PQ net $q(u,v)$. □

Observe that each co-normal vector is orthogonal to the corresponding planar face. So we can write

$$q_1\big(u+\tfrac{1}{2},v\big)\times q_2\big(u,v+\tfrac{1}{2}\big) = \alpha(u,v)\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big) \tag{4}$$

$$q_1\big(u-\tfrac{1}{2},v\big)\times q_2\big(u,v+\tfrac{1}{2}\big) = \beta(u,v)\nu\big(u-\tfrac{1}{2},v+\tfrac{1}{2}\big) \tag{5}$$

$$q_1\big(u-\tfrac{1}{2},v\big)\times q_2\big(u,v-\tfrac{1}{2}\big) = \gamma(u,v)\nu\big(u-\tfrac{1}{2},v-\tfrac{1}{2}\big) \tag{6}$$

$$q_1\big(u+\tfrac{1}{2},v\big)\times q_2\big(u,v-\tfrac{1}{2}\big) = \delta(u,v)\nu\big(u+\tfrac{1}{2},v-\tfrac{1}{2}\big), \tag{7}$$

for some real maps α, β, γ, δ. We say that (q,ν) is *oriented* if the maps α, β, γ, δ are all positive. We shall consider in this paper only oriented pairs (q,ν) (see Figure 1).

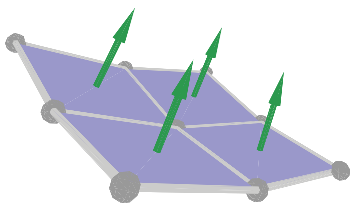

FIG. 1: Four faces of the PQ net with the co-normal vectors.

In terms of the co-normals we have

$$\begin{aligned}
\alpha &= \big[\nu\big(u+\tfrac{1}{2},v-\tfrac{1}{2}\big),\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big),\nu\big(u-\tfrac{1}{2},v+\tfrac{1}{2}\big)\big]\\
\beta &= \big[\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big),\nu\big(u-\tfrac{1}{2},v+\tfrac{1}{2}\big),\nu\big(u-\tfrac{1}{2},v-\tfrac{1}{2}\big)\big]\\
\gamma &= \big[\nu\big(u-\tfrac{1}{2},v+\tfrac{1}{2}\big),\nu\big(u-\tfrac{1}{2},v-\tfrac{1}{2}\big),\nu\big(u+\tfrac{1}{2},v-\tfrac{1}{2}\big)\big]\\
\delta &= \big[\nu\big(u-\tfrac{1}{2},v-\tfrac{1}{2}\big),\nu\big(u+\tfrac{1}{2},v-\tfrac{1}{2}\big),\nu\big(u+\tfrac{1}{2},v+\tfrac{1}{2}\big)\big].
\end{aligned}$$

One can also verify that

$$\Omega_1 = \alpha\beta\,, \quad \Omega_2 = \beta\gamma\,, \quad \Omega_3 = \gamma\delta\,, \quad \Omega_4 = \delta\alpha\,. \tag{8}$$

We conclude also that the underlying PQ net of an oriented discrete surfaces (q,ν) is necessarily definite. Moreover, it must satisfy the condition

$$\Omega_1\Omega_3 = \Omega_2\Omega_4\,. \tag{9}$$

Next proposition shows that the converse is also true:

Proposition 7. *Consider a definite PQ net q satisfying condition* (9)*. Then there exists a co-normal field ν such that (q,ν) is an oriented net.*

Proof. The idea is to define consistently the parameters α, β, γ and δ and define the co-normal vector field by equations (4), (5), (6) and (7). Given one of the parameters α, β, γ or δ at (u,v), one can determine the other three by solving equations (8).

Now fix an initial value $\alpha(u_0,v_0)$ and define ν at the faces $(u_0 \pm \frac{1}{2}, v_0 \pm \frac{1}{2})$ by

$$\begin{aligned}
\nu(u_0+\tfrac{1}{2},v_0+\tfrac{1}{2}) &= \alpha^{-1}(u_0,v_0)q_1(u_0+\tfrac{1}{2},v_0)\times q_2(u_0,v_0+\tfrac{1}{2})\\
\nu(u_0-\tfrac{1}{2},v_0+\tfrac{1}{2}) &= \beta^{-1}(u_0,v_0)q_1(u_0-\tfrac{1}{2},v_0)\times q_2(u_0,v_0+\tfrac{1}{2})\\
\nu(u_0-\tfrac{1}{2},v_0-\tfrac{1}{2}) &= \gamma^{-1}(u_0,v_0)q_1(u_0-\tfrac{1}{2},v_0)\times q_2(u_0,v_0-\tfrac{1}{2})\\
\nu(u_0+\tfrac{1}{2},v_0-\tfrac{1}{2}) &= \delta^{-1}(u_0,v_0)q_1(u_0+\tfrac{1}{2},v_0)\times q_2(u_0,v_0-\tfrac{1}{2})\,.
\end{aligned}$$

Observe that Lelieuvre's equations hold. For example,

$$\nu\big(u_0+\tfrac{1}{2},v_0+\tfrac{1}{2}\big)\times\nu\big(u_0-\tfrac{1}{2},v_0+\tfrac{1}{2}\big) = \alpha^{-1}\beta^{-1}(u_0,v_0)\Omega_1(u_0,v_0)q_2\big(u_0,v_0+\tfrac{1}{2}\big)\,,$$

which, from equations (8), equals $q_2(u_0,v_0+\frac{1}{2})$.

The co-normals at the faces $\big(u_0 \pm \frac{1}{2}, v_0 \pm \frac{1}{2}\big)$ determine the value of α, β, γ and δ at $(u_0+\epsilon_1, v+\epsilon_2)$, where $\epsilon_i = -1,0,1$. With this values we can extend ν to the 12 faces that touches the faces $\big(u_0 \pm \frac{1}{2}, v_0 \pm \frac{1}{2}\big)$, and as above, Lelieuvre's equation still holds at each edge. In this way, we can define the co-normal vector field as far as the orientability condition permits, thus proving the proposition. A more formal proof can be done by induction. □

Example 8. Let $q(u,v) = \big(u, v, \frac{u^2+v^2}{2}\big)$, $(u,v)\in\mathbb{Z}^2$. Then

$$q_1(u+\tfrac{1}{2},v) = (1,0,u+\tfrac{1}{2})$$

$$q_2(u,v+\tfrac{1}{2}) = (0,1,v+\tfrac{1}{2})\,.$$

Taking $\nu\big(u+\frac{1}{2},v+\frac{1}{2}\big) = \big(-\big(u+\frac{1}{2}\big), -\big(v+\frac{1}{2}\big), 1\big)$, one can verify that Lelieuvre's equations (2) and (3) hold. An illustration of this paraboloid is shown in Figure 2. Since

$$\frac{q_{11}+q_{22}}{2} = (0,0,1)\,,$$

this discrete paraboloid is in fact a discrete improper affine sphere, as we shall see in next section.

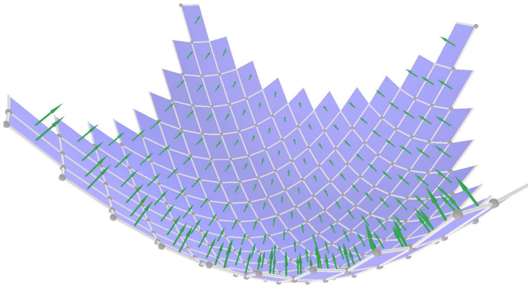

FIG. 2: Discrete paraboloid with co-normal vector field in green.

Discrete proper affine spheres. A definition of discrete proper affine spheres is proposed in [3], and it can be seen as a particular case of the above construction.

A pair (q,ν) is a proper affine sphere if

$$\nu_1(u,v+\tfrac{1}{2}) = q(u,v+1)\times q(u,v)$$

$$\nu_2(u+\tfrac{1}{2},v) = q(u,v)\times q(u+1,v)\,.$$

These equations together with equations (2) and (3) show that, in the case of proper affine spheres, q and ν have symmetric roles. Moreover, one can easily show that $q(u,v)\cdot\nu\big(u\pm\frac{1}{2},v\pm\frac{1}{2}\big)=1$.

We can calculate the above parameters from the ν-net or from the q-net. For example, one can verify that

$$\begin{aligned}
\alpha(u,v) &= [q(u,v),q(u+1,v),q(u,v+1)]\\
\beta(u,v) &= [q(u,v),q(u,v+1),q(u-1,v)]\\
\gamma(u,v) &= [q(u,v),q(u-1,v),q(u,v-1)]\\
\delta(u,v) &= [q(u,v),q(u,v-1),q(u+1,v)]\,.
\end{aligned}$$

We have also that

$$\begin{aligned}
q_{11}+q_{22} &= -H(u,v)q(u,v)\\
\nu_{11}+\nu_{22} &= -H^*(u+\tfrac{1}{2},v+\tfrac{1}{2})\nu(u+\tfrac{1}{2},v+\tfrac{1}{2})\,,
\end{aligned}$$

where

$$\begin{aligned}
H(u,v) &= \alpha(u,v)+\gamma(u,v) = \beta(u,v)+\delta(u,v)\\
H^*(u+\tfrac{1}{2},v+\tfrac{1}{2}) &= \alpha(u,v)+\gamma(u+1,v+1) = \beta(u+1,v)+\delta(u,v+1)\,.
\end{aligned}$$

Bobenko and Schief ([3]) also proposed a method for obtaining discrete affine spheres by solving a discrete Cauchy problem: Begin with one line of points $q(u,0)$ and one line of co-normals $\nu\big(u+\frac{1}{2},\frac{1}{2}\big)$ and then extend then to a domain of $\mathbb{Z}^2$ by using Lelieuvre's and dual Lelieuvre's equations. One continues this extension while α, β, γ and δ remain positive.

4. Discrete minimal surfaces and improper affine spheres

We begin with a usual definition of discrete holomorphic functions. Consider a pair of discrete harmonic functions $A\colon \mathbb{Z}^2 \to \mathbb{R}$ and $B\colon (\mathbb{Z}^2)^* \to \mathbb{R}$. We say that (A, B) is discrete holomorphic if

$$A(u+1,v) - A(u,v) = B\big(u+\tfrac{1}{2}, v+\tfrac{1}{2}\big) - B\big(u+\tfrac{1}{2}, v-\tfrac{1}{2}\big)$$
$$A(u,v+1) - A(u,v) = B\big(u-\tfrac{1}{2}, v+\tfrac{1}{2}\big) - B\big(u+\tfrac{1}{2}, v+\tfrac{1}{2}\big)\,.$$

It is easy to see that any discrete harmonic function $A\colon \mathbb{Z}^2 \to \mathbb{R}$, respectively $B\colon (\mathbb{Z}^2)^* \to \mathbb{R}$, admits a unique, up to a constant, discrete harmonic function $A\colon \mathbb{Z}^2 \to \mathbb{R}$, resp. $B\colon (\mathbb{Z}^2)^* \to \mathbb{R}$, such that the pair (A, B) is holomorphic.

4.1. **Affine minimal surfaces.** We define a pair (q, ν) to be a *discrete affine minimal surface* if the co-normal vector field ν is discrete harmonic, i.e, if it satisfies

$$\nu_{11}\big(u+\tfrac{1}{2}, v+\tfrac{1}{2}\big) + \nu_{22}\big(u+\tfrac{1}{2}, v+\tfrac{1}{2}\big) = 0 \tag{10}$$

(see Figure 3).

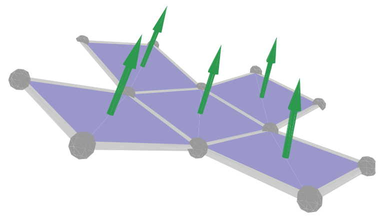

Fig. 3: Five faces of the PQ net with a discrete harmonic co-normal vector field.

It is clear that, starting from the co-normal vector field, we can obtain the PQ net by using Lelieuvre's equations. Also, given $\nu\colon (\mathbb{Z}^2)^* \to \mathbb{R}^3$ harmonic, there exists a unique, up to translations, harmonic function $\eta\colon \mathbb{Z}^2 \to \mathbb{R}^3$ such that each coordinate of $\Psi = (\eta, \nu)$ is holomorphic. So we can also obtain the PQ net from the holomorphic data Ψ. We can think of the formula that represents q in terms of Ψ as a discrete Weierstrass representation formula.

4.2. **Improper affine spheres.** We say that a discrete minimal surface is an improper affine sphere if the vectors $\nu(u, v)$, $(u, v) \in D \subset \mathbb{Z}^2$, are co-planar.

Proposition 9. *Let (q, ν) be a discrete improper affine sphere. Then*

(1) *The vector field $q_{11} + q_{22}\colon \mathbb{Z}^2 \to \mathbb{R}^3$ is parallel to a fixed direction.*

(2) *Let $\eta = (q^2, -q^1, 0)$. Then the pairs (q^2, ν^1) and $(-q^1, \nu^2)$ are holomorphic.*

Proof. We can assume without loss of generality that the third coordinate of ν is 1. We write $\nu = (\phi, 1)$ and $q = (p, f)$, where p a planar vector consisting of the first two components of q. From Lelieuvre's equations we obtain that

$$\begin{aligned} p_1\big(u+\tfrac{1}{2},v\big) &= \phi_2^{\mathrm{ort}}\big(u+\tfrac{1}{2},v\big) \\ p_2\big(u,v+\tfrac{1}{2}\big) &= -\phi_1^{\mathrm{ort}}\big(u,v+\tfrac{1}{2}\big) \\ -p_1\big(u-\tfrac{1}{2},v\big) &= -\phi_2^{\mathrm{ort}}\big(u-\tfrac{1}{2},v\big) \\ -p_2\big(u,v-\tfrac{1}{2}\big) &= \phi_1^{\mathrm{ort}}\big(u,v-\tfrac{1}{2}\big)\,, \end{aligned}$$

where $(A,B)^{\mathrm{ort}}=(-B,A)$. Summing these equations, we conclude that $p_{11}+p_{22}$ is zero, and so $q_{11}+q_{22}$ points in the direction of the z axis. The second assertion also follows from the above equations. □

Remark. In [11], an improper affine sphere is defined as a PQ net with $q_{11}+q_{22}$ parallel. From the above proposition, we conclude that the PQ nets that satisfy our definition must necessarily satisfy the definition of [11], but the reciprocal is not true.

Discrete Monge-Ampère equation. Let $q(u,v)=(p(u,v),f(u,v))$ be an improper affine sphere as above. Then it is proved in [9] that

$$f_{11}+f_{22}=\big[p(u+1,v)-p(u-1,v),p(u,v+1)-p(u,v-1)\big]\,.$$

This discretization may be seen as a discretization of equation (1), which is equivalent to the Monge-Ampère differential equation $\det(D^2(f))=1$.

4.3. **Basic construction of discrete minimal affine surfaces.** We now describe a basic algorithm to generate examples of discrete affine minimal surfaces. This algorithm will be adapted in next sections to solve discrete Cauchy problems for improper affine spheres and affine minimal surfaces.

We start with two lines of co-normal vectors $\nu\big(u+\frac{1}{2},\pm\frac{1}{2}\big)$ satisfying α, β, γ and δ positive. Since ν must be harmonic, we calculate $\nu\big(u+\frac{1}{2},v+\frac{1}{2}\big)$, for any $(u,v)\in\mathbb{Z}^2$, by formula (10) and deduce $q(u,v)$ from Lelieuvre's equations (2) and (3). Then we consider the maximal domain $D\subset(\mathbb{Z}^2)*$ containing $\big(u+\frac{1}{2},\pm\frac{1}{2}\big)$ such that α, β, γ and δ remains positive.

5. The Cauchy Problem for improper affine spheres

We consider the discrete Cauchy problem for improper affine spheres. Given $q(u)=(p(u),f(u))$, $u\in\mathbb{Z}$, and $\nu\big(u+\frac{1}{2}\big)=\big(\phi\big(u+\frac{1}{2}\big),1\big)$ satisfying $q_1\cdot\nu=0$, find an improper affine sphere (q,ν) such that $q(u,0)=q(u)$ and $\nu\big(u+\frac{1}{2},-\frac{1}{2}\big)=\nu\big(u+\frac{1}{2}\big)$.

From the initial data, calculate $\nu\big(u+\frac{1}{2},\frac{1}{2}\big)$ by the condition that the pairs (p^1,ϕ^2) and $(p^2,-\phi^1)$ are holomorphic. We must also assume that

$$\Big[q_1\big(u+\tfrac{1}{2}\big)-q_1\big(u-\tfrac{1}{2}\big)\Big]\cdot\nu\big(u\pm\tfrac{1}{2},\pm\tfrac{1}{2}\big)>0\,, \tag{11}$$

which can be thought as being discrete conditions equivalent to the non degeneracy condition $\nu\cdot q_{ss}>0$. These conditions may also be written as α,β,γ and δ, defined in section 3, being positive. Thus they guarantee the correct orientation of the surface.

The solution to the Cauchy problem is now straightforward: extend ν to a domain $D\subset(\mathbb{Z}^2)^*$ by using the fact that ν is discrete harmonic. This extension is done while α, β, γ and δ remains positive. Then calculate q from Lelieuvre's equations. In Figure 4 we can see an improper affine sphere obtained by this procedure.

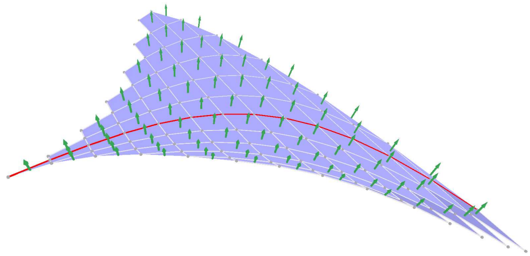

FIG. 4: Solution of the Cauchy problem for improper affine spheres: Original curve in red, co-normal vector field in green.

Improper affine spheres with singular sets. We consider here discrete examples analogous to example 2. Starting from a convex plane polygon, define $\nu\left(u+\frac{1}{2},-\frac{1}{2}\right)=(0,0,1)$ and apply the above algorithm. Although condition (11) is not satisfied, the algorithm works well. At the end we obtain a discrete surface with the original polygon as its singular set. For details of this construction, see [9]. In Figure 5 we can see an improper affine sphere with the original curve as a singular set.

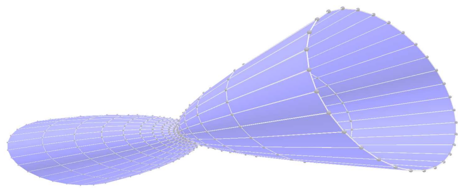

FIG. 5: Solution of the Cauchy problem for improper affine spheres with the original curve (circle) as a singular set.

6. The affine Cauchy problem for minimal surfaces

We consider now the discrete analogous of Problem II of Section 2.3: Assume that we are given a poligonal curve $q(u,0)$ and two lines of co-normals $\nu\left(u+\frac{1}{2},\pm\frac{1}{2}\right)$ satisfying

$$q_1\left(\tfrac{1}{2},0\right)=\nu\left(u+\tfrac{1}{2},\tfrac{1}{2}\right)\times\nu\left(u+\tfrac{1}{2},-\tfrac{1}{2}\right).$$

The non degeneracy condition can be stated as α,β,γ and δ being positive at $(u,0)$. Then there exists a unique discrete minimal surface (q,ν) extending $q(u,0)$ and $\nu(u+\frac{1}{2},\pm\frac{1}{2})$.

The calculation of this minimal surface is straightforward: Extend ν to a domain of $(\mathbb{Z}^2)^*$ by the harmonic condition, while the parameters α,β,γ and δ remain positive. Then calculate q from Lelieuvre's equations. A discrete affine minimal surface obtained by this procedure is shown in Figure 6.

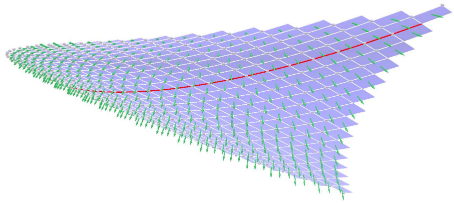

FIG. 6: Solution of the Cauchy problem for affine minimal surfaces: Original curve in red, co-normal vector field in green.

Minimal surfaces with singular sets. We can also obtain discrete minimal surfaces with a given curve as its singular set. Start with a planar polygon and define $\nu\big(u+\frac{1}{2},-\frac{1}{2}\big) = (0,0,1)$. Define also $\nu\big(u+\frac{1}{2},\frac{1}{2}\big) = (-q_1^2, q_1^1, h)$, for an arbitrary map h. If $h \neq 0$, then the minimal discrete surface obtained with the above algorithm has the original polygon as its singular set. An illustration of this procedure is shown in Figure 7.

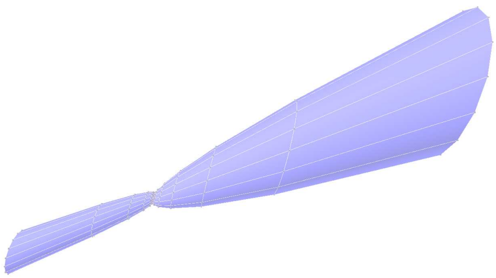

FIG. 7: Solution of the Cauchy problem for affine minimal surfaces with the original curve as a singular set.

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