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CANONICAL BASES FOR $\mathfrak{sl}(2,\mathbb{C})$-MODULES OF SPHERICAL MONOGENICS IN DIMENSION 3

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Abstract. Spaces of homogeneous spherical monogenics in dimension 3 can be considered naturally as $\mathfrak{sl}(2,\mathbb{C})$-modules. As finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules, they have canonical bases which are, by construction, orthogonal. In this note, we show that these orthogonal bases form the Appell system and coincide with those constructed recently by S. Bock and K. Gürlebeck in [3]. Moreover, we obtain simple expressions of elements of these bases in terms of the Legendre polynomials.

1. Introduction

The main aim of this paper is to present an easy way to construct explicitly orthogonal bases for spaces of homogeneous spherical monogenics in dimension 3. Such bases were recently obtained by I. Cação in [9] and by S. Bock and K. Gürlebeck in [3]. In [9], orthogonal bases are constructed from systems of spherical monogenics which are obtained by applying the adjoint Cauchy-Riemann operator to elements of the standard bases of spherical harmonics. In [3], this idea is used for producing another orthogonal bases of spherical monogenics forming, in addition, the Appell system. In [4], it is observed that these bases forming the Appell system can be seen as the so-called Gelfand-Tsetlin bases. Moreover, in [4], it is shown that the Gelfand-Tsetlin bases could be obtained in quite a different way using the Cauchy-Kovalevskaya method and a characterization of the bases is given there. In [14, Theorem 2.2.3, p. 315], the Cauchy-Kovalevskaya method was already explained. But this method is not used in [14] for a construction of orthogonal bases of spherical monogenics although the construction is obvious not only in dimension 3 but in an arbitrary dimension as well. Actually, in [14, pp. 254-264] and [23, 25], another constructions even in all dimensions are given. By the way, the Cauchy-Kovalevskaya method is applicable in other settings, see [7, 6] and [13]. Finally, let us remark that Appell systems of monogenic polynomials were

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discussed before by H. R. Malonek et al. in [10, 11, 15, 16]. Similar questions were also studied for the Riesz system, see [20, 21, 22, 12, 26].

For an account of Clifford analysis, we refer to [14]. Now we introduce some notations. Let $(e_1, \dots, e_m)$ be the standard basis of the Euclidean space $\mathbb{R}^m$ and let $\mathbb{C}_m$ be the complex Clifford algebra generated by the vectors $e_1, \dots, e_m$ such that $e_j^2 = -1$ for $j = 1, \dots, m$. Recall that the Spin group $\mathrm{Spin}(m)$ is defined as the set of products of an even number of unit vectors of $\mathbb{R}^m$ endowed with the Clifford multiplication. The Lie algebra $\mathfrak{so}(m)$ of the group $\mathrm{Spin}(m)$ can be realized as the space of bivectors of Clifford algebra $\mathbb{C}_m$, that is,

$$\mathfrak{so}(m) = \left\langle \{e_{ij} : 1 \le i < j \le m\} \right\rangle .$$

Here $e_{ij} = e_i e_j$ and $\langle M \rangle$ stands for the span of a set M.

Denote by $\mathcal{H}_k(\mathbb{R}^3)$ the space of complex valued harmonic polynomials P in $\mathbb{R}^3$ which are k-homogeneous. Then the space $\mathcal{H}_k(\mathbb{R}^3)$ of spherical harmonics is an irreducible module under the h-action, defined by

$$[h(s)(P)](x) = P(s^{-1}xs)\,, \quad s \in \mathrm{Spin}(3) \quad \text{and} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Moreover, let S be a basic spinor representation of the group $\mathrm{Spin}(3)$. Then denote by $\mathcal{M}_k(\mathbb{R}^3, S)$ the set of S-valued k-homogeneous polynomials P in $\mathbb{R}^3$ which satisfy the equation $\partial P = 0$ where the Dirac operator ∂ is given by

$$\partial = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} .$$

It is well-known that the space $\mathcal{M}_k(\mathbb{R}^3, S)$ of spherical monogenics is an irreducible module under the L-action, defined by

$$[L(s)(P)](x) = s\,P(s^{-1}xs)\,, \quad s \in \mathrm{Spin}(3) \quad \text{and} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Both spaces $\mathcal{H}_k(\mathbb{R}^3)$ and $\mathcal{M}_k(\mathbb{R}^3, S)$ can be seen naturally as irreducible finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules. As finite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules, they have canonical bases which are, by construction, orthogonal.

In this paper, we study properties of canonical bases of spaces $\mathcal{M}_k(\mathbb{R}^3, S)$. In Theorem 1, we describe their close relation to canonical bases of spherical harmonics, we show that they form the Appell system and we give recurrence formulas for their elements. By the way, in [1, 2] analogous recurrence formulas generate easily elements of the orthogonal bases described in [3]. Moreover, we express elements of the canonical bases in terms of classical special functions (see Theorem 2). As in [4], we can adapt these results easily to quaternion valued spherical monogenics. It turns out that these bases coincide with those constructed recently by S. Bock and K. Gürlebeck in [3]. In Theorem 3, we obtain simple expressions of elements of these bases in terms of the Legendre polynomials. Let us remark that in [18, 19] homogeneous solutions of the Riesz system in dimension 3 forming orthogonal bases are expressed as finite sums of products of the Legendre and Chebyshev polynomials.

2. Spherical harmonics in dimension 3

In this section, we recall the construction of canonical bases for finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules and, as an example, we describe well-known bases of spherical harmonics in dimension 3 by means of classical special functions.

Obviously, the action of $\mathfrak{so}(3)$ on the space $\mathcal{H}_k(\mathbb{R}^3)$ is given by

$$h_{ij} = dh(e_{ij}/2) = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} \quad (i \neq j)\,.$$

Moreover, it is easily seen that

$$[h_{12}, h_{23}] = h_{31}\,, \quad [h_{23}, h_{31}] = h_{12} \quad \text{and} \quad [h_{31}, h_{12}] = h_{23}$$

where $[L, K] = LK - KL$. We can naturally identify the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ with the complexification of $\mathfrak{so}(3)$. Indeed, the operators

$$H = -ih_{12}\,, \quad X^+ = h_{31} + ih_{23} \quad \text{and} \quad X^- = -h_{31} + ih_{23}$$

satisfy the standard $\mathfrak{sl}(2,\mathbb{C})$-relations:

$$[X^+, X^-] = 2H \quad \text{and} \quad [H, X^{\pm}] = \pm X^{\pm}\,.$$

Putting $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, we have that

$$X^+ = -2x_3 \frac{\partial}{\partial z} + \overline{z} \frac{\partial}{\partial x_3} \quad \text{and} \quad X^- = 2x_3 \frac{\partial}{\partial \overline{z}} - z \frac{\partial}{\partial x_3}\,. \tag{1}$$

Furthermore, it is well-known that, as an $\mathfrak{sl}(2,\mathbb{C})$-module, $\mathcal{H}_k(\mathbb{R}^3)$ is irreducible and has the highest weight k. In each finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-module there exists always a canonical basis consisting of weight vectors, see [8, p. 116].

Proposition 1. Let V_l be an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module with the highest weight l. Then

(i) There is a primitive element f_0 of V_l, that is, there is a non-zero element f_0 of V_l such that

$$Hf_0 = lf_0 \quad \text{and} \quad X^+ f_0 = 0\,.$$

(ii) A basis of V_l is formed by the elements

$$f_j = (X^-)^j f_0\,, \quad j = 0, \dots, 2l\,.$$

In addition, for each $j = 0, \dots, 2l$, the element f_j is a weight vector with the weight $l - j$, that is, f_j is a non-zero element of V_l such that

$$Hf_j = (l - j)f_j.$$

Moreover, $X^- f_{2l} = 0$ and each weight vector f_j is uniquely determined up to a non-zero multiple.

(iii) The basis $\{f_0, \dots, f_{2l}\}$ is orthogonal with respect to any inner product $(\cdot,\cdot)$ on V_l which is invariant, that is, for each $L \in \mathfrak{sl}(2,\mathbb{C})$ and each $f, g \in V_l$, we have that

$$(Lf, Lg) = (f, g).$$

By Proposition 1, to construct the canonical basis of the module $\mathcal{H}_k(\mathbb{R}^3)$ it is sufficient to find its primitive.

Proposition 2. The irreducible $\mathfrak{sl}(2,\mathbb{C})$-module $\mathcal{H}_k(\mathbb{R}^3)$ has a basis consisting of the polynomials

$$f_0^k = \frac{1}{k!\,2^k}\,\overline{z}^k \quad \text{and} \quad f_j^k = (X^-)^j f_0^k\,, \quad 0 < j \leq 2k\,.$$

In addition, for each $j = 0, \ldots, 2k$, the polynomial f_j^k is a weight vector with the weight $k - j$, that is, $H f_j^k = (k-j) f_j^k$.

Proof. It is easy to see that f_0^k is a primitive of $\mathcal{H}_k(\mathbb{R}^3)$. □

Following [8], we identify the functions f_j^k with classical special functions. To do this we use spherical co-ordinates

$$x_1 = r\sin\theta\sin\varphi\,, \quad x_2 = r\sin\theta\cos\varphi\,, \quad x_3 = r\cos\theta \tag{2}$$

with $0 \leq r$, $-\pi \leq \varphi \leq \pi$ and $0 \leq \theta \leq \pi$. Let us remark that, in spherical co-ordinates (2), the operators H, X^+ and X^- have the form

$$\begin{aligned} H &= -i\frac{\partial}{\partial\varphi}\,, \\ X^+ &= e^{i\varphi}\Big(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\Big)\,, \\ X^- &= e^{-i\varphi}\Big(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\varphi}\Big)\,. \end{aligned} \tag{3}$$

In [8, pp. 120-121] (with the variables x_1 and x_2 interchanged), the next result is shown.

Proposition 3. Let $\{f_0^k, \ldots, f_{2k}^k\}$ be the basis of $\mathcal{H}_k(\mathbb{R}^3)$ defined in Proposition 2. Using spherical co-ordinates (2), we have then that, for each $j = 0, \ldots, 2k$,

$$f_j^k(r,\theta,\varphi) = i^{k-j} r^k e^{i(k-j)\varphi} P_k^{j-k}(\cos\theta)$$

where

$$P_k^l(s) = \frac{1}{k!\,2^k}\,(1-s^2)^{l/2}\frac{d^{l+k}}{ds^{l+k}}\,(s^2-1)^k\,, \quad s \in \mathbb{R}\,.$$

Here P_k^0 is the k-th Legendre polynomial and P_k^l are its associated Legendre functions.

3. Spherical monogenics in dimension 3

In this section, we study properties of canonical bases of $\mathfrak{sl}(2,\mathbb{C})$-modules of spherical monogenics and, in particular, we express elements of these bases by means of classical special functions. We begin with spinor valued spherical monogenics.

Spinor valued polynomials. In what follows, S stands for a (unique up to equivalence) basic spinor representation of Spin (3) and $\mathfrak{so}(3) = \langle e_{12}, e_{23}, e_{31} \rangle$. As an $\mathfrak{so}(2)$-module, the module S is reducible and decomposes into two inequivalent irreducible submodules

$$S^{\pm} = \{u \in S : -ie_{12}\, u = \pm u\}$$

provided that $\mathfrak{so}(2) = \langle e_{12} \rangle$. Moreover, the spaces $S^{\pm}$ are both one-dimensional. Let $S^{\pm} = \langle v^{\pm} \rangle$. Then each $s \in S$ is of the form $s = s^+ v^+ + s^- v^-$ for some complex numbers $s^{\pm}$. We write $s = (s^+, s^-)$.

Furthermore, the action of $\mathfrak{so}(3)$ on the space $\mathcal{M}_k(\mathbb{R}^3, S)$ is given by

$$L_{ij} = dL(e_{ij}/2) = \frac{e_{ij}}{2} + h_{ij} \quad \text{with} \quad h_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} \quad (i \neq j)\,.$$

It is easily seen that

$$[L_{12}, L_{23}] = L_{31}\,, \quad [L_{23}, L_{31}] = L_{12} \quad \text{and} \quad [L_{31}, L_{12}] = L_{23}\,.$$

Moreover, the operators

$$\tilde{H} = -iL_{12}\,, \quad \tilde{X}^+ = L_{31} + iL_{23} \quad \text{and} \quad \tilde{X}^- = -L_{31} + iL_{23}$$

generate the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Indeed, we have that

$$[\tilde{X}^+, \tilde{X}^-] = 2\tilde{H} \quad \text{and} \quad [\tilde{H}, \tilde{X}^{\pm}] = \pm \tilde{X}^{\pm}\,.$$

Put again $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$. Then it is easy to see that

$$\tilde{X}^{\pm} = X^{\pm} + \omega^{\pm} \quad \text{where} \quad \omega^+ = \frac{1}{2}(e_{31} + ie_{23})\,, \quad \omega^- = \frac{1}{2}(-e_{31} + ie_{23})$$

and $X^{\pm}$ are defined as in (1). Furthermore, as an $\mathfrak{sl}(2, \mathbb{C})$-module, $\mathcal{M}_k(\mathbb{R}^3, S)$ is irreducible and has the highest weight $k + \frac{1}{2}$. We can construct again a canonical basis of this module using Proposition 1.

Proposition 4. The irreducible $\mathfrak{sl}(2, \mathbb{C})$-module $\mathcal{M}_k(\mathbb{R}^3, S)$ has a basis consisting of the polynomials

$$F_0^k = \frac{1}{k!\, 2^k}\, \overline{z}^k v^+ \quad \text{and} \quad F_j^k = (\tilde{X}^-)^j F_0^k\,, \quad 0 < j \leq 2k + 1\,.$$

In addition, for each $j = 0, \dots, 2k + 1$, the polynomial F_j^k is a weight vector with the weight $k + \frac{1}{2} - j$, that is, $\tilde{H} F_j^k = (k + \frac{1}{2} - j) F_j^k$.

Proof. Obviously, the polynomial F_0^k is a primitive of $\mathcal{M}_k(\mathbb{R}^3, S)$. □

By Proposition 1, the basis of $\mathcal{M}_k(\mathbb{R}^3, S)$ constructed in Proposition 4 is orthogonal with respect to any invariant inner product on $\mathcal{M}_k(\mathbb{R}^3, S)$. As is well-known, the Fischer inner product and the standard L^2-inner product on the unit ball of $\mathbb{R}^3$ are examples of invariant inner products on $\mathcal{M}_k(\mathbb{R}^3, S)$, see [14, pp. 206 and 209]. In the next theorem, we show further properties of the constructed bases. Statement (a) of Theorem 1 shows the close relation of the canonical bases of spherical harmonics to those of spherical monogenics. Moreover, by statement (b), the polynomials F_j^k form the so-called Appell system, that is, they satisfy the property (4) below. Finally, statement (c) of Theorem 1 contains the recurrence formula for elements F^k_j of the constructed bases.

Theorem 1. (a) We have that

$$(\tilde{X}^-)^j = (X^-)^j + j(X^-)^{j-1}\omega^- \,, \quad j \in \mathbb{N}\,.$$

In particular, for $j = 0, \dots, 2k+1$, we get that $F^k_j = f^k_j v^+ + j f^k_{j-1}\omega^- v^+$. Here $f^k_{-1} = f^k_{2k+1} = 0$ and $\{f^k_0, \dots, f^k_{2k}\}$ is the basis of $\mathcal{H}_k(\mathbb{R}^3)$ as in Proposition 2.

(b) Moreover, it holds that

$$\Big[\frac{\partial}{\partial x_3}, (\tilde{X}^-)^j\Big] = 2j(\tilde{X}^-)^{j-1}\frac{\partial}{\partial \overline{z}}\,, \quad j \in \mathbb{N}\,.$$

In particular, for each $k \in \mathbb{N}$,

$$\frac{\partial F^k_j}{\partial x_3} = \begin{cases} j\ F^{k-1}_{j-1}\,, & j = 1, \dots, 2k\,;\\ 0\,, & j = 0, 2k+1\,.\end{cases} \tag{4}$$

(c) Finally, we have that

$$[x_3, (\tilde{X}^-)^j] = j(\tilde{X}^-)^{j-1} z\,, \quad j \in \mathbb{N}\,.$$

In particular, for each $k \in \mathbb{N}_0$ and $j = 0, 1, \dots, 2k+1$,

$$F^{k+1}_{j+1} = x_3 F^k_j - jzF^k_{j-1} + \omega^- F^{k+1}_j \quad \text{where} \quad F^k_{-1} = 0\,.$$

Proof. The statements (a) and (b) follow, by induction, from the following facts:

$$(\omega^-)^2 = 0\,, \quad [X^-, \omega^-] = 0\,, \quad \Big[\frac{\partial}{\partial x_3}, X^-\Big] = 2\frac{\partial}{\partial \overline{z}} \quad \text{and} \quad \Big[\frac{\partial}{\partial \overline{z}}, X^-\Big] = 0\,.$$

We show statement (c). We have that $[x_3, X^-] = z$ and $[z, X^-] = 0$ and hence, by induction, we get easily

$$[x_3, (\tilde{X}^-)^j] = j(\tilde{X}^-)^{j-1} z\,.$$

In particular, for $j = 1, \dots, 2k$, we have that

$$x_3 F^k_j - (\tilde{X}^-)^j (x_3 F^k_0) = jzF^k_{j-1}\,,$$

which finishes the proof together with the obvious relation

$$F^{k+1}_1 = x_3 F^k_0 + \omega^- F^{k+1}_0\,.$$

□

Remark 1. (a) We can realize the space S in the Clifford algebra $\mathbb{C}_4$. Indeed, we can put

$$v^+ = \frac{1}{4}(1 - ie_{12})(1 - ie_{34}) \quad \text{and} \quad v^- = \frac{1}{4}(e_1 + ie_2)(e_3 + ie_4)\,.$$

We denote this realization of the space S by S^+_4. In particular, we have that $\omega^- v^+ = v^-$ and $\omega^- v^- = 0$.

(b) There is another realization S^-_4 of the space S inside $\mathbb{C}_4$ if we put

$$v^+ = \frac{1}{4}(1 - ie_{12})(e_3 + ie_4) \quad \text{and} \quad v^- = \frac{1}{4}(e_1 + ie_2)(1 - ie_{34})\,.$$

In this case, we have that $\omega^- v^+ = -v^-$ and $\omega^- v^- = 0$. Let us remark that although, as $\mathfrak{so}(3)$-modules, S_4^+ and S_4^- are of course equivalent to each other they are different as $\mathfrak{so}(4)$-modules. See [14, pp. 114-118] for details.

(c) Let $\{F_0^{k,\pm}, \dots, F_{2k+1}^{k,\pm}\}$ be the basis of $\mathcal{M}_k(\mathbb{R}^3, S_4^\pm)$ defined in Proposition 4. By statement (a) of Theorem 1, it is easy to see that, for $j = 0, \dots, 2k+1$, we get

$$F_j^{k,\pm} = (f_j^k,\ \pm j f_{j-1}^k)\,.$$

Here $\{f_0^k, \dots, f_{2k}^k\}$ is the basis of $\mathcal{H}_k(\mathbb{R}^3)$ defined in Proposition 2.

Using the observation (c) of Remark 1 and Proposition 3, we can easily express the functions $F_j^{k,\pm}$ in terms of classical special functions.

Theorem 2. Let $\{F_0^{k,\pm}, \dots, F_{2k+1}^{k,\pm}\}$ be the basis of $\mathcal{M}_k(\mathbb{R}^3, S_4^\pm)$ defined in Proposition 4. Using spherical co-ordinates (2), we then have that

$$F_j^{k,\pm}(r,\theta,\varphi) = i^{k-j} r^k e^{i(k-j)\varphi}\ \left(P_k^{j-k}(\cos\theta),\ \pm i j e^{i\varphi} P_k^{j-k-1}(\cos\theta)\right)$$

for each $j = 0, \dots, 2k+1$. Here $P_k^{k+1} = 0 = P_k^{-k-1}$.

Now we are going to deal with quaternion valued spherical monogenics.

Quaternion valued polynomials. In what follows, $\mathbb{H}$ stands for the skew field of real quaternions q with the imaginary units i_1, i_2 and i_3, that is,

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1 \quad \text{and} \quad q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3, (q_0, q_1, q_2, q_3) \in \mathbb{R}^4\,.$$

For a quaternion q, put $\overline{q} = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3$. We realize $\mathbb{H}$ as the subalgebra of complex 2×2 matrices of the form

$$q = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix}. \tag{5}$$

In particular, we have that

$$i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Furthermore, we identify $\mathfrak{so}(3)$ with $\langle i_1, i_2, i_3 \rangle$ as follows: $e_{12} \simeq i_3$, $e_{23} \simeq i_1$ and $e_{31} \simeq i_2$. Then we can realize the basic spinor representation S of $\mathfrak{so}(3)$ as the space $\mathbb{C}^2$ of column vectors

$$s = \begin{pmatrix} q_0 + iq_3 \\ q_2 + iq_1 \end{pmatrix}.$$

Here the action of $\mathfrak{so}(3)$ on S is given by the matrix multiplication from the left.

Now we are interested in quaternion valued polynomials $g = g(y)$ in the variable $y = (y_0, y_1, y_2)$ of $\mathbb{R}^3$. Let us denote by $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ the space of $\mathbb{H}$-valued k-homogeneous polynomials g satisfying the Cauchy-Riemann equation $Dg = 0$ with

$$D = \frac{\partial}{\partial y_0} + i_1 \frac{\partial}{\partial y_1} + i_2 \frac{\partial}{\partial y_2}.$$

It is easy to see that both columns of an element $g \in \mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ belong to the space $\tilde{\mathcal{M}}_k(\mathbb{R}^3, S)$ of S-valued solutions h of the equation $Dh = 0$ which are k-homogeneous.

Moreover, we can consider naturally $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ as a right $\mathbb{H}$-linear Hilbert space with the $\mathbb{H}$-valued inner product

$$(Q, R)_{\mathbb{H}} = \int_{B_3} \overline{Q} R \, dV$$

where B_3 is the unit ball and dV is the Lebesgue measure in $\mathbb{R}^3$. In [3], orthogonal bases of spaces $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ forming, in addition, the Appell system are constructed. In [4], the following characterization of these bases is given.

Proposition 5. For each $k \in \mathbb{N}_0$, there exists an orthogonal basis

$$\{g_j^k \mid j = 0, \dots, k\} \tag{6}$$

of the right $\mathbb{H}$-linear Hilbert space $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ such that:

(i) Let $j = 0, \dots, k$ and let h_j^k and h_{2k+1-j}^k be the first and the second column of the (matrix valued) polynomial g_j^k, respectively. Then we have that

$$H h_j^k = \Big(k + \frac{1}{2} - j\Big) h_j^k \quad \text{and} \quad H h_{2k+1-j}^k = -(k + \frac{1}{2} - j) h_{2k+1-j}^k$$

with

$$H = -i\Big(\frac{i_3}{2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}\Big)\,.$$

(ii) We have that

$$\frac{\partial g_j^k}{\partial y_0} = \begin{cases} k g_{j-1}^{k-1}\,, & j = 1, \dots, k\,; \\ 0\,, & j = 0\,. \end{cases}$$

(iii) For each $k \in \mathbb{N}_0$, we have that $g_0^k = (y_1 - i_3 y_2)^k$.

Moreover, the polynomials g_j^k are determined uniquely by the conditions (i), (ii) and (iii). Finally, for each $k \in \mathbb{N}_0$, the S-valued polynomials

$$h_0^k, h_1^k, \dots, h_{2k+1}^k$$

form the canonical basis of the $\mathfrak{sl}(2, \mathbb{C})$-module $\tilde{\mathcal{M}}_k(\mathbb{R}^3, S)$.

In [3] and [4], quite explicit formulas for the polynomials g_j^k are given in the cartesian coordinates y_0, y_1, y_2. We now construct these polynomials in yet another way using Theorem 2. Indeed, in Theorem 3 below, we express the polynomials g_j^k in spherical co-ordinates

$$y_0 = r\cos\theta\,, \quad y_1 = r\sin\theta\cos\varphi\,, \quad y_2 = r\sin\theta\sin\varphi \tag{7}$$

with $0 \le r$, $-\pi \le \varphi \le \pi$ and $0 \le \theta \le \pi$.

Theorem 3. Let the set $\{g_j^k \mid j = 0, \dots, k\}$ be the basis of $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ as in Proposition 5. Using spherical co-ordinates (7), we have then that

$$g_j^k(r, \theta, \varphi) = (k!/j!)(-2)^{k-j} r^k \, (g_{j,0}^k + g_{j,1}^k \, i_1 + g_{j,2}^k \, i_2 + g_{j,3}^k \, i_3) \ \text{ where}$$

$$g_{j,0}^k = P_k^{j-k}(\cos\theta)\cos(j-k)\varphi\,, \qquad g_{j,1}^k = -jP_k^{j-k-1}(\cos\theta)\cos(j-k-1)\varphi\,,$$
$$g_{j,2}^k = jP_k^{j-k-1}(\cos\theta)\sin(j-k-1)\varphi\,, \quad g_{j,3}^k = P_k^{j-k}(\cos\theta)\sin(j-k)\varphi\,.$$

Here P_k^0 is the k-th Legendre polynomial and P_k^l are its associated Legendre functions (see Proposition 3 for the formulas of P_k^l).

Proof. (a) Let $k \in \mathbb{N}_0$ and $j = 0, \dots, k$. It is easy to see that the S-valued polynomial

$$\hat{h}_j^k(y_0, y_1, y_2) = F_j^{k,-}(-y_2, y_1, y_0)$$

solves the equation $D\hat{h}_j^k = 0$. Here $F_j^{k,-}$ are as in Remark 1 (c).

(b) We can find non-zero complex numbers $c_j^k \in \mathbb{C}$ such that the polynomials $h_j^k = c_j^k \hat{h}_j^k$ satisfy, in addition, that $h_0^k = ((y_1 - iy_2)^k, 0)$ and

$$\frac{\partial h_j^k}{\partial y_0} = \begin{cases} k h_{j-1}^{k-1}\,, & j = 1, \dots, k\,; \\ 0\,, & j = 0\,. \end{cases}$$

Indeed, by Theorem 1, it is sufficient to put $c_j^k = (2i)^{k-j} k!/j!$.

(c) Using spherical co-ordinates (7), we obviously have that

$$h_j^k(r, \theta, \varphi) = c_j^k\ F_j^{k,-}(r, \theta, -\varphi)$$

where $F_j^{k,-}$ are as in Theorem 2. In particular, putting $d_j^k = (k!/j!)(-2)^{k-j}$, we have that $h_j^k = (h_{j,0}^k, h_{j,1}^k)$ with

$$h_{j,0}^k = d_j^k r^k e^{i(j-k)\varphi} P_k^{j-k}(\cos\theta) \quad \text{and} \quad h_{j,1}^k = d_j^k r^k (-i) j e^{i(j-k-1)\varphi} P_k^{j-k-1}(\cos\theta)\,.$$

(d) Finally, we define an $\mathbb{H}$-valued polynomial g_j^k corresponding to the S-valued polynomial $h_j^k = (h_{j,0}^k, h_{j,1}^k)$ by

$$g_j^k = \operatorname{Re} h_{j,0}^k + i_1 \operatorname{Im} h_{j,1}^k + i_2 \operatorname{Re} h_{j,1}^k + i_3 \operatorname{Im} h_{j,0}^k\,.$$

Here, for a complex number z, we write $\operatorname{Re} z$ for its real part and $\operatorname{Im} z$ for its imaginary part. Obviously, the polynomials g_j^k satisfy the conditions (i), (ii) and (iii) of Proposition 5, which easily completes the proof. □

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