ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 389–394

CONFORMALLY GEODESIC MAPPINGS SATISFYING A CERTAIN INITIAL CONDITION

Hana Chudá and Josef Mikeš

Abstract. In this paper we study conformally geodesic mappings between pseudo-Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$, i.e. mappings $f\colon M \to \bar{M}$ satisfying $f = f_1 \circ f_2 \circ f_3$, where f_1, f_3 are conformal mappings and f_2 is a geodesic mapping. Suppose that the initial condition $f^*\bar{g} = kg$ is satisfied at a point $x_0 \in M$ and that at this point the conformal Weyl tensor does not vanish. We prove that then f is necessarily conformal.

1. Introduction

One may say that the pioneering work in conformal and projective geometry was done by H. Weyl [13] and T. Thomas [12]. Corresponding Weyl tensors for these structures are known for many decades. In that period many monographs and research papers were devoted to these topics. Let us mention, e.g. [1] and [3] which are closely connected to this paper.

Composing the conformal mapping first with a geodesic and then with a conformal mapping give rise to the so called *conformally geodesic mapping* to which we focus our attention. These mappings were studied e.g., in the papers of Hinterleitner [4], Hinterleitner, Mikeš [5] and of Mikeš, Vanžurová and Hinterleitner [9]. In this paper we prove that under a certain condition if two manifolds are related by a conformally geodesic mapping the mapping is already conformal. This is a kind of rigidity result.

Let us mention that geodesic mappings were studied under a certain additional condition based on the proportionality of the metrics which was suppose to be valid in a certain subset of the underlying manifold. It turns out that even under this condition, the mapping is a homothety. See e.g. [2, 7]. We shall suppose the condition is satisfied at a single point only. We prove the homothety result under this milder condition in this text.

Although one can be quite precise about the degrees of differentiability of the mappings, manifolds, tensor fields etc., we will suppose all the objects we work with are smooth, i.e. of the class $\mathcal{C}^\infty$, for simplicity. Because our statements are local, the computations e.g., in the proofs, are supposed to be valid locally as well. We will not always write that explicitly. Although in the statements we stress

2010 *Mathematics Subject Classification*: primary 53B20; secondary 53B30, 53C21.
Key words and phrases: conformal mappings, geodesic mappings, conformally geodesic mappings.

it. Due the signature is not for our reasonings, we decided to formulate them for pseudo-Riemannian manifolds.

2. Main properties of geodesic and conformal mappings

2.1. **Geodesic mappings.** Let us recall that a diffeomorphism f between pseudo-Riemannian manifolds V_n and $\bar{V}_n$ is called a *geodesic mapping*, if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$. Let us stress that we consider geodesics as unparameterized curves. Thus actually when talking about geodesic mappings, we work in the realm of projective differential geometry. Because f is a diffeomorphism, we can actually suppose that $V_n = (M, g)$ and $\bar{V}_n = (M, \bar{g})$, where g and $\bar{g}$ denote the metrics defined on the manifold M. See e.g. [1, 3, 6, 7, 9, 10, 11, 12, 13] where these objects are investigated.

It is known that a diffeomorphism of M is geodesic if and only if so called the *Levi-Civita equation* ([6]), i.e.

$$(\bar{\nabla} - \nabla)_X X = 2\psi(X) \cdot X \tag{1}$$

or equivalently

$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) \tag{2}$$

holds. Here, ∇ and $\bar{\nabla}$ are the Levi-Civita connections on V_n and $\bar{V}_n$, ψ is a differential 1-form and X, Y, Z are vector fields tangent to M. If $\psi = 0$, then the geodesic mapping is called *affine* or sometimes, *trivial*. The latter name is used because the diffeomorphism preserves not only the geodesics but also the geodesics considered with their “preferred” parameterizations.

It is also known that if the equations above are satisfied, there exists a function Ψ on M such that $\psi_i = \dfrac{\partial \Psi}{\partial x^i}$. To prove this, take e.g. $\Psi = \dfrac{1}{2(n+1)} \ln \left| \dfrac{\det \bar{g}}{\det g} \right|$.

There is the so called projective Weyl tensor W, which measures the projective features of (M, g). Let us recall the definition and describe its meaning at once. If there is a geodesic mapping from V_n onto $\bar{V}_n$, the *projective Weyl tensor* defined by

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right), \tag{3}$$

remains invariant (i.e. $\bar{W} = W$). Here, R^h_{ijk} are the components of the Riemannian tensor of (M, g) and $R_{ij} = R^\alpha_{ij\alpha}$ are the components of the corresponding Ricci tensor with respect to orthogonal basis.

It is known, that a pseudo-Riemannian manifold V_n $(n > 2)$ is a space of constant curvature if and only if the projective Weyl tensor vanishes $(W = 0)$. For $n = 2$, the projective Weyl tensor W vanishes identically.

An analysis of the Levi-Civita equations (2) gives to the following theorem.

Theorem 1 (Chudá, Mikeš [2]). *Let f be a geodesic mapping between pseudo-Riemannian manifolds (M, g) and $(M, \bar{g})$, $x_0 \in M$ and $\bar{x}_0 = f(x_0)$. Suppose that the initial condition $\bar{g}(\bar{x}_0) = k \cdot g(x_0)$ is satisfied for a $k \in \mathbb{R}$. If the projective Weyl tensor does not vanish at x_0, then the mapping f provides a homothety between (M, g) and $(M, \bar{g})$, i.e. $\bar{g} = k \cdot g$.*

2.2. **Conformal mappings.** Now, let us turn our attention to conformal structures. A diffeomorphism f between pseudo-Riemannian manifolds V_n and $\bar{V}_n$ is called a *conformal mapping*, if f preserves angles between all (smooth) curves on V_n. We will again suppose, $M = \bar{M}$. Equivalently, a mapping f of $V_n = (M, g)$ onto $\bar{V}_n = (M, \bar{g})$ is conformal if and only if

$$\bar{g} = \rho \cdot g \,, \tag{4}$$

where ρ is a nowhere zero function on M.

From the equation (4) it follows that

$$(\bar{\nabla} - \nabla)_X X = 2\,\sigma(X) \cdot X - g(X, X) \cdot \Sigma \,, \tag{5}$$

where $\sigma(X) = \frac{1}{2}\,\nabla_X \ln|\rho|$, $\sigma(X) = g(X, \Sigma)$ and X is an arbitrary tangent vector.

As in the projective situation, we have the so called *conformal Weyl tensor* at our disposal. Let us recall its definition

$$C^h_{ijk} = R^h_{ijk} + \delta^h_j L_{ik} - \delta^h_k L_{ij} + L^h_j g_{ik} + L^h_k g_{ij} \,, \tag{6}$$

where $L_{ij} = \dfrac{1}{n-2}\,(R_{ij} - \dfrac{R}{2\,(n-1)}\,g_{ij})$, $L^h_i = g^{h\alpha} L_{\alpha i}$, $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature and g^{ij} are components of the inverse matrix of g_{ij}. As in the case of the geodesic mappings, a parallel theorem is known. If there is a conformal mapping $V_n \to \bar{V}_n$ $(n > 2)$, the conformal Weyl tensor remains invariant (i.e. $\bar{C} = C$). The converse is not true. The round spheres as well as its suitable quotients satisfy $C = 0$ but they are even not diffeomorphic, thus certainly not globally conformally equivalent.

For $n > 3$, a pseudo-Riemannian space is locally *conformally flat* if and only if, the conformal Weyl tensor vanishes $(C = 0)$. For $n = 3$ the conformal Weyl tensor C always vanishes identically. (But ofcourse this does not mean that all three dimensional manifolds are conformally flat, but rather that the Weyl tensor defined above is not a suitable tool for recognizing conformal flatness in this dimension. Let us remark that the flatness in this case is measured by the so called Cotton-York tensor.)

Spaces of constant curvature are characterized by the vanishing of the projective Weyl tensor $(W = 0)$. These spaces form a special subclass of conformally flat spaces.

Using the Weyl symmetry analysis, we obtain the following claim.

Lemma 1. *Let x_0 be a fixed point on M and $n > 2$. If $W(x_0) = 0$, then $C(x_0) = 0$.*

Proof. Suppose $W(x_0) = 0$. We suppose the tensors are evaluated at this point not writing it explicitly. It follows from (3) that

$$R^h_{ijk} = \frac{1}{n-1}\,(\delta^h_k R_{ij} - \delta^h_j R_{ik}) \,.$$

After a contraction of the previous equation by g^{ij}, we obtain that the Ricci tensor has the following form

$$R_{ij} = \frac{R}{n}\,g_{ij} \,.$$

Substituting the Riemann and Ricci tensors into (6), we find that $C^h_{ijk} = 0$. □

Remark 1. *One can easily see that the converse is not true.*

3. Conformally geodesic mappings

After we have sketched some basic properties of geodesic and conformal mappings, let us focus our attention to the already mentioned conformally geodesic ones. In papers [4, 5] of Hinterleitner the so called conformally projective mappings were studied. These mappings are closely related to our subject. Inspired by her observations, we will derive some further results on them. We say that $f\colon V_n \to \bar{V}_n$ is *conformally geodesic* if $f = f_1 \circ f_2 \circ f_3$, where

$$f_1\colon V_n = (M, g) \to \overset{1}{V}_n = (M, \overset{1}{g}) \qquad \text{is a conformal mapping,}$$

$$f_2\colon \overset{1}{V}_n = (M, \overset{1}{g}) \to \overset{2}{V}_n = (M, \overset{2}{g}) \qquad \text{is a geodesic mapping and}$$

$$f_3\colon \overset{2}{V}_n = (M, \overset{2}{g}) \to \bar{V}_n = (M, \bar{g}) \qquad \text{is a conformal mapping.}$$

We may again suppose that all of the three pseudo-Riemannian manifolds coincide (as smooth manifolds) and differ by the metrics only.

First, let us derive the following consequence of Theorem 1 (conformally geodesic Levi-Civita relations).

Theorem 2. *A diffeomorphism $f\colon V_n = (M, g) \to \bar{V}_n = (M, \bar{g})$ is a conformally geodesic mapping if and only if for each vector field X the following condition hold*

$$(\bar{\nabla} - \nabla)_X X = 2\psi(X) \cdot X + g(X, X) \cdot \Sigma + \bar{g}(X, X) \cdot \Omega\,, \tag{7}$$

where ψ is a differential 1-form, Σ and Ω are vector fields and there exist functions ϱ_1^*, ϱ_2^* and ϱ_3^* on M such that for each field X,

$$\nabla_X \varrho_1^* = g(X, \Sigma)\,, \quad \nabla_X \varrho_2^* = \bar{g}(X, \Omega)\,, \quad \nabla_X \varrho_3^* = \psi(X)\,.$$

Proof. The necessity of (7) and the existence of the functions ϱ_1^*, ϱ_2^* and ϱ_3^* follows from the relations (1) and (5). The conditions are sufficient due to the following observation. Suppose the conditions are satisfied. Then one may construct metrics

$$\overset{1}{g} = \exp(-2\,\varrho_1^*) \cdot g\,, \quad \text{and} \quad \overset{2}{g} = \exp(2\,\varrho_2^*) \cdot \bar{g}\,.$$

Computing the difference between the Levi-Civita connections associated to $\overset{1}{g}$ and $\overset{2}{g}$, we get

$$(\overset{2}{\nabla} - \overset{1}{\nabla})_X X = (2\psi(X) - \nabla_X \varrho_1^* - \nabla_X \varrho_2^*) \cdot X\,.$$

Thus according to (2), the spaces $\overset{1}{V}_n$ and $\overset{2}{V}_n$ are in geodesic correspondence. □

It is evident that the relation of 'being conformally geodesically equivalent' is symmetric and reflexive. Unfortunately, the conformal geodesic mappings do not form a group because of lack of transitivity - the relation is not an equivalence relation. Nevertheless, we found the following solution to the appropriate 'equivalence problem' though, only partial.

Theorem 3. *Let f be a conformally geodesic mapping between two pseudo-Riemannian manifolds (M, g) and $(M, \bar{g})$ Suppose the metrics are homothetic at x_0, i.e. $\bar{g}(f(x_0)) = k \cdot g(x_0)$, $k \in \mathbb{R}$ and $C(x_0) \neq 0$. Then f is a conformal mapping.*

Proof. Let $V_n = (M, g)$ admit a conformally geodesic mapping f onto $\bar{V}_n = (M, \bar{g})$ and at the point $x_0 \in M$ the following equation holds

$$\bar{g}_{ij}(f(x_0)) = k \cdot g_{ij}(x_0)\,.$$

Because of the existence of conformal equivalences between, we have $\overset{1}{g} = \overset{1}{\sigma} \cdot g$ and $\bar{g} = \overset{2}{\sigma} \cdot \overset{2}{g}$ and in particular,

$$\overset{1}{g}(x_0) = \overset{1}{\sigma}(x_0) \cdot g(x_0) \qquad \text{and} \qquad \bar{g}(x_0) = \overset{2}{\sigma}(x_0) \cdot \overset{2}{g}(x_0)\,.$$

Combining the three last written equation, we get

$$\overset{2}{\sigma}(x_0) \cdot \overset{2}{g}(x_0) = k \cdot (\overset{1}{\sigma}(x_0))^{-1} \cdot \overset{1}{g}(x_0)\,,$$

i.e.

$$\overset{2}{g}(x_0) = \overset{\star}{C} \cdot \overset{1}{g}(x_0)\,, \tag{8}$$

where $\overset{\star}{C} = k \cdot (\overset{1}{\sigma}(x_0) \cdot \overset{2}{\sigma}(x_0))^{-1}$.

We know that $C(x_0) \neq 0$. Because the tensor C of the conformal Weyl tensor is conformally invariant, it does not vanish after translating it to the space $\overset{1}{V}(x_0)$ via a diffeomorphis, i.e. $\overset{1}{C}(x_0) \neq 0$. Using Lemma 1 we know that the projective Weyl tensor of $\overset{1}{V_n}$ does not vanish as well, i.e.

$$\overset{1}{W}(x_0) \neq 0\,.$$

Using Theorem 1 we obtain that the geodesic mappings $f_2\colon \overset{1}{V_n} \to \overset{2}{V_n}$ is a homothety. Therefore

$$\overset{2}{g} = \text{const} \cdot \overset{1}{g} \quad \text{globally.}$$

Using relation (6) we find

$$\bar{g} = \overset{1}{\sigma} \cdot \text{const} \cdot \overset{2}{\sigma} \cdot g\,.$$

This means that the mapping $f = f_1 \circ f_2 \circ f_3$ is conformal. □

We hope that this is a beginning of the study of equivalence problems for structures not forming a group in general.

Acknowledgement. The paper was supported by grant P201/11/0356 of The Czech Science Foundation and by the Council of the Czech Government MSM 6198959214.

References

[1] Aminova, A. V., *Projective transformations of pseudo-Riemannian manifolds*, J. Math. Sci. (New York) **113** (3) (2003), 367–470.

[2] Chudá, H., Mikeš, J., *On geodesic mappings with certain initial conditions*, Acta Math. Acad. Paedagog. Nyházi. **26** (2) (2010), 337–341.

[3] Eisenhart, L. P., *Non-Riemannian Geometry*, Amer. Math. Soc. Colloq. Publ. 8, 1990, reprint of the 1927 original.

[4] Hinterleitner, I., *Special mappings of equidistant spaces*, J. Appl. Math. **2** (2008), 31–36.

[5] Hinterleitner, I., *Selected Special Vector Fields and Mappings in Riemannian Geometry*, Ph.D. thesis, VUT Brno, 2009.

[6] Levi-Civita, T., *Sulle transformationi delle equazioni dinamiche*, Ann. Mat. Milano **24** Ser. 2 (1886), 255–300.

[7] Mikeš, J., Kiosak, V., Vanžurová, A., *Geodesic mappings of manifolds with affine connection*, Palacky University Press, Olomouc, 2008.

[8] Mikeš, J., Strambach, K., *Differentiable structures on elementary geometries*, Results Math. **53** (1–2) (2009), 153–172.

[9] Mikeš, J., Vanžurová, A., Hinterleitner, I., *Geodesic mappings and some generalizations*, Palacky University Press, Olomouc, 2009.

[10] Petrov, A. Z., *New methods in the general theory of relativity*, Nauka, Moscow, 1966.

[11] Sinyukov, N. S., *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, 1979.

[12] Thomas, T. Y., *The differential invariants of generalized spaces*, III, Cambridge Univ. Press, 1934.

[13] Weyl, H., *Zur Infinitesimalgeometrie. Einordnung der projektiven und der konformen Auffassung*, Göttinger Nachrichten (1921), 99–112.

Dept. of Mathematics, Faculty of Applied Informatics,
Tomas Bata University
T.G. Masaryka 5555, 760 01 Zlín, Czech Republic
E-mail: chuda@fai.utb.cz

Department of Algebra and Geometry,
Faculty of Science, Palacky University,
17. listopadu 12, 779 00 Olomouc, Czech Republic
E-mail: josef.mikes@upol.cz