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FINITE GROUPS WITH A UNIQUE NONLINEAR NONFAITHFUL IRREDUCIBLE CHARACTER

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Abstract. In this paper, we consider finite groups with precisely one nonlinear nonfaithful irreducible character. We show that the groups of order 16 with nilpotency class 3 are the only p-groups with this property. Moreover we completely characterize the nilpotent groups with this property. Also we show that if G is a group with a nontrivial center which possesses precisely one nonlinear nonfaithful irreducible character then G is solvable.

Introduction

Di Martino and Tamburini in [2] proved that the nonlinear irreducible characters of a p-group G are all faithful and of degree $|G\colon Z(G)|^{1/2}$ if and only if $Z(G)$ is cyclic and $|G'| = p$. It is easy to see that this characterization remains valid for finite groups with nontrivial center (see 1.6 below). On the other hand if G is a group with a trivial center, then Isaacs in [6, Lemma 12.3] showed that it is a Frobenius group with an elementary abelian kernel and a cyclic complement. So finite solvable groups all of whose nonlinear irreducible characters are faithful are characterized. The aim of this paper is to characterize finite groups with a unique nonlinear nonfaithful irreducible character. One may observe that groups of order 16 with nilpotency class 3 satisfy this property. It is clear that if G is a group with a unique nonlinear nonfaithful irreducible character χ, then $G/\ker\chi$ is a group with a unique nonlinear irreducible character. So it is natural to consider groups with exactly one nonlinear irreducible character. These groups have already been studied by Seitz [8]. It is proved that these groups are extra-special 2-groups or Frobenius groups of order $m(m-1)$ with an abelian kernel of order m where m is a power of a prime. It is interesting that if we add a single group to the collection of these groups then we have reached the entire groups with distinct nonlinear irreducible character degrees. This fact is proved in [1]. Indeed the authors of [1] have proved that if G is a group with distinct nonlinear irreducible character degrees, then it has one or two nonlinear irreducible characters. Also they proved that in the latter case, G is a Frobenius group of order 72. A generalization of this result has

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recently been expressed by Loukaki [7] where she has studied groups G with a normal subgroup N such that the degrees of the elements of $\mathrm{Irr}(G|N)$ consisting of all irreducible characters of G not containing N in their kernel are distinct.

In this paper all groups are finite. If G is nilpotent, then $c(G)$ denotes its nilpotency class.

1. Preliminary results

In this section, we state some preliminary results. We start with the following definition which is the central notion of this paper.

Definition 1.1. Let G be a finite group. We say that the pair (G, K) satisfies the condition $(*)$ if and only if G contains a unique nonlinear nonfaithful irreducible character and K is the kernel of its nonlinear character. If K is not involved in the context we simply write G satisfies the condition $(*)$.

Theorem 1.2 ([1]). *Let G be a finite group. Then G has precisely one nonlinear irreducible character if and only if one of the following holds:*

(i) *G is an extra-special 2-group.*

(ii) *G is a Frobenius group of order $m(m-1)$ for some prime power m with an abelian Frobenius kernel of order m and a cyclic Frobenius complement.*

Remark 1. If G is a Frobenius group with an abelian complement H, then it is well-known that H is indeed cyclic.

Lemma 1.3. *Let G satisfies the condition $(*)$ and $N \lhd G$ be nontrivial. If $G' \nleq N$, then G/N contains exactly one nonlinear irreducible character.*

Proof. Let $\hat{\psi}$ be an arbitrary nonlinear irreducible character of G/N and ψ be its corresponding character in G. Then N is contained in $\ker\psi$ and consequently ψ is not faithful. That is $\psi = \chi$, where χ is the unique nonlinear non faithful irreducible character of G. Hence $\hat{\chi}$ is the only nonlinear irreducible character of G/N and the proof is complete. □

Lemma 1.4. *Let G be a group with a unique nonlinear irreducible character χ. Then χ is faithful.*

Proof. Let $K = \ker\chi$. If $K \neq 1$, then the pair (G, K) satisfies the condition $(*)$. So by previous lemma G/K has precisely one nonlinear irreducible character. Let $\hat{\chi}$ be the corresponding character of χ in G/K. Then we can write:

$$|G| - |G : G'| = \chi(1)^2 = \hat{\chi}(1)^2 = |G/K| - |G/K : G/G'K| \ .$$

Now an easy computation shows that:

$$(K/G' \cap K)\,(|G'| - 1) = (G'/G' \cap K) - 1\,.$$

Hence $(|G'| - 1) \mid (G'/G' \cap K) - 1$. That is $G' \cap K = 1$. But it forces K to be identity which is clearly a contradiction. □

Corollary 1.5. *Assume that the group G has a unique nonlinear irreducible character. Then G is an M-group.*

Proof. We may assume that G is not nilpotent. By Theorem 1.2 G is a Frobenius group with an abelian kernel N. Let χ be the nonlinear irreducible character of G. Since χ is faithful, then [6, Theorem 6.34] implies that $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(N)$. But N is abelian; hence χ is monomial and the proof is complete. □

Lemma 1.6. *Let G be a nonabelian solvable group with a nontrivial center. Then the following are equivalent:*

(i) *All of the nonlinear irreducible characters of G are faithful.*

(ii) *G' is the unique minimal normal subgroup of G.*

(iii) *G is a p-group, $Z(G)$ is cyclic and $|G'| = p$.*

Proof. (ii) → (iii) and (iii) → (i) hold by [6, Lemma 12.3] and [2, Lemma 1], respectively. Now suppose that (i) holds and let N be a normal subgroup of G not containing G'. Then N is contained in the kernel of some nonlinear irreducible character of G. That is, $N = 1$ and the proof is completed. □

2. Main results for p-groups

In this section we study p-groups satisfying the condition $(*)$. Throughout the section G is a finite p-group. Also set $\Re(G)$ to be the collection of nontrivial normal subgroups of G not containing G'.

Definition 2.1 ([3])**.** G is said to satisfy the strong condition on normal subgroups if every normal subgroup of G not containing G' is contained in $Z(G)$.

Lemma 2.2. *Let G satisfies the condition $(*)$. Then for an arbitrary element $N \in \Re(G)$ we have $|G : N| = 2\chi(1)^2$. In particular any two elements of $\Re(G)$ have equal order.*

Proof. Since $N \in \Re(G)$, then by Lemma 1.3, G/N has precisely one nonlinear irreducible character. Hence by Theorem 1.2, G/N is an extra-special 2-group. Now we have:

$$|G : N| = \left|(G/N) : (G/N)'\right| + \chi(1)^2 = (1/2)\,|G : N| + \chi(1)^2\,.$$

Hence $|G : N| = 2\chi(1)^2$. □

Proposition 2.3. *If G satisfies the condition $(*)$, then $c(G) \geq 3$.*

Proof. Since G is not an abelian group, then it suffices to prove that the nilpotency class of G is not 2. By contradiction, assume that the nilpotency class of G is 2. We claim that G satisfies the strong condition on normal subgroups. Let $N \in \Re(G)$. We have to show that N is a central subgroup of G. Note that by Theorem 1.2, $|G' : G' \cap N| = \left|(G/N)'\right| = 2$ because G/N has only one nonlinear irreducible character. Now if $|G'| > 2$, then it has a subgroup of order 2 which is certainly in $\Re(G)$. Hence $|N| = 2$ and consequently it is a central subgroup of G. So G satisfies the strong condition on normal subgroups. Also if $|G'| = 2$, then $G' \cap N$ must be trivial. That is, $N \leq Z(G)$. Hence the claim is proved in each case. By [3, Theorem 4.3] G' is an elementary abelian group. If $Z(G)$ is cyclic, then $|G'| = 2$ which is impossible by Lemma 1.6. Also if $Z(G)$ is not cyclic, then G has no faithful irreducible characters. Then it has precisely one nonlinear irreducible character. So by Theorem 1.2, G is an extra-special group which is a contradiction. □

Corollary 2.4. *The p-group G satisfies the condition $(*)$ if and only if $|G| = 16$ and $\mathrm{c}(G) = 3$.*

Proof. Suppose that G satisfies the condition $(*)$. Since the nilpotency class of G is greater than 2, then by Lemma 2.2 $Z(G)$ and its subgroup(s) of order 2 lie in $\Re(G)$. Hence $|Z(G)| = 2$. Particularly G satisfies the strong conditions on normal subgroups. So by [3, Theorem 6.1], $|G : Z(G)| = 8$. Conversely, assume that $|G| = 16$ and $\mathrm{c}(G) = 3$. We use GAP [5] to verify that G satisfies the condition $(*)$. It is well-known that a 2-group of maximal class is dihedral, semidihedral or generalized quaternion. In particular $G \in \{\mathrm{D}_{16}, \mathrm{SD}_{16}, \mathrm{Q}_{16}\}$. The character table of these groups is the same:

χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	-1
χ_3	1	1	-1	1	1	-1	-1
χ_4	1	1	1	1	1	1	1
χ_5	2	0	0	-2	2	0	0
χ_6	2	0	0	0	-2	A	$-A$
χ_7	2	0	0	0	-2	$-A$	A

Now it is clear that χ_5 is the unique nonlinear nonfaithful irreducible character of G. □

3. Generalizing the results

In this section, we study groups satisfying $(*)$ in general. First we characterize nilpotent groups with this property. Finally we get some information about these groups in general. Lets start with the following lemma:

Lemma 3.1 ([6, Problem 4.3])**.** *Let $G = H \times K$, $\varphi \in \mathrm{Irr}(H)$ and $\vartheta \in \mathrm{Irr}(K)$ be faithful. Then $\varphi \times \vartheta$ is faithful if and only if* $\gcd(|Z(\varphi)|, |Z(\vartheta)|) = 1$

Theorem 3.2. *Let G be a decomposable group. Then G satisfies the condition $(*)$ if and only if $G \cong C \times K$ where K is a group with precisely one nonlinear irreducible character and C is a group of prime order.*

Proof. Let $G = H \times K$ where H and K are nontrivial groups and assume that G satisfies the condition (*). Without loss of generality we may assume that K is not abelian. Note that K has exactly one nonlinear irreducible character; since otherwise using the principal character of H, we can find two distinct nonfaithful nonlinear irreducible characters for G. A similar argument shows that all of the nonprincipal irreducible characters of H are faithful. That is, H is a simple group. But if H is not abelian then it contains at least two nonlinear irreducible characters. Using the principal character of K we can find two distinct nonfaithful nonlinear irreducible characters for G which is impossible. Hence, $|H| = p$ for a prime p and the "only if" part is proved. Conversely assume that $G \cong C \times K$ where C is the cyclic group of order a prime p and K is a group with precisely one nonlinear irreducible character (say χ). We show that G has a unique nonlinear nonfaithful irreducible character. Obviously $\lambda_C \times \chi$ is a nonlinear nonfaithful irreducible character of G where λ_C is the principal character of C. So we must prove that the other nonlinear irreducible characters of G are faithful. Let λ be a nonprincipal character of C. We know that K is either a 2-group or a Frobenius group. In the first case p must be an odd prime and in the latter case $Z(\chi)$ is trivial. Hence in both cases $\gcd(|Z(\lambda)|, |Z(\chi)|) = 1$. We deduce by Lemma 3.1 that $\lambda \times \chi$ is faithful and the proof completes. □

Corollary 3.3. *Let G be a nilpotent group. Then G satisfies the condition $(*)$ if and only if either of the following holds:*

(i) $|G| = 16$ *and* $\mathrm{c}(G) = 3$

(ii) *$G \cong C \times E$ where C is a cyclic group of order an odd prime and E is an extra-special 2-group.*

Proof. If (i) or (ii) hold, then the result follows by Corollary 2.4 and Theorem 3.2 respectively. Conversely assume that G satisfies the condition $(*)$. If G is a p-group, then Corollary 2.4 implies that $|G| = 16$ and $\mathrm{c}(G) = 3$. So assume that G is not a p-group. In particular G is decomposable and by Theorem 3.2, $G \cong C \times E$ where C is the cyclic group of order p and E is a group with precisely one nonlinear irreducible character. Now by Theorem 1.2 and the fact that G is a nilpotent group we obtain E is an extra-special 2-group. Certainly $p \neq 2$ since G is not a p-group and the proof is completed. □

Proposition 3.4. *Let G be a nonnilpotent group and assume that the pair (G, K) satisfies the condition $(*)$. Then K is a minimal normal subgroup of G. Moreover G has no other minimal normal subgroups except possibly G'.*

Proof. We prove the proposition in four steps:

Step 1. Normal subgroups of G which are not containing G' are contained in K.

Let L be a normal subgroup of G which is not containing G'. Then it is contained in the kernel of some irreducible character of G. Since the pair (G, K) satisfies the condition $(*)$ we have $L \leq K$.

Step 2. K either is contained in G' or $G' \cap K = 1$.

Let $L = G' \cap K$ and $L \neq 1$. We must show that $L = K$. Since neither L nor K contain G' we have:

$$|G : K| - |G : G'K| = |G : L| - |G : G'| .$$

In particular $L = K$.

Step 3. If $K \cap G' = 1$, then $K = Z(G)$ and $|K|$ is a prime.

It is obvious that $K \leq Z(G)$. On the other hand since G is not nilpotent, $G' \nleq Z(G)$. So by step 1, $Z(G) \leq K$. Now let N be a subgroup of K of order a prime p. Then N is a normal subgroup of G and we have:

$$|G:K| - |G:G'K| = |G:N| - |G:G'N|$$
$$\frac{|G'|-1}{|G'|\,|N|} = \frac{|G'|-1}{|G'|\,|K|}$$
$$|K| = p$$

Step 4. Proof of the proposition.

By Step 2, $K \cap G' = 1$ or $K < G'$. In the former case K is a minimal normal subgroup of G (Step 3). So let $K < G'$. If $N \leq K$ is a nontrivial normal subgroup of G. We have:

$$|G:K| - |G:G'| = |G:N| - |G:G'|$$
$$N = K$$

So, we proved that K is a minimal normal subgroup of G. Now an easy observation shows that if $K \cap G' = 1$, then K and G' are the only minimal normal subgroups of G and if $K < G'$, then K is the unique minimal normal subgroup of G. This completes the proof. □

Lemma 3.5. *Let G be a nonnilpotent group and assume that the pair (G, K) satisfies the condition $(*)$. Then G/K is a Frobenius group with precisely one nonlinear irreducible character.*

Proof. It is clear that G/K is either an extra-special 2-group or a Frobenius group with precisely one nonlinear irreducible character. So our task is now to prove that the former case fails. By contradiction let G/K be an extra-special 2-group. Note that $K < G'$ because if $G' \cap K = 1$, then $K = Z(G)$ that contradicts the hypothesis of the theorem which implies that G is not nilpotent. Also if χ is the nonlinear irreducible character of G, then $G' \leq Z(\chi)$. Indeed if $G' \nleq Z(\chi)$, then $1 = Z(\chi)/\ker\chi = Z(G/K)$ which is impossible. So by [6, Lemma 2.31], $\chi(1)^2 = |G : Z(\chi)|$. On the other hand G/K is a group with a unique nonlinear irreducible character and we can write:

$$|G:K| = |G:G'| + \chi(1)^2 = |G:G'| + |G:Z(\chi)| \ .$$

Now let $|Z(\chi)| = m|G'|$ for an integer m. Then $\frac{1}{|L|} = \frac{1}{|G'|} + \frac{1}{m|G'|}$ and we conclude that $(m+1) = m\,|G' : L|$. That is, $G' = Z(\chi)$ and we can write:

$$|G:K| = |G:G'|\,(1 + |G:G'|)$$

which is impossible and therefore G/K is not a 2-group. □

Lemma 3.6. *Let G be a nonnilpotent group and K be a normal subgroup of G. Then the pair (G, K) satisfies the condition $(*)$ if and only if all of the following conditions hold.*

(i) *K is maximal with the property that G/K is not abelian.*

(ii) *K is a minimal normal subgroup of G and G has no other minimal normal subgroups except possibly G'.*

(iii) *G is solvable of order $m(m-1)$, for a prime power m.*

Proof. We have already proved that if the pair (G, K) satisfies the condition $(*)$, then the above conditions hold. Conversely, let (i), (ii) and (iii) hold. Condition (i) implies that $(G/K)'$ is the unique minimal normal subgroup of G/K. Hence by condition (iii) and [6, Lemma 12.3], G/K is a Frobenius group with an abelian Frobenius kernel and a cyclic complement. So by Theorem 1.2 G/K has precisely one nonlinear irreducible character. Hence by condition (ii), G has exactly one nonlinear nonfaithful irreducible character. Furthermore this character has K in its kernel. Now (i) implies that the pair (G, K) satisfies the condition $(*)$. It completes the proof. □

In the rest of the paper we consider the following conjecture:

Conjecture. *Let G be a group which satisfies the condition $(*)$. Then G is solvable.*

Corollary 3.7. *Assume that the pair (G, K) satisfies the condition $(*)$. Then G is solvable if and only if K is solvable. In particular, if $G' \cap K = 1$, then G is solvable.*

Proof. It suffices to show that G/K is solvable. But it is clear by 1.3, 1.5 and the fact that every M-group is solvable. □

Theorem 3.8. *Let G be a nonsolvable group that satisfies the condition $(*)$. Then there exists an irreducible character χ of G such that $\chi(1)^2 \nmid |G|$.*

Proof. Assume that $\chi(1)^2 \mid |G|$ for all irreducible character of G. Then by the main theorem of [4], G has a nontrivial abelian normal subgroup. Equivalently the Fitting subgroup of G is not trivial. On the other hand, since G is not solvable, K is the unique minimal normal subgroup of G. Hence K is contained in the Fitting subgroup of G. In particular, K is solvable which is a contradiction by Corollary 3.7. □

We close this paper with the following proposition:

Proposition 3.9. *Assume that the pair (G, K) satisfies the condition $(*)$. Then $\mathrm{dl}(G) \leq 3$.*

Proof. If G' is either an abelian or a perfect group then we obtain the desired result. Assume that $G'' < G'$ is not trivial. Hence $G'' = K$. On the other hand, K is either abelian or perfect. So G''' is either trivial or equals to G''. This completes the proof. □

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