ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 45–59

AN EXTRAGRADIENT APPROXIMATION METHOD FOR VARIATIONAL INEQUALITY PROBLEM ON FIXED POINT PROBLEM OF NONEXPENSIVE MAPPINGS AND MONOTONE MAPPINGS

Alongkot Suvarnamani and Mongkol Tatong

Abstract. We introduce an iterative sequence for finding the common element of the set of fixed points of a nonexpansive mapping and the solutions of the variational inequality problem for tree inverse-strongly monotone mappings. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. Moreover, using the above theorem, we also apply to finding solutions of a general system of variational inequality and a zero of a maximal monotone operator in a real Hilbert space. As applications, at the end of paper we utilize our results to study the zeros of the maximal monotone and some convergence problem for strictly pseudocontractive mappings. Our results include the previous results as special cases extend and improve the results of Ceng et al., [*Math. Meth. Oper. Res.*, 67:375–390, 2008] and many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$, and respectively and let C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into R, where R is the set of real number. The equilibrium problem for $F \colon C \times C \longrightarrow R$ is to find $x \in C$ such that

$$F(x,y) \geq 0\,, \quad \forall x, y \in C\,. \tag{1.1}$$

The set of solution of (1.1) is denoted by $EP(F)$. Give a mapping $T \colon C \to H$, let $F(x,y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tx, y-x \rangle \geq 0$ for all $y \in C$, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In 1997 Combettes and Hirstoaga introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

2010 *Mathematics Subject Classification*: primary 47J05; secondary 47J25, 47H09, 47H10.

Key words and phrases: nonexpansive mapping, fixed point problems, Variational inequality, relaxed extragradient approximation method, maximal monotone.

Received January 6, 2011, revised May 2011. Editor A. Pultr.

DOI: http://dx.doi.org/10.5817/AM2012-1-45

Let $A\colon C \to H$ be a mapping. The classical variational inequality, denote by $VI(A,C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \parallel Au - Av \parallel^2$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed point of S. For finding an element of $F(S) \cap VI(A,C)$, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP(x_n - \lambda_n A x_n) \tag{1.2}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, α_n is a sequence in $(0,1)$, and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [10] and Zeng and Yao [24] proposed some new iterative schemes for finding element in $F(S) \cap VI(A,C)$. In 2006, Yao and Yao [22] introduced the following iterative scheme.

Let C be a closed convex subset of real Hilbert space H. Let A be an α-inverse--strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are give by

$$\begin{cases} y_n = PC(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n) \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequenced in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ converges strongly to common element of the set of fixed point of a nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone monotone mappings under some parameters controlling condition. Moreover, Takahashi and Takahashi [15] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. They also proved a strong convergence theorem which is connected with Combettes and Hirtoaga's result [4] and Wittmann's result.

In this paper motivated by the iterative schemes, we will introduce a new iterative process below for finding a common element of the set of fixed point of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for an α-inverse-strongly monotone monotone mappings in a real Hilbert space. Then, we prove a strong convergence theorem which is connected with Yao and Yao's result [22] and Takahashi and Takahashi's result [15].

A mapping $A\colon C \to H$ is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \qquad \forall x, y \in C$$

(see Browder and Petryshyn 1967 [2]; Liu and Nashed 1998 [9]). It is obvious that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping $S\colon C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|\,, \qquad \forall x, y \in C\,.$$

We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C. Recall that the classical variational inequality, denoted by $VI(A, C)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0\,, \quad \forall x \in C\,.$$

The set of solutions of $VI(A, C)$ is denoted by Γ. The variational inequality has been widely studied in the literature; see, e.g. [1], [8], [20], [21], [24] and the references therein.

For finding an element of $F(S)\cap\Gamma$, Takahashi and Toyoda (2003) [16] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (-\alpha_n) SP_C(x_n + \lambda_n Ax_n)\,, \tag{1.4}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $0, 2\alpha$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbf{R}^n$, Korpelevich (1976) [7] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C\,, \\ y_n = P_C(x_n - \lambda_n Ax_n)\,, \\ x_{n+1} = P_C(x_n - \lambda_n Ay_n) \end{cases} \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \in (0, \frac{1}{k})$. Recently, Nadezhkina and Takahashi (2006) [10] and Zeng and Yao (2006) [23] proposed some iterative schemes for finding elements in $F(S) \cap \Gamma$ by combining (1.4) and (1.5). Further, these iterative schemes are extended in Yao and Yao (2007) [22] to develop a new iterative scheme for finding elements in $F(S) \cap \Gamma$.

Consider the following problem of finding $(x^*, y^*) \in C \times C$ such that (see cf. Ceng et al. (2008) [3]).

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0\,, & \forall x \in C\,, \end{cases} \tag{1.6}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0\,, & \forall x \in C\,, \end{cases} \tag{1.7}$$

which is defined by Verma (1999) [17] and Verma (2001) [18], and is called the new system of variational inequalities. Further, if $x^* = y^*$, then problem (1.7) reduces to the classical variational inequality $VI(C, A)$.

In 2008 Ceng et al. [3], introduced a relaxed extragradient method for finding solutions of problem (1.6). Let the mappings $A, B\colon C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S\colon C \to C$ be a nonexpansive mapping. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n)\,, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n + \lambda_n A y_n)\,, \end{cases} \tag{1.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequence in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$. First, problem (1.6) is proven to be equivalent to a fixed point problem of nonexpansive mapping.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|\,, \quad \forall y \in C\,.$$

P_C is call the metric projection of H onto C.

Lemma 2.1 (see Zhang, Lee and Chan [25])**.** *The metric projection P_C has the following properties:*

(i) *$P_C\colon H \to C$ is nonexpansive;*

(ii) *$P_C\colon H \to C$ is firmly nonexpansive i.e.,*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle\,, \quad \forall x, y \in H\,;$$

(iii) *for each $x \in H$,*

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0] = \,, \quad \forall y \in C\,.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. Let A, B, $C\colon C \to H$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda A z^* + x^* - z^*, x - x^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \mu B y^* + z^* - y^*, x - z^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \tau C x^* + y^* - x^*, x - y^*\rangle \geq 0\,, & \forall x \in C\,, \end{cases} \tag{2.1}$$

which is called a general system of variational inequalities where $\lambda > 0$, $\mu > 0$ and $\tau > 0$ are three constants.

In particular, if $A = B = C$, then problem (2.1) reduces to finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda A z^* + x^* - z^*, x - x^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \mu A y^* + z^* - y^*, x - z^*\rangle \geq 0\,, & \forall x \in C\,, \\ \langle \tau A x^* + y^* - x^*, x - y^*\rangle \geq 0\,, & \forall x \in C\,. \end{cases} \tag{2.2}$$

Lemma 2.2. *For given x^*, y^*, $z^* \in C \times C \times C$, (x^*, y^*, z^*) is a solution of problem* (2.1) *if and only if x^* is a fixed point of the mapping $G\colon C \to C$ defined by*

$$\begin{aligned} G(x) = P_C\{&P_C[P_C(z - \lambda Az) - \mu BP_C(z - \lambda Az)] \\ &- \tau CP_C[P_C(z - \lambda Az) - \mu BP_C(z - \lambda Az)]\}, \quad \forall x \in C, \end{aligned}$$

where $y^* = P_C(z^* - \lambda Az^*)$.

Lemma 2.3 (see Osilike and Igbokwe [11])**.** *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all x, y, $z \in E$ and α, β, $\gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2 .$$

Lemma 2.4 (see Suzuki [14])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5 (see Xu [19])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n , \quad n \ge 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (Goebel and Kirk [5])**.** *Demi-closedness Principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, then $I - T$ is demi-closed; that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I-T)x_n\}$ converges strongly to some y (for short, $(I-T)x_n \to y$), it follows that $(I-T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.7. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle , \quad \forall x, y \in H .$$

Remark 2.8. We also have that, for all u, $v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)u - (I-\lambda A)v\|^2 &= \|(u-v) - \lambda(Au - Av)\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle + \lambda^2\|Au-Av\|^2 \\ &\le \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|Au-Av\|^2 . \end{aligned} \tag{2.3}$$

So, if $\lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

3. Main results

In this section, we introduce an iterative precess by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for two inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above three sets.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping A, B, $C\colon C \longrightarrow H$ be α-inverse-strongly monotone, β-inverse-strongly monotone and γ-inverse-strongly monotone, respectively. Let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$. Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0 \\ y_n = (1-\gamma_n)u_n + \gamma_n P_c(u_n - \lambda_n A u_n) \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_c(x_n - \lambda_n A y_n) \end{cases} \tag{3.1}$$

where $\lambda_n \in (0, 2\alpha)$, $\mu_n \in (0, 2\beta)$, $\tau_n \in (0, 2\gamma)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap \Omega$, where $\overline{x} = P_{F(S)\cap\Omega} f(x)$.

Proof.

Step 1. x_n is bounded. Indeed, put $t_n = P_C(x_n - \lambda_n A y_n)$. Let $x^* \in F(S) \cap \Omega$. Then $x^* = P_C(x^* - \lambda_n A x^*)$.

$$\begin{aligned} \|t_n - x^*\| &= \|P_C(x_n - \lambda_n A y_n) - P_C(x^* - \lambda_n A x^*)\| \\ &\leq \|(x_n - \lambda_n A y_n) - (x^* - \lambda_n A x^*)\| \\ &\leq \|x_n - x^*\| + \lambda_n \|A y_n - A x^*\| . \end{aligned} \tag{3.2}$$

We observe that

$$\begin{aligned} \|f(y_n) - x^*\| &= \|f(y_n) - f(x^*) + f(x^*) - x^*\| \\ &\leq \|f(y_n) - f(x^*)\| + \|f(x^*) - x^*\| \leq \alpha\|y_n - x^*\| + \|f(x^*) - x^*\| \\ &= \alpha\big[\|(1-\gamma_n)(u_n - x^*) + \gamma_n\big(P_C(u_n - \lambda_n A u_n) \\ &\quad - P_C(x^* - \lambda_n A x^*)\big)\|\big] + \|f(x^*) - x^*\| \\ &\leq \alpha\big[(1-\gamma_n)\|u_n - x^*\| + \gamma_n\|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|\big] \\ &\quad + \|f(x^*) - x^*\| \leq \alpha\big[(1-\gamma_n)\|u_n - x^*\| + \gamma_n\big(\|u_n - x^*\| \\ &\quad + \lambda_n\|A u_n - A x^*\|\big)\big] + \|f(x^*) - x^*\| \\ &= \alpha\|u_n - x^*\| + \alpha\gamma_n\lambda_n\|A u_n - A x^*\| + \|f(x^*) - x^*\| . \end{aligned}$$

By (3.2) and, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|(1-\alpha_n-\beta_n)x_n+\alpha_n f(y_n)\beta_n St_n - x^*\| \\
&\leq (1-\alpha_n-\beta_n)\|x_n-x^*\|+\alpha_n\|f(y_n)-x^*\|+\beta_n\|t_n-x^*\| \\
&= (1-\alpha_n-\beta_n)\|x_n-x^*\|+\alpha_n\big[\alpha\|u_n-x^*\|+\alpha\gamma_n\|(Au_n-Ax^*)\| \\
&\quad +\|f(x^*)-x^*\|\big]+\beta_n\big[\|x_n-x^*\|+\lambda_n\|Ay_n-Ax^*\|\big] \\
&= (1-\alpha_n-\beta_n)\|x_n-x^*\|+\alpha_n\alpha\|u_n-x^*\|+\alpha_n\alpha\gamma_n\|(Au_n-Ax^*)\| \\
&\quad +\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|+\beta_n\lambda_n\|Ay_n-Ax^*\| \\
&= (1-\alpha_n)\|x_n-x^*\|+\alpha_n\alpha\|u_n-x^*\|+\alpha_n\alpha\gamma_n\|(Au_n-Ax^*)\| \\
&\quad +\alpha_n\|f(x^*)-x^*\|+\beta_n\lambda_n\|Ay_n-Ax^*\| \\
&\leq \max\big\{\|x_n-x^*\|,\|f(x^*)-x^*\|\big\}+\alpha_n\alpha\|u_n-x^*\| \\
&\quad +\alpha_n\alpha\|Au_n-Ax^*\|+\beta_n x_n\|Ay_n-Ax^*\|\,.
\end{aligned}$$

Therefore, $\|x_n\|$ is bounded, the set $\{t_n\}$, $\{St_n\}$, $\{Ax_n\}$ and $\{Ay_n\}$ are also bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \|P_C(x_{n+1}-\lambda_{n+1}Ay_{n+1})-P_C(x_n-\lambda_nAy_n)\| \\
&\leq \|(x_{n+1}-\lambda_{n+1}Ay_{n+1})-(x_n-\lambda_nAy_n)\| \\
&= \|(x_{n+1}-\lambda_{n+1}Ay_{n+1})-(x_n-\lambda_nAy_n)+\lambda_{n+1}Ay_n-\lambda_{n+1}Ay_n)\| \\
&= \|(x_{n+1}-\lambda_{n+1}Ay_{n+1})-(x_n-\lambda_{n+1}Ay_n)+(\lambda_nAy_n-\lambda_{n+1}Ay_n)\| \\
&\leq \|(x_{n+1}-\lambda_{n+1}Ay_{n+1})-(x_n-\lambda_{n+1}Ay_n)\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\| \\
&= \|x_{n+1}-\lambda_{n+1}Ay_{n+1}-x_n+\lambda_{n+1}Ay_n\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\| \\
&\leq \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}\|Ay_n\| \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|(1-\gamma_{n+1})u_{n+1}+\gamma_{n+1}P_C(u_{n+1}-\lambda_{n+1}Au_{n+1}) \\
&\qquad -(1-\gamma_{n+1})u_n+\gamma_nP_C(u_n-\lambda_nAu_n)\| \\
&= \|(1-\gamma_{n+1})u_{n+1}-(1-\gamma_n)u_n+\gamma_{n+1}P_C(u_{n+1}-\lambda_{n+1}Au_{n+1}) \\
&\quad +\gamma_nP_C(u_n-\lambda_nAu_n)\| \\
&= \|u_{n+1}-\gamma_{n+1}u_{n+1}-u_n+\gamma_nu_n+\gamma_{n+1}u_n-\gamma_{n+1}u_n \\
&\quad +\gamma_{n+1}P_C(u_{n+1}-\lambda_{n+1}Au_{n+1})+\gamma_{n+1}P_C(u_n-\lambda_nAu_n) \\
&\quad +\gamma_{n+1}P_C(u_n-\lambda_nAu_n)-\gamma_nP_C(u_n-\lambda_nAu_n)\| \\
&= \|(1-\gamma_{n+1})(u_{n+1}-u_n)-(\gamma_{n+1}-\gamma_n)u_n \\
&\quad +\gamma_{n+1}\big(P_C(u_{n+1}-\lambda_{n+1}Au_{n+1})-P_C(u_n-\lambda_nAu_n)\big) \\
&\quad +(\gamma_{n+1}-\gamma_n)P_C(u_n-\lambda_nAu_n)\|
\end{aligned}$$

$$
\begin{aligned}
&= \|(1-\gamma_{n+1})(u_{n+1}-u_n) - (\gamma_{n+1}-\gamma_n)u_n \\
&\quad + \gamma_{n+1}P_C(u_{n+1}-\lambda_{n+1}Au_{n+1}) - \gamma_{n+1}P_C(u_n-\lambda_n Au_n) \\
&\quad + \gamma_{n+1}P_C(u_n-\lambda_n Au_n) - \gamma_n P_C(u_n-\lambda_n Au_n)\| \\
&= \|(1-\gamma_{n+1})(u_{n+1}-u_n) - (\gamma_{n+1}-\gamma_n)\big(P_C(u_n-\lambda_n Au_n)-u_n\big) \\
&\quad + \gamma_{n+1}\big(P_C(u_{n+1}-\lambda_{n+1}Au_{n+1}) - P_C(u_n\lambda_n Au_n)\big)\| \\
&\le (1-\gamma_{n+1})\|(u_{n+1}-u_n)\| + |\gamma_{n+1}-\gamma_n|\lambda_n\|Au_n\| \\
&\quad + \gamma_{n+1}\big(\|u_{n+1}-u_n\| + \lambda_{n+1}\|Au_{n+1}\| + \lambda_n\|Au_n\|\big) \\
&\le \|(u_{n+1}-u_n)\| + \lambda_n\|Au_n\| + \lambda_{n+1}\|Au_{n+1}\| + \lambda_n\|Au_n\| \\
&= \|(u_{n+1}-u_n)\| + 2\lambda_n\|Au_n\| + \lambda_{n+1}\|Au_{n+1}\|\,.
\end{aligned}
$$

Define a sequence z_n by

$$x_{n+1} = \varrho_n x_n + (1-\varrho_n)z_n \qquad n \ge 0$$

where $\varrho_n = 1\alpha_n - \beta_n$, $n \ge 0$. Then we have

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\varrho_{n+1}x_{n+1}}{1-\varrho_{n+1}} - \frac{x_{n+1}-\varrho_n x_n}{1-\varrho_n} \\
&= \frac{\varrho_{n+1}x_{n+1}+\alpha_{n+1}f(y_{n+1})+\beta_{n+1}St_{n+1}-\varrho_{n+1}x_{n+1}}{1-\varrho_{n+1}} \\
&\quad - \frac{\varrho_n x_n + \alpha_n f(y_n) + \beta_n St_n - \varrho_n x_n}{1-\varrho_n} \\
&= \frac{\alpha_{n+1}f(y_{n+1})+\beta_{n+1}St_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n f(y_n)+\beta_n St_n}{1-\varrho_n} \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}}f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}}St_{n+1} - \frac{\alpha_n}{1-\varrho_n}f(y_n) + \frac{\beta_n}{1-\varrho_n}St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}}f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}}St_{n+1} - \frac{\alpha_n}{1-\varrho_n}f(y_n) + \frac{1-\alpha_n-\varrho_n}{1-\varrho_n}St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}}f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}}St_{n+1} \\
&\quad - \frac{\alpha_n}{1-\varrho_n}f(y_n) + \frac{\alpha_n}{1-\varrho_n}St_n - \frac{1-\varrho_n}{1-\varrho_n}St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}}f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}}St_{n+1} - \frac{\alpha_n}{1-\varrho_n}f(y_n) + \frac{\alpha_n}{1-\varrho_n}St_n - St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}}f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}}St_{n+1} - \frac{\alpha_n}{1-\varrho_n}f(y_n) \\
&\quad + \frac{\alpha_n}{1-\varrho_n}St_n - \Big(\frac{1-\varrho_{n+1}}{1-\varrho_{n+1}}\Big)St_n
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}} St_{n+1} - \frac{\alpha_n}{1-\varrho_n} f(y_n) \\
&\quad + \frac{\alpha_n}{1-\varrho_n} St_n - \Big(\frac{\beta_{n+1}+\alpha_{n+1}}{1-\varrho_{n+1}}\Big) St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_{n+1}) + \frac{\beta_{n+1}}{1-\varrho_{n+1}} St_{n+1} - \frac{\alpha_n}{1-\varrho_n} f(y_n) \\
&\quad - \frac{\beta_{n+1}}{1-\varrho_{n+1}} St_n + \frac{\alpha_n}{1-\varrho_n} St_n - \frac{\alpha_{n+1}}{1-\varrho_{n+1}} St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_{n+1}) - \frac{\alpha_n}{1-\varrho_n} f(y_n) + \frac{\beta_{n+1}}{1-\varrho_{n+1}} (St_{n+1} - St_n) \\
&\quad + \Big(\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\Big) St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_{n+1}) - \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_n) + \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_n) \\
&\quad - \frac{\alpha_n}{1-\varrho_n} f(y_n) + \frac{\beta_{n+1}}{1-\varrho_{n+1}} (St_{n+1} - St_n) + \Big(\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\Big) St_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} (f(y_{n+1}) - f(y_n)) - \Big(\frac{\alpha_{n+1}}{1-\varrho_{n+1}} + \frac{\alpha_n}{1-\varrho_n}\Big) f(y_n) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} (St_{n+1} - St_n) + \Big(\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\Big) St_n \,. \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\le \frac{\alpha_{n+1}}{1-\varrho_{n+1}} \|f(y_{n+1}) - f(y_n)\| + \Big|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\Big| \|f(y_n)\| \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} \|St_{n+1} - St_n\| + \Big|\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\Big| \|St_n\| \\
&\le \frac{\alpha\alpha_{n+1}}{1-\varrho_{n+1}} \|y_{n+1} - y_n\| + \Big|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\Big| \big(\|f(y_n)\| - \|St_n\|\big) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} \|t_{n+1} - t_n\| \\
&\le \frac{\alpha\alpha_{n+1}}{1-\varrho_{n+1}} \big[\|u_{n+1} - u_n\| + 2\lambda_n \|Au_n\| + \lambda_{n+1} \|Au_{n+1}\|\big] \\
&\quad + \Big|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\Big| \big(\|f(y_n)\| - \|St_n\|\big) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} \big[\|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}| \|Ay_n\|\big] \\
&\le \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}| \|Ay_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\Big| \big(\|f(y_n)\| - \|St_n\|\big) \\
&\quad + \frac{\alpha\alpha_{n+1}}{1-\varrho_{n+1}} \big[\|u_{n+1} - u_n\| + 2\lambda_n \|Au_n\| + \lambda_{n+1} \|Au_{n+1}\|\big]
\end{aligned}
$$

which implies that

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\leq |\lambda_n-\lambda_{n+1}\|Ay_n\| \\ &\quad+\Big|\frac{\alpha_{n+1}}{1-\varrho_{n+1}}-\frac{\alpha_n}{1-\varrho_n}\Big|\big(\|f(y_n)\|-\|St_n\|\big) \\ &\quad+\frac{\alpha\alpha_{n+1}}{1-\varrho_{n+1}}\big[\|u_{n+1}-u_n\|+2\lambda_n\|Au_n\|+\lambda_{n+1}\|Au_{n+1}\|\big]\end{aligned}$$

This together with (ii) and (v) imply that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0$$

by lemma, we obtain $\|z_n-x_n\|\to 0$ as $n\to\infty$. Consequently

$$\lim_{n\to\infty}(\|x_{n+1}-x_n\|)=\lim_{n\to\infty}(1-\varrho_n)\|z_n-x_n\|=0\,. \tag{3.5}$$

Step 3. $\lim_{n\to\infty}\|Sx_n-x_n\|=\lim_{n\to\infty}\|St_n-t_n\|$

$$\begin{aligned}\|y_n-t_n\| &= \big\|(1-\gamma_n)\big(P_Cu_n-P_C(x_n-\lambda_nAy_n)\big) \\ &\qquad+\gamma_n\big(P_C(u_n-\lambda_nAu_n)-P_C(x_n-\lambda_nAy_n)\big)\big\| \\ &\leq (1-\gamma_n)\big\|P_Cu_n-P_C(x_n-\lambda_nAy_n)\big\| \\ &\qquad+\gamma_n\big\|P_C(u_n-\lambda_nAu_n)-P_C(x_n-\lambda_nAy_n)\big\| \\ &\leq \big\|u_n-(x_n-\lambda_nAy_n)\big\|+\big\|(u_n-\lambda_nAu_n)-(x_n-\lambda_nAy_n)\big\| \\ &= \big\|(u_n-x_n)+\lambda_nAy_n\big\|+\big\|(u_n-x_n)+(-\lambda_n)(Au_n-Ay_n)\big\| \\ &\leq \|u_n-x_n\|+\lambda_n\|Ay_n\|+\|u_n-x_n\|+\lambda_n\|Au_n-Ay_n\|\to 0\end{aligned}$$

$$\begin{aligned}\|t_n-x_n\| &= \|t_n-y_n+y_n-x_n\| \\ &\leq \|t_n-y_n\|+\|y_n-x_n\|\to 0\end{aligned}$$

and hence

$$\begin{aligned}\|Sy_n-x_{n+1}\| &= \|Sy_n-St_n+St_n-x_{n+1}\| \\ &= \|(Sy_n-St_n)+(St_n-x_{n+1})\| \\ &\leq \|Sy_n-St_n\|+\|St_n-x_{n+1}\| \\ &\leq \|y_n-t_n\|+\|St_n-\big[(1-\alpha_n-\beta_n)x_n+\alpha_nf(y_n)+\beta_nSt_n\big]\| \\ &\leq \|y_n-t_n\|+(1-\alpha_n-\beta_n)\|St_n-x_n\|+\alpha_n\|St_n-f(y_n)\| \\ &= \|y_n-t_n\|+\alpha_n\|St_n-f(y_n)\|+(1-\alpha_n-\beta_n)\|St_n-Sx_n+Sx_n-x_n\| \\ &\leq \|y_n-t_n\|+\alpha_n\|St_n-f(y_n)\|+(1-\alpha_n-\beta_n)\big[\|St_n-Sx_n\|+\|Sx_n-x_n\|\big] \\ &\leq \|y_n-t_n\|+\alpha_n\|St_n-f(y_n)\|+\|t_n-x_n\|+(1-\beta_n)\|Sx_n-x_n\|\end{aligned}$$

thus from the last three inequalities we conclude that

$$\begin{aligned}
\|Sx_n - x_n\| &= \|Sx_n - Sy_n + Sy_n - x_{n+1} + x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|Sy_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|Sy_n - \big[(1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n St_n\big]\| \\
&\quad + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + (1-\alpha_n-\beta_n)\|Sy_n - x_n\| + \alpha_n\|Sy_n - f(y_n)\| \\
&\quad + \beta_n\|Sy_n - St_n\| + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n\|Sy_n - f(y_n)\| \\
&\quad + (1-\alpha_n-\beta_n)\|Sy_n - Sx_n + Sx_n - x_n\| + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n\|Sy_n - f(y_n)\| \\
&\quad + (1-\alpha_n-\beta_n)\big[\|Sy_n - Sx_n\| + \|Sx_n - x_n\|\big] + \|x_{n+1} - x_n\| \\
&= \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n\|Sy_n - f(y_n)\| \\
&\quad + (1-\alpha_n-\beta_n)\|Sy_n - Sx_n\| \\
&\quad + (1-\alpha_n-\beta_n)\|Sx_n - x_n\| + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n\|Sy_n - f(y_n)\| + \|y_n - x_n\| \\
&\quad + (1-\beta_n)\|Sx_n - x_n\| + \|x_{n+1} - x_n\| \\
&\le 2\|x_n - y_n\| + \|y_n - t_n\| + \alpha_n\|Sy_n - f(y_n)\| \\
&\quad + (1-\beta_n)\|Sx_n - x_n\| + \|x_{n+1} - x_n\|
\end{aligned}$$

since

$$0 \le \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1\,, \qquad \|Sx_n - x_n\| \to 0\,.$$

Consequently

$$\begin{aligned}
\|St_n - t_n\| &= \|St_n - Sx_n + Sx_n - x_n + x_n - t_n\| \\
&\le \|St_n - Sx_n\| + \|Sx_n - x_n\| + \|x_n - t_n\| \\
&\le \|t_n - x_n\| + \|Sx_n - x_n\| + \|x_n - t_n\| \\
&= 2\|t_n - x_n\| + \|Sx_n - x_n\| \to \infty
\end{aligned}$$

Step 4. $\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \le 0$. Pick a subsequence x_{n_i} of x_n so that

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \limsup_{n\to\infty}\langle f(q) - q, x_{n_i} - q\rangle\,.$$

Let $x_{n_i} \rightharpoonup \hat{x} \in C$. Then Let $x_{n_i} \rightharpoonup \hat{x} \in C$. Then

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \limsup_{n\to\infty}\langle f(q) - q, \hat{x} - q\rangle$$

to show $\langle f(q) - q, \hat{x} - q\rangle \le 0$, and to show $\hat{x} \in F(S) \cap \Omega$. By Lemma 2.2 and Step 3, we have $\hat{x} \cap \Omega$. Let

$$Tv = \begin{cases} Av + N_C v\,, & \text{if} \quad v \in C\,, \\ \emptyset\,, & \text{if} \quad v \notin C\,. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$. Let $(v, w) \in G(T)$. Then $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. Therefore $\langle v - u, w - Av\rangle \geq 0$ for all $u \in C$. Taking $u = x_{n_i}$, we have

$$\begin{aligned}
\langle v - \hat{x}, w\rangle &= \liminf_{i\to\infty} \langle v - x_{n_i}, w\rangle \geq \liminf_{i\to\infty} \langle v - x_{n_i}, Av\rangle \\
&= \liminf_{i\to\infty} [\langle v - x_{n_i}, Av - Ax_{n_i} + \langle v - x_{n_i}, Ax_{n_i}\rangle] \\
&\geq \liminf_{i\to\infty} \langle v - x_{n_i}, Ax_{n_i}\rangle \geq \liminf_{i\to\infty} \langle v - x_{n_i}, Ax_n\rangle \geq 0
\end{aligned}$$

and so $\langle v - \hat{x}, w\rangle \geq 0$. Since T is maximal monotone, $\hat{x} \in \Omega$. Thus $\hat{x} \in F(S) \cap \Omega$. Therefore by property of the metric projection, $\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle \leq 0$.

Step 5. $\limsup_{n\to\infty} \|x_n - q\| = 0$ where $q = P_{F(s)\cap\Omega} f(q)$ we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1 - \alpha_n - \beta_n)(x_n - q) + \alpha_n(f(y_n) - q) + \beta_n(St_n - q)\|^2 \\
&\leq \|(1 - \alpha_n - \beta_n)(x_n - q) + \beta_n(St_n - q)\|^2 \\
&\quad + 2\alpha_n\langle f(y_n) - q, 1 - \alpha_n - \beta_n(x_n - q) + \beta(st_n - q) + \alpha_n f(y_n) - q\rangle \\
&\leq \|(1 - \alpha_n - \beta_n)(x_n - q) + \beta_n(t_n - q)\|^2 + 2\alpha_n\langle f(y_n) - q, x_{n+1} - q\rangle \\
&\leq \big[(1 - \alpha_n - \beta_n)\|x_n - q\| + \beta_n\|t_n - q\|\big]^2 + 2\alpha_n\langle f(y_n) - q, x_{n+1} - q\rangle \\
&= \big[(1 - \alpha_n - \beta_n)\|x_n - q\| + \beta_n\|P_C(x_n - \lambda_n Ay_n) - q\|\big]^2 \\
&\quad + 2\alpha_n\langle f(y_n) - q, x_{n+1} - q\rangle \\
&\leq \big[(1 - \alpha_n - \beta_n)\|x_n - q\| + \beta_n(\|x_n - q\| + \lambda_n Ay_n)\big]^2 \\
&\quad + 2\alpha_n\langle f(y_n) - q, x_{n+1} - q\rangle \\
&= \big[(1 - \alpha_n)\|x_n - q\| + \lambda_n Ay_n\big]^2 + 2\alpha_n\big[\langle f(y_n) - f(x_n), x_{n+1} - q\rangle \\
&\quad + \langle f(x_n) - f(q), x_{n+1} - q\rangle + \langle f(q) - q, x_{n+1} - q\rangle\big] \\
&\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2(1 - \alpha_n)\|x_n - q\|\lambda_n Ay_n + (\lambda_n Ay_n)^2 \\
&\quad + 2\alpha_n\big[\alpha\|y_n - x_n\|\|x_{n+1} - q\| \\
&\quad + \alpha\|x_n - q\|\|x_{n+1} - q\| + \langle f(q) - q, x_{n+1} - q\rangle\big] \\
&= (1 - \alpha_n)^2\|x_n - q\|^2 + \lambda_n Ay_n\big(2(1 - \alpha_n)\|x_n - q\| + \lambda_n Ay_n\big) \\
&\quad + 2\alpha_n\big[\alpha\|y_n - x_n\|\|x_{n+1} - q\| \\
&\quad + \alpha\|x_n - q\|\|x_{n+1} - q\| + \langle f(q) - q, x_{n+1} - q\rangle\big] \\
&= (1 - \alpha_n)^2\|x_n - q\|^2 + \alpha\alpha_n\|x_n - q\|^2 + \alpha\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\big[\alpha\|y_n - x_n\|\|x_{n+1} - q\| + \langle f(q) - q, x_{n+1} - q\rangle\big] \\
&\quad + \lambda_n Ay_n(2\|x_n - q\| + \lambda_n Ay_n)
\end{aligned}$$

which implies that

$$\begin{aligned}
(1 - \alpha\alpha_n)\|x_{n+1} - q\|^2 \leq{} & \big((1 - \alpha_n)^2 + \alpha\alpha_n\big)\|x_n - q\|^2 + 2\alpha_n\big[\alpha\|y_n - x_n\|\|x_{n+1} - q\| \\
& + \langle f(q) - q, x_{n+1} - q\rangle\big] + \lambda_n Ay_n(2\|x_n - q\| + \lambda_n Ay_n)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \frac{(1-\alpha_n)^2+\alpha\alpha_n}{1-\alpha\alpha_n}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\big[\alpha\|y_n-x_n\|\,\|x_{n+1}-q\| \\
&\quad +\langle f(q)-q,x_{n+1}-q\rangle\big]+\frac{\lambda_n Ay_n}{1-\alpha\alpha_n}(2\|x_n-q\|+\lambda_n Ay_n) \\
&= \Big(1-2\alpha_n+2\alpha\alpha_n+\frac{\alpha_n^2+2\alpha^2\alpha_n^2-2\alpha\alpha_n^2}{1-\alpha\alpha_n}\Big)\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\big[\alpha\|y_n-x_n\|\,\|x_{n+1}-q\|+\langle f(q)-q,x_{n+1}-q\rangle\big] \\
&\quad +\frac{\lambda_n Ay_n}{1-\alpha\alpha_n}(2\|x_n-q\|+\lambda_n Ay_n) \\
&= \Big(1-2(1-\alpha)\alpha_n\frac{\alpha_n^2+2\alpha^2\alpha_n^2-2\alpha\alpha_n^2}{1-\alpha\alpha_n}\Big)\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\big[\alpha\|y_n-x_n\|\,\|x_{n+1}-q\|+\langle f(q)-q,x_{n+1}-q\rangle\big] \\
&\quad +\frac{\lambda_n Ay_n}{1-\alpha\alpha_n}(2\|x_n-q\|+\lambda_n Ay_n) \\
&\leq \Big(1-2(1-\alpha)\alpha_n\frac{\alpha_n^2}{1-\alpha\alpha_n}\Big)\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\big[\alpha\|y_n-x_n\|\|x_{n+1}-q\|+\langle f(q)-q,x_{n+1}-q\rangle\big] \\
&\quad +\frac{\lambda_n Ay_n}{1-\alpha\alpha_n}(2\|x_n-q\|+\lambda_n Ay_n) \\
&= \big(1-2(1-\alpha)\alpha_n\big)\|x_n-q\|^2+2(1-\alpha)\alpha_n\frac{1}{(1-\alpha)(1-\alpha\alpha_n)} \\
&\quad \times\Big[\frac{\alpha_n}{2}\|x_n-q\|^2+\alpha\|y_n-x_n\|\,\|x_{n+1}-q\|+\langle f(q)-q,x_{n+1}-q\rangle\Big] \\
&\quad +\frac{\lambda_n Ay_n}{1-\alpha\alpha_n}(2\|x_n-q\|+\lambda_n Ay_n)
\end{aligned}$$

but $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}2(1-\alpha)\alpha_n=\infty$. Since $\limsup_{n\to\infty}\langle f(q)-q,x_{n+1}-q\rangle\leq 0$, $\lim_{n\to\infty}\|y_n-x_n\|=0$ and $\|x_n-q\|$ is bounded. We imply that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{(1-\alpha)(1-\alpha\alpha_n)} \\
&\qquad\times\Big[\frac{\alpha_n}{2}\|x_n-q\|^2+\alpha\|y_n-x_n\|\|x_{n+1}-q\|+\langle f(q)-q,x_{n+1}-q\rangle\Big]\leq 0
\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{1-\alpha\alpha_n}\lambda Ay_n(2\|x_n-q\|+\lambda_n Ay_n)<\infty\,,$$

by Lemma 2.1 we conclude $\|x_n-q\|\to 0$. □

Acknowledgement. I would like to thank Dr. Poom Kumam for drawing my attention to the subject and for many useful discussions. This work was supported by the Faculty of Sciences and Technology, RMUTT, Thailand.

References

[1] Blum, E., Oettli, W., *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[2] Browder, F. E., Petryshyn, W. V., *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[3] Ceng, L.-C., Wang, C.-Y., Yao, J.-C., *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res. **67** (2008), 375–390.

[4] Combettes, P. L., Hirstoaga, S. A., *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (1) (2005), 117–136.

[5] Goebel, K., Kirk, W. A., *Topics on Metric Fixed–Point Theory*, Cambridge University Press, 1990.

[6] Haugazeau, Y., *Surles in équations variationnelles et la minimisation de fonctionnelles convexes, Thèse*, Master's thesis, Université de Paris, 1968.

[7] Korpelevich, G. M., *An extragradient method for finding saddle points and for other problems*, Ékonom. i Mat. Metody **12** (940) (1976), 747–756, Russian.

[8] Kumam, P., *Strong convergence theorems by an extragradient method for solving variational inequalities and equilibrium problems in a Hilbert space*, Turkish J. Math. **33** (1) (2009), 85–98.

[9] Liu, F., Nashed, M. Z., Takahashi, W., *Regularization of nonlinear ill–posed variational inequalities and convergence rates*, Set–Valued Anal. **6** (1998), 313–344.

[10] Nadezhkina, N., Takahashi, W., *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[11] Osilike, M. O., Igbokwe, D. I., *Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations*, Comput. Math. Appl. **40** (2000), 559–567.

[12] Plubtieng, S., Punpaeng, R., *A new iterative method for equilibrium problems 3 and fixed point problems of nonexpansive mappings and 4 monotone mappings*, Appl. Math. Comput. (2007). DOI: `http://dx.doi.org/10.1016/j.amc.2007.07.075`

[13] Su, Y. et al., *An iterative method of solution for equilibrium and optimization problems*, Nonlinear Anal. (2007). DOI: `http://dx.doi.org/10.1016/j.na.2007.08.045`

[14] Suzuki, T., *Strong convergence of Krasnoselskii and Mann's type sequences for one–parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305** (2005), 227–239.

[15] Takahashi, S., Takahashi, W., *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (1) (2007), 506–515.

[16] Takahashi, W., Toyoda, M., *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[17] Verma, R. U., *On a new system of nonlinear variational inequalities and associated iterative algorithms*, Math. Sci. Res. Hot–Line **3** (8) (1999), 65–68.

[18] Verma, R. U., *Iterative algorithms and a new system of nonlinear quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **4** (1) (2001), 117–127.

[19] Xu, H. K., *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[20] Yao, J-C., Chadli, O., *Handbook of Generalized Convexity and Monotonicity*, ch. Pseudomonotone complementarity problems and variational inequalities, pp. 501–558, Springer, Netherlands, 2005.

[21] Yao, Y., C., Liou Y., Yao, J.-C., *An extragradient method for fixed point problems and variational inequality problems*, Journal of Inequalities and Applications **2007** (2007), 12, article ID 38752. DOI: http://dx.doi.org/10.1155/2007/38752

[22] Yao, Y., Yao, J.-C., *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput. **186** (2007), 1551–1558.

[23] Zeng, L. C., Wong, N. C., Yao, J.-C., *Strong convergence theorems for strictly pseudocontractive mapping of Browder–Petryshyn type*, Taiwanese J. Math. **10** (4) (2006), 837–849.

[24] Zeng, L. C., Yao, J.-C., *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwanese J. Math. **10** (2006), 1293–1303.

[25] Zhang, S., Lee, J., Chan, C., *Algorithms of common solutions to quasi variational inclusion and fixed point problems*, Appl. Math. Mech. (English Ed.) **29** (5) (2008), 571–581.

Department of Mathematics, Faculty of Science and Technology,
Rajamangala University of Technology Thanyaburi, (RMUTT)
Thanyaburi, PathumThani, 12110, Thailand
E-mail: kotmaster2@rmutt.ac.th kotmaster2@hotmail.com tatong_max1222@hotmail.com