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TRANSVERSAL BIWAVE MAPS

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Abstract. In this paper, we prove that the composition of a transversal biwave map and a transversally totally geodesic map is a transversal biwave map. We show that there are biwave maps which are not transversal biwave maps, and there are transversal biwave maps which are not biwave maps either. We prove that if f is a transversal biwave map satisfying certain condition, then f is a transversal wave map. We finally study the transversal conservation laws of transversal biwave maps.

1. Introduction

Following the theory of harmonic maps of Riemannian manifolds established by Eells, Sampson and Lemaire [9, 10, 11], biharmonic maps were introduced by Jiang [15, 16] in 1986. In this decade, there has been progress in biharmonic maps made by Caddeo, Montaldo, Loubeau, Oniciuc, Piu [1, 2, 24, 26], Chiang, Wolak, Sun [5, 6, 7], Chang, L. Wang and Yang [3], C. Wang [36], etc. Wave maps are harmonic maps on Minkowski spaces. In recent years, there have been new developments in wave maps achieved by Klainerman and Macghedon [19, 20], Shatah and Struwe [29, 30], Tao [31, 32], Tataru [33, 34], Nahmod, Stefanov, Uhlenbeck [27], etc. Moreover, Chiang and Yang have studied exponential wave maps in [8].

Transversal harmonic maps between foliated Riemannian manifolds were introduced by Konderak and Wolak [21, 22] in 2003. Transversal harmonic maps between foliated manifolds with one manifold foliated by points were first studied by Eells and Verjovsky [12], and Kacimi and Gomez [18]. Biwave maps are biharmonic maps on Minkowski spaces, which generalize wave maps, have been first studied by Chiang [4] recently. In this paper, we investigate transversal biwave maps from foliated Minkowski spaces into foliated Riemannian manifolds. Transversal biwave maps whose equations are the fourth order hyperbolic systems of PDEs on transverse manifolds, which are different than transversal biharmonic maps [7] whose equations are the fourth order elliptic systems of PDEs on transverse manifolds.

In Section 2, we introduce semi-Riemannian (resp. Minkowskian, Lorentzian) foliations following Riemannian foliations, and recall transversal tension fields and transversal biharmonic maps. In Section 3, we prove in Theorem 3.3 that if $f\colon R^{m,1} \to (M_1, \mathcal{F}_1)$ is a transversal biwave map and $f_1\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is

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a transversally totally geodesic map, then $f_1 \circ f\colon R^{m,1} \to (M_2, \mathcal{F}_2)$ is a transversal biwave map. Thus for any target foliated manifold $(M_2, \mathcal{F}_2)$, we can provide many transversal biwave maps associated to the transversal geodesics of $(M_2, \mathcal{F}_2)$. We show that there are biwave maps which are not transversal biwave maps in Example 3, and there are transversal biwave maps which are not biwave maps in Example 4 either. Afterwards, we prove in Theorem 3.4 that *if f is a transversal biwave map from a compact foliated domain in a foliated Minkowski space into a foliated manifold such that*

$$-|\tau_\Box \bar{f}|_t^2 + \sum_{i=1}^{q} |\tau_\Box \bar{f}|_{x^i}^2 - {R'}^{\alpha}_{\beta\gamma\mu}\Big(-\bar{f}_t^\beta \bar{f}_t^\gamma + \sum_{i=1}^{q} \bar{f}_i^\beta \bar{f}_i^\gamma\Big)\tau_\Box(\bar{f})^\mu \geq 0\,,$$

then f is a transversal wave map. This theorem is different than the theorem obtained in [7]: *If f is a transversal biharmonic map from a compact foliated Riemannian manifold into a foliated manifold with non-positive transversal Riemannian curvature, then f is a transversal harmonic map.*

In Section 4, we study the transversal conservation laws of transversal biwave maps associated to stress bi-energy tensors in Theorem 4.3 and Corollary 4.4. We finally investigate stable transversal biwave maps in Theorem 4.5.

2. Preliminaries

2.1. **Foliations.** Let $\mathcal{F}$ be a foliation on a Riemannian n-manifold (M, g). Then $\mathcal{F}$ is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i\in I}$ modeled on a q-manifold N_0, where

(1) $\{U_i\}_{i\in I}$ is an open covering of M,

(2) $f_i\colon U_i \to N_0$ are submersions with connected fibres,

(3) $g_{ij}\colon N_0 \to N_0$ are local diffeomorphisms of N_0 such that $f_i = g_{ij} f_j$ on $U_i \cap U_j$.

The connected components of the trace of any leaf of $\mathcal{F}$ on U_i consist of the fibres of f_i. The open subsets $N_i = f_i(U_i) \subset N_0$ form a q-manifold $N_{\mathcal{U}} = \amalg N_i$, which can be considered as a transverse manifold of the foliation $\mathcal{F}$. The pseudogroup $\mathcal{H}_{\mathcal{U}}$ of local diffeomorphisms of $N_{\mathcal{U}}$ generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold $(M, \mathcal{F})$ defined by the cocycle $\mathcal{U}$. If the foliation $\mathcal{F}$ is Riemannian for the Riemannian metric g, then it induces a Riemannian metric $\bar{g}$ on $N_{\mathcal{U}}$ such that the submersions f_i are Riemannian submersions and the elements of the holonomy group are isometries. The foliation $\mathcal{F}$ is transversally semi-Riemannian (resp. Minkowskian, Lorentzian) if its normal bundle admits a semi-Riemannian (resp. Minkowskian, Lorentzian) metric h such that for any vector field X tangent to the leaves of $\mathcal{F}$ we have $L_X h = 0$ (where $L_X h(X, Y) = X \cdot h(Y, Z) - h([X, Y], Z) - h(Y, [X, Z])$ for vector fields Y, Z tangent to the leaves of $\mathcal{F}$). This condition is equivalent to the existence of an $\mathcal{H}_{\mathcal{U}}$-invariant semi-Riemannian (resp. Minkowskian, Lorentzian) metric $\bar{h}$ on the transverse manifold $N_{\mathcal{U}}$, cf. [37].

Let $\phi\colon U \to R^p \times R^q$, $\phi = (\phi^1, \phi^2) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ be an adapted chart on the foliated manifold $(M, \mathcal{F})$. Then on U the vector fields $\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation $\mathcal{F}$, the equivalence classes denoted by $\frac{\bar\partial}{\partial y_1}, \dots \frac{\bar\partial}{\partial y_q}$ of $\frac{\partial}{\partial y_1}, \dots \frac{\partial}{\partial y_q}$ span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$, which is isomorphic to the subbundle $T\mathcal{F}^\perp$. This bundle and the others considered in the paper are naturally foliated by foliations whose leaves are covering spaces of leaves of $\mathcal{F}$ and whose defining cocycles can be derived in the obvious way from the cocycle $\mathcal{U}$, cf. [37]. In the non-Riemannian case we can take any subbundle Q supplementary to $T\mathcal{F}$ and for simplicity we shall denote it by the same symbol.

The sheaf $\Gamma_b(T\mathcal{F}^\perp)$ of foliated sections of the vector bundle $T\mathcal{F}^\perp \to M$ may be described as follows: Let U be an open subset of M. Then $X \in \Gamma_b(U, T\mathcal{F}^\perp)$ if and only if for each local Riemannian submersion $\phi\colon U \to \bar U$ defining $\mathcal{F}$, the restriction of X to U is projectable via the map ϕ on a vector field $\bar X$ on $\bar U$.

Definition 2.1 ([25])**.** A *basic partial connection* on $(M, \mathcal{F}, g)$ is a sheaf operator D such that for each open subset U of M

$$D\colon \Gamma_b(U, T\mathcal{F}^\perp) \times \Gamma_b(U, T\mathcal{F}^\perp) \to \Gamma_b(U, T\mathcal{F}^\perp)$$

and for any X, Y, $Z \in \Gamma_b(U, T\mathcal{F}^\perp)$ and any f, $h \in C_b^\infty(U)$:
1. $D_{fX+hY}Z = fD_XY + hD_XZ$,
2. D_X is R-linear,
3. $D_X fY = X(f)Z + fD_XY$ (the transversal Leibniz rule).

Let ∇ be the Levi-Civita connection of g. Then for any open subset U of M and $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$ we define D as

$$D_XY = (\nabla_XY)^\perp\,, \tag{2.1}$$

where $(\nabla_XY)^\perp$ is a local foliated section of $T\mathcal{F}^\perp$. It is easy to check that D is a basic partial connection on $(M,\, g,\, \mathcal{F})$. Let $\phi\colon U \to \bar U$ be a Riemannian submersion defining the foliation $\mathcal{F}$ on an open set U. Let us assume that $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$, and $\bar X$, $\bar Y$ be the push forward vector fields via the map ϕ. Then there is a well-known property of Riemannian foliations from [35] that

$$d\phi(D_XY) = \nabla^{\bar g}_{\bar X}\bar Y\,, \tag{2.2}$$

where $\nabla^{\bar g}$ is the Levi-Civita connection of the metric $\bar g$.

The operator D can be defined using the induced metric on the normal bundle via the well-known formula for the Levi-Civita connection restricted to normal vectors. Foliated semi-Riemannian (resp. Minkowskian, Lorentzian) metrics in the normal bundle define basic partial connections in the standard way.

2.2. **Transversal tension fields.** Let $(M_1, \mathcal{F}_1, g_1)$ and $(M_2, \mathcal{F}_2, g_2)$ be two Riemannian manifolds with Riemannian foliations. Let ∇^i be the Levi-Civita connections of the respective metrics and D^i be the induced basic partial connections on the orthogonal complement bundles $T\mathcal{F}_i^\perp \to M_i$, $i = 1, 2$. Suppose that $f\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a smooth foliated leaf-preserving map, i.e., $df(T\mathcal{F}_1) \subset T\mathcal{F}_2$. Then there are given natural bundle maps

$$I_i\colon T\mathcal{F}_i^\perp \to TM_i\,, \quad P_i\colon TM_i \to T\mathcal{F}_i^\perp \quad \text{for} \quad i = 1, 2\,,$$

where I_i is the inclusion of $T\mathcal{F}_i^\perp$ in TM_i and P_i is the orthogonal projection of TM_i onto $T\mathcal{F}_i^\perp$. Let X be a local foliated section of $T\mathcal{F}_1^\perp \to M_1$, and then $P_2df(X)$ is a foliated section of the bundle $f^{-1}T\mathcal{F}_2^{\perp}$. Thus $P_2 df I_1$ is a foliated section of the bundle $(T\mathcal{F}_1^{\perp})^* \otimes f^{-1}T\mathcal{F}_2^{\perp}$. We define *transversal second fundamental form* $S_b(f)$ of f as the covariant derivative $D(P_2 df I_1)$, which is a global section of the bundle

$$(T\mathcal{F}_1^{\perp})^* \otimes (T\mathcal{F}_1^{\perp})^* \otimes f^{-1}T\mathcal{F}_2^{\perp} \to M_1\,,$$

where the connection D on the bundle $\left(T\mathcal{F}_1^{\perp}\right)^* \otimes f^{-1}T\mathcal{F}_2^{\perp} \to M_1$ is induced by D_1 and D_2.

Let $(M_1, g_1, \mathcal{F}_1)$ and $(M_2, g_2, \mathcal{F}_2)$ be two foliated Riemannian manifolds defined by cocycles $\mathcal{U} = \{U_i, \phi_i, g_{ij}\}$ and $\mathcal{V} = \{V_\alpha, \psi_\alpha, h_{\alpha\beta}\}$, respectively. Suppose that $f\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a smooth leaf-preserving map. Let $U \subset M_1$ and $V \subset M_2$ be open subsets. Let $\phi\colon (U, g_1) \to (\bar{U}, \bar{g}_1)$ be a Riemannian submersion on U and let $\psi\colon (V, g_2) \to (\bar{V}, \bar{g}_2)$ be a Riemannian submersion on V, which define locally the Riemannian foliations $\mathcal{F}_i$ for $i = 1, 2$. Suppose that $f(U) \subset V$. Then there exists the unique map $\bar{f}\colon \bar{U} \to \bar{V}$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ \bar{U} \arrow[r, "\bar{f}"'] & \bar{V} \end{tikzcd}$$

Diagram 1

commutes. If the cocycles $\mathcal{U}$ and $\mathcal{V}$ are such that for any U_i there exists V_α for which $f(U_i) \subset V_\alpha$, we say that these cocycles are f-related. Then f induces a map $\bar{f}\colon N_{\mathcal{U}} \to N_{\mathcal{V}}$ such that f and $\bar{f}$ commute locally in Diagram 1.

Lemma 2.2 ([21]). *Let Z_1, Z_2 be two local foliated vector fields on U which project, via the map ϕ, onto vector fields $\bar{Z}_1$, $\bar{Z}_2$ on $\bar{U}$. Then*

$$d\psi(D(P_2 df I_1)(Z_1, Z_2)) = (\nabla d\bar{f})(\bar{Z}_1, \bar{Z}_2)\,,$$

where $\bar{f}$ is the induced map between $\bar{U}$ and $\bar{V}$.

The trace of the transversal second fundamental form is called *transversal tension field of* f, and it is denoted by $\tau_b(f)$. If $X_{1x}, \ldots, X_{q_1 x}$ is an orthonormal basis of the space $T_x\mathcal{F}_1^{\perp}$, then

$$\tau_b(f)_x = \mathrm{trace}_{T\mathcal{F}_1^{\perp}}\, D(\Pi_2 d_x f I_1) = \sum_{\alpha=1}^{q_1} D(\Pi_2 d_x f I_1)(X_{\alpha x}, X_{\alpha x}) \tag{2.3}$$

is a section of the bundle $f^{-1}T\mathcal{F}_2^{\perp} \to M_1$. Please see more details in [21].

We shall also study one parameter families of foliated maps $f_s\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$, $s \in R$. In order to use variational arguments, we need to refine the cocycles defining foliations. Let $\mathcal{U} = \{U_i, \phi_i, g_{ij}\}_{i\in I}$ and $\hat{\mathcal{U}} = \{\hat{U}_i, \hat{\phi}_i, \hat{g_{ij}}\}_{i\in I}$ be two cocycles defining the foliation $\mathcal{F}_1$ such that U_i is a relatively compact subset of $\hat{U}_i$, $\phi_i = \hat{\phi}_i | U_i$ and g_{ij} is also the suitable restriction of $\hat{g_{ij}}$. Let $\mathcal{V} = \{V_\alpha, \psi_\alpha, h_{\alpha\beta}\}_{\alpha\in A}$ and $\hat{\mathcal{V}} = \{\hat{V_\alpha}, \hat{\psi_\alpha}, \hat{h_{\alpha\beta}}\}_{\alpha\in A}$ be two cocycles defining the foliation $\mathcal{F}_2$ such that V_α is a relatively compact subset of $\hat{V_\alpha}$, $\psi_\alpha = \hat{\psi_\alpha} | V_\alpha$ and $h_{\alpha\beta}$ is also the suitable restriction of $\hat{h_{\alpha\beta}}$. If the cocycles $\hat{\mathcal{U}}$ and $\mathcal{V}$ are f-related, $f = f_0$, then the cocycles $\mathcal{U}$ and $\mathcal{V}$ are f_s-related for any sufficiently small s.

2.3. **Transversal biharmonic maps.** Let X, Y, ξ be the foliated sections of $T\mathcal{F}_2^{\perp}$, and $D' = D^2$ be the basic partial connection on $T\mathcal{F}_2^{\perp}$. Then the Riemannian curvature

$$R'(X,Y)\xi = D'_X D'_Y \xi - D'_Y D'_X \xi - D'_{[X,Y]}\xi$$

is a section of the bundle $T\mathcal{F}_2^{\perp} \to M_2$. Following the notion of transversal tension field in Section 2.2, we define transversal bi-tension field as

$$(\tau_2)_b(f) = \triangle\tau_b(f) + R'(df, df)\tau_b(f)\,, \tag{2.4}$$

where $\triangle\xi = D^*D(\xi)$ is an operator from a section of $f^{-1}T\mathcal{F}_2^{\perp}$ to a section of $f^{-1}T\mathcal{F}_2^{\perp}$, D is the connection on $T\mathcal{F}_1^{\perp *} \otimes f^{-1}T\mathcal{F}_2^{\perp}$. Therefore, $(\tau_2)_b(f)$ is a section of the bundle $f^{-1}T\mathcal{F}_2^{\perp} \to M_1$.

We consider a one-parameter family of maps $\{f_t\} \in C^\infty((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2)), t \in I_\epsilon = (-\epsilon, \epsilon)$ from a compact foliated Riemannian manifold $(M_1, \mathcal{F}_1)$ into a foliated Riemannian manifold $(M_2, \mathcal{F}_2)$ such that $f_t(x)$ is the endpoint of the segment starting at $f(x)$ determined in length and direction by the vector field $\dot{f}$ along f. If we choose the defining cocycles $\mathcal{U}$ and $\mathcal{V}$ as at the end of Section 2.2, these foliated maps induce a one-parameter family of maps $\{\bar{f}_t\} \in C^\infty(N_{\mathcal{U}}, N_{\mathcal{V}})$ such that $\bar{f}_t(x)$ is the end point of the segment starting at $\bar{f}(x)$ determined in length and direction by the vector field $\dot{\bar{f}}$ along $\bar{f}$. The transversal bi-energy of f is

$$E_2(\bar{f}) = \frac{1}{2}\int_{N_{\mathcal{U}}} \|(d+d^*)^2\bar{f}\|^2\,dv = \frac{1}{2}\int_{\amalg\bar{U}_i} \|d^*d\bar{f}\|^2\,dv = \frac{1}{2}\int_{\amalg\bar{U}_i} \|\tau\bar{f}\|^2\,dv\,, \tag{2.5}$$

Then by [7] we have

$$\frac{d}{dt}E_2(\bar{f}_t)|_{t=0} = \int_{\amalg\bar{U}_i} \big(J(\tau\bar{f}), \tau(\bar{f})\big)\,dv\,, \tag{2.6}$$

where

$$\tau_2(\bar{f}) = J(\tau\bar{f}) = \triangle\tau(\bar{f}) + \bar{R}'(d\bar{f}, d\bar{f})\tau(\bar{f}) \tag{2.7}$$

$\triangle = \nabla^*\nabla$ is an operator between local sections of $\bar{f}^{-1}TN_{\mathcal{V}} \to N_{\mathcal{U}}$, ∇ is the connection on $T^*N_{\mathcal{U}} \otimes \bar{f}^{-1}TN_{\mathcal{V}}$, and the Riemannian curvature $\bar{R}'$ is the transverse Riemannian curvature of $(M_2, \mathcal{F}_2)$.

Following the notions of transversal harmonic maps [21], there is a close relationship between the transversal bi-tension field of f and the bi-tension fields of the induced maps $\bar{f}$, obtained by using the local submersions defining the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Then by Diagram 1

$$d\psi(\tau_2)_b(f)_x = \tau_2(\bar{f})_{\phi(x)} \tag{2.8}$$

holds for each of the foliation defining local submersions $\phi\colon U \to \bar{U}$, $\psi\colon V \to \bar{V}$.

Theorem 2.3 ([7])**.** *Let $f\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a smooth foliated map between two foliated Riemannian manifolds. Then f is transversal biharmonic if and only if the induced map $\bar{f}$ is biharmonic.*

Remarks: (1) The notion of "transversal biharmonic map" is independent of cocycles defining the foliations. It is a transverse property, cf. [21, 7]. (2) The variational description is also independent of cocycles chosen. However, we have to be careful, for the state-of-art definitions, properties, and discussion of equivalences of pseudogroups (see [23]).

3. Transversal biwave maps

Let $R^{m,1}$ be an $m+1$-dimensional Minkowski space $R \times R^m$ with the metric $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ and the coordinates $x_0 = t, x_1, x_2, \ldots, x_m$ foliated by planes parallel to $\{0\} \times R^p \subset R \times R^m$, $(p+q=m)$. Then $(R^{m,1}, \mathcal{H}^p)$ is a transversal Minkowski foliation defined by the global submersion $\iota \times \psi \colon R \times R^m \to R \times R^q$; $R \times R^q$ can be considered as its complete transverse manifold. Let $(M, g, \mathcal{F})$ be a Riemannian foliated manifold of dimension n defined by a cocycle $\mathcal{U} = \{U_i, \phi_i, g_{ij}\}$, which induces a Riemannian metric $\bar{g}$ on a $q_1(p_1 + q_1 = n)$ dimensional transverse manifold $N_{\mathcal{U}} = \amalg_i \bar{U}_i$. Let $f : (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ be a smooth foliated map from a foliated Minkowski space into the foliated Riemannian manifold. Form $W_i = f^{-1}(U_i) \subset R^{m,1}$ for each i. Let $\bar{W}_i$ be the quotient of W_i for each i, which is an open subset of $R^{q,1}$. Refining the covering W_i, if necessary, we get a cocycle $\mathcal{W}$ defining the foliation $\mathcal{H}$. Then f induces a map $\bar{f} = \amalg_i \bar{f}_i \colon N_{\mathcal{W}} = \amalg_i \bar{W}_i \to N_{\mathcal{U}} = \amalg_i \bar{U}_i$ with $\bar{f}_i \colon \bar{W}_i \to \bar{U}_i$ such that the diagram (for the sake of convenience, we drop "i" from $\bar{f}_i$ if there is no confusion)

$$\begin{tikzcd}
W_i \subset R^{m,1} \arrow[r, "f"'] \arrow[d, "\iota\times\psi_i"'] & U_i \arrow[d, "\phi_i"] \\
\bar{W}_i \subset R^{q,1} \arrow[r, "\bar{f}"'] & \bar{U}_i
\end{tikzcd}$$

Diagram 2

commutes, i.e. $\bar{f} \circ (\iota \times \psi_i) = \phi_i \circ f$, where $\iota \times \psi_i \colon W_i \to \bar{W}_i$ is a submersion defined by the foliation $\mathcal{H}$ on an open subset W_i, $\psi_i \colon U_i \to \bar{U}_i$ is a Riemannian submersion defining the foliation $\mathcal{F}$ on an open set U_i, and $\iota(t) = t$. By taking a smaller W_i, we can assume that $W_i = T_i \times W_i' \subset R \times R^m$ and $\bar{W}_i = T_i \times \bar{W}_i' \subset R \times R^q$, where T_i is an open interval of R, W_i' is an open subset of R^m, and $\bar{W}_i'$ is an open subset of R^q. We assume that two such cocycles are f-related.

A transversal biwave map $f \colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ is a transversal biharmonic map on the Minkowski space $R^{m,1}$ with the transversal bi-energy functional, following from (2.5),

$$E(\bar{f}) = \frac{1}{2} \int_{N_{\mathcal{W}}} \tau_{\Box}(\bar{f})\, dt\, dx = \frac{1}{2} \int_{\amalg \bar{W}_i} \Box \bar{f}^k + \bar{\Gamma}^k_{rs} \Big(-\bar{f}^r_t \bar{f}^s_t + \sum_{a=1}^{q} \bar{f}^r_a \bar{f}^s_a \Big)\, dt\, dx\,, \tag{3.1}$$

where $\Box = -\frac{\partial^2}{\partial t^2} + \sum_{a=1}^q \frac{\partial^2}{\partial x_a^2}$ is the wave operator and $\bar{\Gamma}^k_{rs}$ are the Christoffel symbols of $\bar{U}_i$ for each i.

Similar to (2.8) there is a close relationship between the transversal bi-wave field of f and the bi-wave fields of the induced maps $\bar{f}$, and by Diagram 2 we have

$$d\phi(\tau_2)_{\square_b}(f)_x = (\tau_2)_{\square}(\bar{f})_{\iota\times\psi(x)}\,.$$

Definition 3.1. The map $f\colon (R^{m,1},\mathcal{H}) \to (M,\mathcal{F})$ is a transversal biwave map iff

$$\begin{aligned}(\tau_2)_{\square}(\bar{f}) &= J_{\bar{f}}(\tau_{\square}\bar{f}) = \triangle\tau_{\square}(\bar{f}) + \bar{R}'(d\bar{f},d\bar{f})\tau_{\square}(\bar{f})\\ &= \square\tau_{\square}(\bar{f})^k + {\Gamma'}^k_{rs}(-\tau_{\square}(\bar{f})^r_t\tau_{\square}(\bar{f})^s + \sum_{a=1}^{q}\tau_{\square}(\bar{f})^\mu_a\tau_{\square}(\bar{f})^\gamma_a)\\ &\quad + {\bar{R}'}^k_{rsl}(-\bar{f}^r_t\bar{f}^s_t + \sum_{a=1}^{q}\bar{f}^r_a\bar{f}^s_a)\tau_{\square}(\bar{f})^l = 0\,,\end{aligned}\tag{3.2}$$

where $\bar{R}'$ is the Riemannian curvature of the transverse manifold $N_{\mathcal{U}}$.

Since Diagram 2 commutes, the definition of a transversal biwave map does not depend on the choices of local Riemannian submersions defining the Riemannian foliations, and thus the choices of cocycles defining the foliations.

Example 1. Let $u\colon (R^{m,1},\mathcal{H}^p) \to R$ be a transversal biwave function, i.e., a transversal biwave map into R foliated by points, which satisfies $\square^2u(t,x) = \square(\square u) = 0$ with initial data $u_0 = u$, $u_1 = \frac{\partial u}{\partial t}$. We have $\square u_0 = \square u$ and $\frac{\partial}{\partial t}\square u = \square\frac{\partial u}{\partial t} = \square u_1$. The transversal biwave function u induces $\bar{u}\colon \bar{V} \subset R^{q,1} \to R$ locally satisfying

$$\square^2\bar{u}(t,x) = \bar{u}_{tttt} - 2\bar{u}_{ttxx} + \bar{u}_{xxxx} = 0,\ (t,x) \in (0,\infty)\times R^q\,,$$

$$\bar{u}_0 = \bar{u}, \bar{u}_1 = \frac{\partial\bar{u}}{\partial t}, \square\bar{u}_0 = \square\bar{u},\ \frac{\partial}{\partial t}\square\bar{u} = \square\bar{u}_1, (t,x) \in \{t=0\}\times R^q\,,$$

where the initial data $\bar{u}_0$, $\bar{u}_1$ are given. This is a fourth order homogeneous linear equation with constant coefficients. It is well-known that $\bar{u}(t,x)$ can be solved by [13, 29] in each $\bar{W} \subset R^{q,1}$.

Let $(M_1,\mathcal{F}_1,g_1)$ and $(M_2,\mathcal{F}_2,g_2)$ be two foliated Riemannian manifolds defined by cocycles $\mathcal{U}$ and $\mathcal{V}$, respectively. Suppose that $f_1\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ is a smooth foliated leaf-preserving map, i.e., $df_1(T\mathcal{F}_1) \subset T\mathcal{F}_2$, and $f_1(U_i) \subset V_\alpha$ for some α such that the cocycles are f_1-related. Based on the notion of [21], there is a closed relationship between the transversal second fundamental form of f_1 and the second fundamental forms of the induced maps $\bar{f}_1$ of the transverse manifolds. It follows from Section 2, Diagram 1 and Lemma 2.2 that for any i

$$d\psi_\alpha S_b(f_1)_x = S(\bar{f}_1)_{\phi_i(x)}\,.\tag{3.3}$$

Definition 3.2. $f_1\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ is a *transversally totally geodesic* map if $S(\bar{f}_1)_{\bar{x}} = \nabla d(\bar{f}_1)_{\bar{x}} = 0$ for any $\bar{x} \in N_{\mathcal{U}}$, where ∇ is the connection on $T^*N_{\mathcal{U}} \otimes \bar{f}_1^{-1}TN_{\mathcal{V}}$.

Let $f\colon (R^{m,1},\mathcal{H}) \to (M_1,\mathcal{F}_1)$ be a smooth map from a foliated Minkowski space to a foliated Riemannian manifold such that the foliations are defined by f-related cocycles $\mathcal{W}$ and $\mathcal{U}$, respectively. Let $f_1\colon (M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ be a smooth map between two foliated Riemannian manifolds such that the foliation $\mathcal{F}_2$ is defined by an f_1-related cocycle $\mathcal{V}$. Since $R^{m,1}$ is a semi-Riemannian manifold, by O'Neill [28] we can define a Levi-Civita connection on $R^{m,1}$, and then we can define a Levi-Civita connection on each $\bar{W}_i\subset R^{q,1}$, and thus on $N_{\mathcal{W}}$. Let ∇, ∇', $\bar\nabla$, $\bar\nabla'$, $\bar\nabla''$, $\hat\nabla$, $\hat\nabla'$, $\hat\nabla''$ be the connections on $TN_{\mathcal{W}}$, $TN_{\mathcal{U}}$, $\bar f^{-1}TN_{\mathcal{U}}$, $\bar f_1^{-1}TN_{\mathcal{V}}$, $(\bar f_1\circ\bar f)^{-1}TN_{\mathcal{V}}$, $T^*N_{\mathcal{W}}\otimes\bar f^{-1}TN_{\mathcal{U}}$, $T^*N_{\mathcal{U}}\otimes\bar f_1^{-1}TN_{\mathcal{V}}$, $T^*N_{\mathcal{W}}\otimes(\bar f_1\circ\bar f)^{-1}TN_{\mathcal{V}}$, respectively. We have

$$\bar\nabla''_X d(\bar f_1\circ\bar f)Y=\hat\nabla'_{d\bar f(X)}d\bar f_1(Y)+d\bar f_1\circ\bar\nabla_X d\bar f(Y)\,, \tag{3.4}$$

for X, $Y\in TN_{\mathcal{V}}$.

Theorem 3.3. *If $f\colon (R^{m,1},\mathcal{H})\to(M_1,\mathcal{F}_1)$ is a transversal biwave map and $f_1\colon(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ is a transversally totally geodesic between two foliated Riemannian manifolds $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$, then the composition $f_1\circ f\colon(R^{m,1},\mathcal{H})\to(M_2,\mathcal{F}_2)$ is a transversal biwave map.*

Proof. The transversal biwave map $f\colon(R^{m,1},\mathcal{H})\to(M_1,\mathcal{F}_1)$ induces $\bar f\colon N_{\mathcal{W}}\to N_{\mathcal{U}}$ such that Diagram 2 commutes locally. The transversally totally geodesic map $f_1\colon(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ induces $\bar f_1\colon N_{\mathcal{U}}\to N_{\mathcal{V}}$ such that Diagram 1 commutes locally. Let $x_0=t$, $x_1,\dots,x_q$ be the coordinate of a point p in $\bar V\subset R^{q,1}$, $e_0=\frac{\partial}{\partial t}$, $e_1=(1,0,\dots,0)$, $e_2=(0,1,0,\dots,0),\dots$, $e_q=(0,\dots,0,1)$ be the frame at p and $\bar\nabla^*\bar\nabla=\bar\nabla''_{e_k}\bar\nabla''_{e_k}-\bar\nabla_{\nabla_{e_k}e_k}$ by [15]. Because f_1 is transversally totally geodesic, i.e., $\hat\nabla' d\bar f_1=0$, it follows from [11] that $\tau_\Box(\bar f_1\circ\bar f)=d\bar f_1\circ\tau_\Box(\bar f)$. Thus we have

$$\begin{aligned}\bar\nabla^*\bar\nabla\tau_\Box(\bar f_1\circ\bar f)&=\bar\nabla^*\bar\nabla\big(d\bar f_1\circ\tau_\Box(\bar f)\big)\\&=\bar\nabla''_{e_k}\bar\nabla''_{e_k}\big(d\bar f_1\circ\tau_\Box(\bar f)\big)-\bar\nabla''_{\nabla_{e_k}e_k}\big(d\bar f_1\circ\tau_\Box(\bar f)\big)\,.\end{aligned}\tag{3.5}$$

Since f_1 is transversally totally geodesic, we derive from (3.4) that

$$\begin{aligned}\bar\nabla''_{e_k}(df_1\circ\tau_\Box(\bar f))&=\bar\nabla''_{e_k}\big(d\bar f_1\circ\hat\nabla_{e_j}d\bar f(e_j)\big)\\&=(\hat\nabla'_{\hat\nabla_{e_j}d\bar f(e_k)}d\bar f_1)\big(\hat\nabla_{e_j}d\bar f(e_j)\big)+d\bar f_1\circ\bar\nabla_{e_k}\big(\hat\nabla_{e_j}d\bar f(e_j)\big)\\&=d\bar f_1\circ\bar\nabla_{e_k}\tau_\Box(\bar f)\,,\end{aligned}$$

where $\tau_\Box(\bar f)=\hat\nabla_{e_j}d\bar f(e_j)$. Therefore, we get

$$\bar\nabla''_{e_k}\bar\nabla''_{e_k}\big(d\bar f_1\circ\tau_\Box(\bar f)\big)=\bar\nabla''_{e_k}\big(d\bar f_1\circ\bar\nabla_{e_k}\tau(\bar f)\big)=d\bar f_1\circ\bar\nabla_{e_k}\bar\nabla_{e_k}\tau_\Box(\bar f)\,,\tag{3.6}$$

$$\bar\nabla''_{\nabla_{e_k}e_k}\big(d\bar f_1\circ\tau(\bar f)\big)=d\bar f_1\circ\bar\nabla_{\nabla_{e_k}e_k}\tau_\Box(\bar f)\,.\tag{3.7}$$

Substituting (3.6), (3.7) into (3.5), we arrive at

$$\bar\nabla\bar\nabla^*\tau_\Box(\bar f_1\circ\bar f)=d\bar f_1\circ\bar\nabla^*\bar\nabla\tau_\Box(\bar f)\,.\tag{3.8}$$

Let $R^{\mathcal{V}}(\ ,\ ),R^{\bar f_1^{-1}\mathcal{V}}(\ ,\ )$ be the curvatures on $TN_{\mathcal{V}},\bar f_1^{-1}TN_{\mathcal{V}}$, respectively. We have

$$R^{\bar{\mathcal{V}}}\big(df_1(X'),\,df_1(Y')\big)\,df_1(Z')=R^{f_1^{-1}T\bar{\mathcal{V}}}(X',Y')\,df_1(Z')\,.$$

for X', Y', $Z' \in TN_{\mathcal{U}}$. By the above formula we derive

$$\begin{aligned} &\bar{R}^{\mathcal{V}}\big(d(\bar{f}_1 \circ \bar{f})(e_i), \tau_{\Box}(\bar{f}_1 \circ \bar{f})\big)\, d(\bar{f}_1 \circ \bar{f})(e_i) \\ &\quad = R^{\bar{f}_1^{-1}\mathcal{V}}\big(d\bar{f}(e_i), \tau_{\Box}(\bar{f})\big)\, d\bar{f}_1\big(d\bar{f}(e_i)\big) \\ &\quad = d\bar{f}_1 \circ \bar{R}^{\mathcal{U}}\big(d\bar{f}(e_i), \tau_{\Box}(\bar{f})\big)\, d\bar{f}(e_i)\,. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9) we obtain

$$\begin{aligned} &\bar{\nabla}^*\bar{\nabla}(\bar{f}_1 \circ \bar{f}) + \bar{R}^{\mathcal{V}}\big(d(\bar{f}_1 \circ \bar{f})(e_i), \tau_{\Box}(\bar{f}_1 \circ \bar{f})\big)\, d(\bar{f}_1 \circ \bar{f})(e_i) \\ &\quad = d\bar{f}_1 \circ \big[\bar{\nabla}^*\bar{\nabla}\tau_{\Box}(\bar{f}) + \bar{R}^{\mathcal{U}}(d\bar{f}(e_i), \tau_{\Box}(\bar{f})d\bar{f}(e_i))\big]\,, \end{aligned} \tag{3.10}$$

i.e., $(\tau_2)_{\Box}(\bar{f}_1 \circ \bar{f}) = d\bar{f}_1 \circ (\tau_2)_{\Box}(\bar{f})$. Hence, if f is a transversal biwave map and f_1 is transversally totally geodesic, then $f_1 \circ f$ is a transversal biwave map. $\Box$

Example 2. Let $(M_1, \mathcal{F}_1)$ be a foliated submanifold of $(M_2, \mathcal{F}_2)$ such that the traces of leaves of $\mathcal{F}_2$ on M_1 are leaves of $\mathcal{F}_1$. This condition implies that for suitable choices of foliation cocycles the transverse manifold $N_{\mathcal{U}}$ is a submanifold of the transverse manifold $N_{\mathcal{V}}$. Are the transversal biwave maps into $(M_1, \mathcal{F}_1)$ also transversal biwave maps into $(M_2, \mathcal{F}_2)$? By Theorem 3.3 the answer is affirmative if $(M_1, \mathcal{F}_1)$ is a transversally totally geodesic foliated submanifold of $(M_2, \mathcal{F}_2)$, i.e., $N_{\mathcal{U}}$ is a totally geodesic submanifold of $N_{\mathcal{V}}$, that is, $N_{\mathcal{U}}$ geodesics are also $N_{\mathcal{V}}$ geodesics. Locally, if γ is a transversal geodesic of $(M_1, \mathcal{F}_1)$, i.e., $\bar{\gamma} = \phi_i \circ \gamma\colon R \to U_i \to \bar{U}_i$ is a $N_{\mathcal{U}}$ geodesic, then $\bar{\gamma}$ is also a $N_{\mathcal{V}}$ geodesic. For a map $v\colon R^{m,1} \to R$, let $u = \gamma \circ v\colon R^{m,1} \to R \to U_i$, which induces $\bar{u} = \bar{\gamma} \circ \bar{v}\colon N_{\mathcal{W}} \to R \to \bar{U}_i$. By (3.10) we have

$$(\tau_2)_{\Box}(\bar{f}) = d\gamma \circ (\tau_2)_{\Box}(\bar{v}) = d\gamma \circ \Box^2 \bar{v}\,, \tag{3.11}$$

since $\bar{\gamma}$ is a geodesic. Therefore, u is a transversal biwave map iff $\bar{v}$ solves the fourth order homogeneous linear biwave equation $\Box^2 \bar{v} = 0$. Hence, with respect to the arc length parameterization, the transversal biwave map equation into $\bar{\gamma}$ is equivalent to linear biwave equation by (3.11). Then for any target foliated manifold $(M_2, \mathcal{F}_2)$ we can provide many transversal biwave maps associated to the transversal geodesics of $(M_2, \mathcal{F}_2)$.

We can construct an example of a biwave map, which is not a transversal biwave map using a warped product of two manifolds in Example 3 based on (A). We also show that there are transversal biwave maps, which are not biwave maps in Example 4 based on (B).

(A) By O'Neill [28] a warped product can be defined on semi-Riemannian manifolds or Riemannian manifolds. Let (B, g), (F, h) be semi-Riemannian manifolds or Riemannian manifolds and $\alpha\colon B \to R$ be a smooth map. On the product manifold $B \times F$, we define a metric tensor $k = g \oplus e^{2\alpha}h$. Let ∇^g, ∇^h be the Levi-Civita connections on (B, g) and (F, h), respectively. The Levi-Civita connection ∇^k on $B \times F$ can be related to those of B and F as follows:

$\nabla^k_X Y = \nabla^g_X Y$, where X and Y are vector fields on B.

$\nabla^k_X V = \nabla^k_V X = X(\alpha)V$, where V is a vector field on F.

$$\nabla^k_V W = -h(V,W)\,\mathrm{grad}\,_g\alpha + \nabla^h_V(W), \text{ where } V,\ W \text{ are vector fields on } F.$$

(B) Let (B_1,g_1), (B_2,g_2), (F_1,h_1) and (F_2,h_2) be Riemannian manifolds. Consider the foliations on the Riemannian manifolds $B_1\times F_1$ and $B_2\times F_2$ given by the projections on the first component $\pi_1\colon B_1\times F_1\to B_1$, $\pi_2\colon B_2\times F_2\to B_2$, respectively. The projections π_1 and π_2 are Riemannian submersions, and the foliations defined by them are Riemannian. Let $h\colon B_1\times F_1\to B_2\times F_2$ be a smooth map which preserves the leaves of the foliations. Then h must be of the form $h(x,y)=(h_1(x),h_2(x,y))$, $x\in B_1$, $y\in F_1$, where $h_1\colon B_1\to B_2$, $h_2\colon B_1\times F_1\to F_2$ are smooth. For the product Riemannian metrics on $B_1\times F_1$ and $B_2\times F_2$, the connection of dh is equal to

$$\nabla d(h) = \left(\nabla d(h_1), \nabla d(h_2|_{B_1}) + \nabla d(h_2|_{F_1})\right), \tag{3.12}$$

where $\nabla d(h_1)$ is the connection derivative of dh_1 at x of $h_1\colon B_1\to B_2$, $\nabla d(h_2|_{B_1})$ is the connection derivative of dh_2 at x of the map $x\to h_2(x,y)$ while y is fixed, and $\nabla d(h_2|_{F_1})$ is the connection derivative of dh_2 at y of the map $y\to h_2(x,y)$ while x is fixed.

Example 3. Let $f\colon B_1\times F_1\to B_2\times F_2$ be a smooth map preserving the leaves such that $f(t,x,y)=(f_1(t,x),f_2(t,x,y))$, i.e. $\bar f=f_1$, where $B_1=R\times R=R^{1,1}$, $F_1=R$, $B_2=F_2=R$, $f_1\colon B_1\to B_2$, $f_2\colon B_1\times F_1\to F_2$. Based on (A), let $\alpha_1(x)=0$, $\alpha_2(x)=x$, $f_1(t,x)=t+\frac{4}{3}x^4$, $f_2(t,x,y)=2x^2$ By [21] we have

$$\begin{aligned}\tau_\Box(f) &= \tau_\Box(f_1)+\tau_\Box(f_2|_{B_1})+\tau_\Box(f_2|_{F_1}) - \|df_2\|^2(\mathrm{grad}_{g_2}\alpha_2)\circ f_1\\ &= 16x^2+4-16x^2 = 4\neq 0\,,\end{aligned}$$

where the third term vanishes. It follows that $(\tau_2)_\Box(f)=0$. However, $(\tau_2)_\Box(f_1)=32\neq 0$. Note that f is a transversal biwave map iff f_1 is a biwave map. Therefore, f is a biwave map, but it is not a transversal biwave map.

Example 4. Based on (B), on one hand, by (3.12) the property "totally geodesic" of $h=(h_1,h_2)$ is equivalent to h_1 being totally geodesic and $\nabla d(h_2|_{B_1})+\nabla d(h_2|_{F_1})=0$, i.e., the vertical and horizontal contributions to the totally geodesic annihilate each other. On the other hand, if h_1 is totally geodesic and $h_2|_{B_1}$, $h_2|_{F_1}$ are totally geodesic for $x\in B_1$, $y\in F_1$, then h is totally geodesic. Therefore, it follows that there are maps h which are transversally totally geodesic, but not totally geodesic. Hence, by Theorem 3.3 there are transversal biwave maps which are not biwave maps.

Let Ω be a compact domain in $R^{m,1}$. We can consider $(\Omega,\mathcal{H}|_\Omega)$ as a compact foliated domain in $(R^{m,1},\mathcal{H})$. Let $f\colon(\Omega,\mathcal{H}|_\Omega)\subset(R^{m,1},\mathcal{H})\to(M,\mathcal{F})$ is a transversal biwave map from a compact foliated space-time domain into a foliated Riemannian manifold which induces $\bar f\colon N_{\mathcal{W}}\to N_{\mathcal{U}}$, where for simplicity we still denote $N_{\mathcal{W}}=\amalg\bar W_i$ the transverse manifold of the restricted foliation to Ω and $\bar W_i$ is an open subset of $R^{q,1}$ for each i.

Theorem 3.4. *If $f\colon (\Omega, \mathcal{H}|_{\Omega}) \to (M, \mathcal{F})$ is a transversal biwave map from a compact foliated space-time domain into a foliated Riemannian manifold such that*

$$- |\tau_{\Box}\bar{f}|_t^2 + \sum_{i=1}^{q} |\tau_{\Box}f|_{x^i}^2 - {R'}^{\alpha}_{\beta\gamma\mu}\Big(- \bar{f}_t^{\beta} f_t^{\gamma} + \sum_{i=1}^{q} f_i^{\beta}\bar{f}_i^{\gamma}\Big)\tau_{\Box}(\bar{f})^{\mu} \geq 0\,, \tag{3.13}$$

then f is a transversal wave map.

Proof. Since $f\colon (\Omega, \mathcal{H}|_{\Omega}) \subset (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ is a transversal biwave map, it induces $\bar{f}\colon N_{\mathcal{W}} \to N_{\mathcal{U}}$ with $\bar{f}\colon \bar{W} \to \bar{U}$ such that Diagram 2 commutes locally. We have

$$(\tau_2)_{\Box}(\bar{f}) = \triangle\tau_{\Box}(\bar{f}) + R'(d\bar{f}, d\bar{f})\tau_{\Box}(\bar{f}) = 0\,,$$

where $\triangle = \nabla^*\nabla$, ∇ is the connection on $T^*N_{\mathcal{U}} \otimes \bar{f}^{-1}TN_{\mathcal{V}}$. Let $x_0 = t, x_1, \ldots, x_q$ be the coordinate of a point p in $\bar{W}$ and $e_0 = \frac{\partial}{\partial t}$, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots e_q = (0, \ldots, 0, 1)$ be the frame at the point. We compute

$$\begin{aligned}
\frac{1}{2}\triangle\|\tau_{\Box}(\bar{f})\|^2 &= \big(\nabla_{e_i}\tau_{\Box}(\bar{f}), \nabla_{e_i}\tau_{\Box}(\bar{f})\big) + \big(\nabla^*\nabla\tau_{\Box}(\bar{f}), \tau_{\Box}(\bar{f})\big) \\
&= \sum_{i=0}^{q} \big(\nabla_{e_i}\tau_{\Box}(\bar{f}), \nabla_{e_i}\tau_{\Box}(\bar{f})\big) \\
&\quad - \Big({R'}^{\alpha}_{\beta\gamma\mu}\Big(- \bar{f}_t^{\beta}\bar{f}_t^{\gamma} + \sum_{i=1}^{q} f_i^{\beta}\bar{f}_i^{\gamma}\Big)\tau_{\Box}(\bar{f})^{\mu}, \tau_{\Box}(\bar{f})\Big) \\
&= -|\tau_{\Box}\bar{f}|_t^2 + \sum_{i=1}^{q} |\tau_{\Box}\bar{f}|_{x^i}^2 \\
&\quad - \Big({R'}^{\alpha}_{\beta\gamma\mu}\Big(- \bar{f}_t^{\beta}\bar{f}_t^{\gamma} + \sum_{i=1}^{q} \bar{f}_i^{\beta}\bar{f}_i^{\gamma}\Big)\tau_{\Box}(\bar{f})^{\mu}, \tau_{\Box}(\bar{f})\Big)\,. \qquad (3.14)
\end{aligned}$$

By applying the Bochner's techniques from (3.13) and the assumption that two defining cocycles are f-related, we know that $\|\tau_{\Box}(\bar{f})\|^2$ is constant, i.e., $d\tau_{\Box}(\bar{f}) = 0$. If we use the identity

$$\int_{\amalg\bar{W}_i} \mathrm{div}\,\big(d\bar{f}, \tau_{\Box}(\bar{f})\big)\,dz = \int_{\amalg\bar{W}_i} \big(|\tau_{\Box}(\bar{f})|^2 + (d\bar{f}, d\tau_{\Box}(\bar{f}))\big)\,dz,\ z = (t, x)\,,$$

and the fact $d\tau_{\Box}(\bar{f}) = 0$, then by the divergence theorem we can conclude that $\tau_{\Box}(\bar{f}) = 0$ for each i. Hence, f is a transversal wave map. □

4. Transversal conservation law

In Hilbert's paper [14], the stress-energy tensor associated to a variational problem is a symmetric 2-covariant tensor conserved at critical points, i.e., div $S = 0$. Let $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ be a smooth foliated map from a foliated Minkowski space to a foliated Riemannian manifold $(M, \mathcal{F})$, which induces $\bar{f}\colon N_{\mathcal{W}} = \amalg\bar{W}_i \to N_{\mathcal{U}} = \amalg\bar{U}_i$ with f-related cocycles $\mathcal{W}$ and $\mathcal{U}$. The transversal stress-energy tensor of f is defined by $S_{\bar{f}} = e(\bar{f})\eta - \bar{f}^*\bar{g}$, where $e(\bar{f}) = \frac{1}{2}\|d\bar{f}\|^2$ is the energy density,

$\eta = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$, I is a q by q matrix. The map f satisfies the *transversal conservation law* for S if $=$ div $S_{\bar{f}} = 0$.

Proposition 4.1. *Let $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ be a smooth foliated map from a foliated Minkowski space into a foliated Riemannian manifold with transverse manifolds $N_{\mathcal{W}}$ and $N_{\mathcal{U}}$, respectively, which induces $\bar{f}\colon N_{\mathcal{W}} \to N_{\mathcal{U}}$. Then we have*

$$\text{div } S_{\bar{f}}(X) = -\big(\tau_{\Box}(\bar{f}), d\bar{f}(X)\big)\,, \quad \forall\ X \in TN_{\mathcal{W}}\,. \tag{4.1}$$

Proof. The smooth foliated map $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ induces $\bar{f} = \amalg \bar{f}_i\colon N_{\mathcal{W}} = \amalg \bar{W}_i \to N_{\mathcal{U}} = \amalg \bar{U}_i$ with $\bar{f}\colon \bar{W} \subset R^{q,1} \to \bar{U}$ such that Diagram 2 commutes locally. Let $x^0 = t, x^1 \dots x^q$ be the coordinate in $\bar{W} \subset R^{q,1}$, and $e_0 = \frac{\partial}{\partial t}$, $e_1 = (1, 0, \dots, 0), \dots, e_q = (0, 0, \dots, 1)$. For each $\bar{f}\colon \bar{W} \to \bar{U}$, we compute

$$\begin{aligned}
\text{div } S_{\bar{f}}(X) &= \nabla_{e_i} S_{\bar{f}}(e_i, X) = \nabla_{e_i}\Big(\frac{1}{2}|d\bar{f}|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} - \bar{f}^* \bar{g}\Big)(e_i, X) \\
&= \nabla_{e_i}\Big(\frac{1}{2}|d\bar{f}|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}(e_i, X)\Big) - (\nabla_{e_i} \bar{f}^* \bar{g})(e_i, X) \\
&= \Big(-\Big(\nabla \frac{\partial \bar{f}}{\partial t}, \frac{\partial \bar{f}}{\partial t}\Big)(-1) + \Big(\nabla \frac{\partial \bar{f}}{\partial x_i}, \frac{\partial \bar{f}}{\partial x_i}\Big)(I)\Big)(e_i, X) - \nabla_{e_i}(\bar{f}_* e_i, \bar{f}_* X) \\
&= \Big(\Big(\nabla \frac{\partial \bar{f}}{\partial t}, \frac{\partial \bar{f}}{\partial t}\Big)(e_i, X) + \Big(\nabla \frac{\partial \bar{f}}{\partial x_i}, \frac{\partial \bar{f}}{\partial x_i}\Big)(e_i, X)\Big) \\
&\quad - (\nabla_{e_i} \bar{f}_* e_i, \bar{f}_* X) - (\bar{f}_* e_i, \nabla_{e_i} \bar{f}_* X) \\
&= \big((\nabla_X d\bar{f}) e_i, \bar{f}_* e_i\big) - \big(\tau_{\Box}(\bar{f}), \bar{f}_* X\big) - (\bar{f}_* e_i, \nabla_{e_i} \bar{f}_* X)\,,
\end{aligned}$$

where the first term and the third term are canceled out and $\nabla_{e_i} \bar{f}_* e_i = \tau_{\Box}(\bar{f})$. □

Recall that $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ is a smooth foliated map from a foliated Minkowski space to a foliated Riemannian manifold $(M, \mathcal{F})$, which induces $\bar{f}\colon N_{\mathcal{W}} = \amalg \bar{W}_i \to N_{\mathcal{U}} = \amalg \bar{U}_i$ with f-related cocycles $\mathcal{W}$ and $\mathcal{U}$. Jiang [17] first investigated the conservation law of a biharmonic map in 1987. We apply his technique to study the stress bi-energy tensor and transversal conservation law of a transversal biwave map.

Definition 4.2. The *transversal stress bi-energy tensor of* $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ is defined by

$$\begin{aligned}
S_2(X, Y) = \frac{1}{2}|\tau_{\Box}(\bar{f})|^2(X, Y) &+ \big(d\bar{f}, \nabla \tau_{\Box}(\bar{f})\big)(X, Y) \\
&- \big(d\bar{f}(X), \nabla_Y \tau_{\Box}(\bar{f})\big) - \big(d\bar{f}, \nabla_X \tau_{\Box}(\bar{f})\big)\,,
\end{aligned}$$

for X, $Y \in \Gamma(TN_{\mathcal{W}})$.

Theorem 4.3. *Let $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ is a smooth foliated map from a foliated Minkowski space to a foliated Riemannian manifold $(M, \mathcal{F})$. Then*

$$\text{div } S_2(Y) = (-)\big((\tau_2)_{\Box}(\bar{f}), d\bar{f}(Y)\big)\,, \quad Y \in \Gamma(TN_{\mathcal{W}})\,.$$

(Note that there is a $+$ or $-$ sign convention for $(\tau_\Box)_2(\bar f) = \pm\triangle\tau_\Box(\bar f) \pm R^{\bar U}(d\bar f, d\bar f)\tau_\Box(\bar f)$.)

Proof. The smooth foliated map $f\colon (R^{m,1}, \mathcal{H}) \to (M, \mathcal{F})$ induces $\bar f\colon N_{\mathcal{W}} = \amalg \bar W_i \to N_{\mathcal{U}} = \amalg \bar U_i$. Set $S_2 = Q_1 + Q_2$, where Q_1 and Q_2 are $(0,2)$-tensors defined by

$$Q_1(X,Y) = \frac{1}{2}|\tau_\Box(\bar f)|^2(X,Y) + \big(d\bar f, \nabla\tau_\Box(\bar f)\big)(X,Y)\,,$$

$$Q_2(X,Y) = \big(d\bar f(X), \nabla_Y\tau_\Box(\bar f)\big) - \big(d\bar f, \nabla_X\tau_\Box(\bar f)\big)\,.$$

Let $p \in \bar W$, $x_0 = t$, $x_1, x_2, \dots, x_q$ be the coordinates at the point p, and $\{X_i\}_{i=0}^q = \{e_i\}_{i=0}^q$ be the frame at p, where $e_0 = \frac{\partial}{\partial t}$, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots)$,$\dots e_q = (0, \dots, 0, 1)$. Write $Y = Y^i e_i$ at p. We first compute

$$\begin{aligned}
\text{div } Q_1(Y) &= \sum_i (\nabla_{e_i}Q_1)(e_i, Y) = \sum_i \big(e_i(Q_1(e_i,Y)) - Q_1\big(e_i, \nabla_{e_i}Y\big)\big)\\
&= \sum_i \Big(e_i\Big(\frac{1}{2}|\tau_\Box(\bar f)|^2 Y^i + \sum_j \big(d\bar f(e_j, \nabla_{e_j}\tau_\Box(\bar f)\big)Y^i\Big)\\
&\qquad - \frac{1}{2}|\tau_\Box(\bar f)|^2 Y^i e_i - \sum_j \big(d\bar f(e_j), \nabla_{e_j}\tau_\Box(\bar f)\big)Y^i e_i\Big)\Big)\\
&= \big(\nabla_Y\tau_\Box(\bar f), \tau_\Box(\bar f)\big) + \sum_i \big(d\bar f(Y, e_i), \nabla_{e_i}\tau_\Box(\bar f)\big)\\
&\qquad + \sum_i \big(d\bar f(e_i), \nabla_Y\nabla_{e_i}\tau_\Box(\bar f)\big)\\
&= \big(\nabla_Y\tau_\Box(\bar f), \tau_\Box(\bar f)\big) + \text{trace }\big(\nabla d\bar f(Y, \cdot), \nabla\cdot\tau_\Box(\bar f)\big)\\
&\qquad + \text{trace }\big(d\bar f(\cdot), \nabla^2\tau_\Box(\bar f)(Y, \cdot)\big)\,.
\end{aligned}\tag{4.2}$$

We then compute

$$\begin{aligned}
\text{div } Q_2(Y) &= \sum_i \big(e_i\big(Q_2(e_i,Y)\big) - Q_2\big(e_i, \nabla_{e_i}Y\big)\big)\\
&= -\big(\nabla_Y\tau_\Box(\bar f), \tau_\Box(\bar f)\big) - \sum_i \big(\nabla d\bar f(Y, e_i), \nabla_{e_i}\tau_\Box(\bar f)\big)\\
&\quad - \sum_i \big(d\bar f(e_i), \nabla_{e_i}\nabla_Y\tau_\Box(\bar f) - \nabla_{\nabla_{e_i}Y}\tau_\Box(\bar f)\big) + \big(d\bar f(Y), \triangle\tau_\Box(\bar f)\big)\\
&= -\big(\nabla_Y\tau_\Box(\bar f), \tau_\Box(\bar f)\big) - \text{trace }\big(\nabla d\bar f(Y, \cdot), \nabla\cdot\tau_\Box(\bar f)\big)\\
&\quad - \text{trace }\big(d\bar f(\cdot), \nabla^2\tau_\Box(\bar f)(\cdot, Y)\big) + \big(d\bar f(Y), \triangle\tau_\Box(\bar f)\big)\,.
\end{aligned}\tag{4.3}$$

Adding (4.2) and (4.3), we arrive at

$$\begin{aligned}
\text{div } S_2(Y) &= \big(d\bar f(Y), \triangle\tau_\Box(\bar f)\big) + \sum_i \big(d\bar f(e_i), R(Y, e_i)\tau_\Box(\bar f)\big)\\
&= -\big((\tau_2)_\Box(\bar f), d\bar f(Y)\big)\,.
\end{aligned}$$

$\Box$

Corollary 4.4. *Let $f\colon (R^{m,1},\mathcal{H}) \to (M,\mathcal{F})$ be a non-degenerate map (i.e., $df \neq 0$). Then f satisfies the transversal conservation law for S_2 (i.e., $\operatorname{div} S_2 = 0$) iff f is a transversal biwave map.*

Proof. Since $f\colon (R^{1,m},\mathcal{H}) \to (M,\mathcal{F})$ is non-degenerate, it induces $\bar f\colon N_{\mathcal{W}} \to N_{\mathcal{U}}$ is non-degenerate, i.e., $d\bar f(Y) \neq 0$ for $Y \in \Gamma(TN_{\mathcal{W}})$. Then we have $\operatorname{div} S_2 = 0$ iff $(\tau_2)_\Box(\bar f) = 0$ iff f is a transversal biwave map. □

Let $f\colon (R^{m,1},\mathcal{H}) \to (M,\mathcal{F})$ be a transversal biwave map from a foliated Minkowski to a foliated Riemannian manifolds, which induces $\bar f\colon N_{\mathcal{W}} = \amalg \bar W_i \to N_{\mathcal{U}} = \amalg \bar U_i$ with f-related cocycles $\mathcal{W}$ and $\mathcal{U}$. If $\frac{d^2}{ds^2}E_2(\bar f_s)|_{s=0} \geq 0$, then f is a *stable* transversal biwave map. If we consider a transversal wave map as a transversal biwave map, then by (4.4) we have $\frac{d^2}{ds^2}E_2(\bar f_s)|_{s=0} \geq 0$ and it is automatically stable.

Theorem 4.5. *There does not exist a non-trivial stable transversal biwave map $f\colon (\Omega,\mathcal{H}) \to (M,\mathcal{F})$ from a compact foliated domain into a foliated Riemannian manifold with constant transversal sectional curvature $K > 0$ satisfying the transversal conservation law for stress-energy tensor.*

Proof. Let $f\colon (\Omega,\mathcal{H}) \to (M,\mathcal{F})$ be a transversal biwave map, which induces $\bar f\colon N_{\mathcal{W}} = \amalg \bar W_i \to N_{\mathcal{U}} = \amalg \bar U_i$. By [15] and the concepts of foliated Riemannian manifolds, we can have the following:

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{ds^2}E_2(\bar f_s)|_{s=0} = &\int_{\amalg \bar W_i} \|\triangle \bar\xi_i + R^{\bar U_i}\big(d\bar f(e_k),\bar\xi_i\big)\, d\bar f(e_k)\|^2\, dz \\
&+ \int_{\amalg \bar W_i} \langle \bar\xi_i, \big(\nabla'_{d\bar f(e_k)} R^{\bar U_i}\big(f(e_k),\tau_\Box(\bar f)\big)\bar\xi_i \\
&+ \big(\nabla'_{\tau_\Box(\bar f)} R^{\bar U_i}\big)\big(d\bar f(e_k),\bar\xi_i\big)\, d\bar f(e_k) \\
&+ R^{\bar U_i}\big(\tau_\Box(\bar f),\bar\xi_i\big)\tau(\bar f) + 2R^{\bar U_i}\big(d\bar f(e_k),\bar\xi_i\big)\bar\nabla_{e_k}\tau_\Box(f) \\
&+ 2R^{\bar U_i}\big(d\bar f(e_k),\tau_\Box(\bar f)\big)\bar\nabla_{e_k}\bar\xi_i\rangle dz
\end{aligned}
\tag{4.4}
$$

where $z = (t,x) \in \bar W_i \subset R^{q,1}$, ∇' is the Riemannian connection on $T\bar U_i$, and $\bar\xi_i \in \Gamma(\bar f^{-1}T\bar U_i)$ is the vector field along one-family of maps $\{f_s\}$ with $\frac{\partial f}{\partial s}|_{s=0} = \bar\xi_i$ for each i.

Since M has constant transversal sectional curvature, (4.4) becomes

$$
\begin{aligned}
\frac{d^2}{ds^2}E_2(\bar f_s)|_{s=0} = &\,2\int_{\amalg \bar W_i} \|\triangle \bar\xi + R^{\bar U_i}\big(d\bar f(e_k),\bar\xi\big)\, d\bar f(e_k)\|^2\, dz \\
&+ 2\int_{\amalg \bar W_i} \langle \bar\xi, R^{\bar U_i}\big(\tau(\bar f),\bar\xi\big)\tau(\bar f) + 2R^{\bar U_i}\big(d\bar f(e_k),\bar\xi\big)\nabla_{e_k}\tau(\bar f) \\
&+ 2R^{\bar U_i}\big(d\bar f(e_k),\tau(\bar f)\big)\nabla_{e_k}\bar\xi\rangle\, dz\,.
\end{aligned}
\tag{4.5}
$$

In particular, let $\bar{\xi} = \tau_\Box(\bar{f})$. Because we assume that f is transversal biwave and $(M, \mathcal{F})$ has constant transverse sectional curvature $K > 0$, (4.5) can be reduced to

$$\begin{aligned} \frac{d^2}{df^2} E_2(\bar{f}_t)|_{t=0} &= 8 \int_{\amalg \bar{W}_i} \langle R^{\bar{U}_i}\big(d\bar{f}(e_k), \tau_\Box(\bar{f})\big)\nabla_{e_k}\tau_\Box(\bar{f}), \tau_\Box(\bar{f})\rangle \, dz \\ &= 8K \int_{\amalg \bar{W}_i} \big[\langle d\bar{f}(e_k), \nabla_{e_k}\tau_\Box(\bar{f})\rangle \|\tau_\Box(\bar{f})\|^2 \\ &\qquad - \langle d\bar{f}(e_k), \tau_\Box(\bar{f})\rangle \langle \tau_\Box(\bar{f}), \nabla_{e_k}\tau_\Box(\bar{f})\rangle\big] \, dz \,. \end{aligned} \tag{4.6}$$

Since f satisfies the transverse conservation law for S, by Proposition 4.1 we have

$$\begin{aligned} &\langle d\bar{f}(e_k), \tau_\Box(\bar{f})\rangle = 0 \,, \\ &\langle d\bar{f}(e_k), \nabla_{e_k}\tau_\Box(\bar{f})\rangle = -\langle \nabla_{e_k} d\bar{f}(e_k), \tau_\Box(\bar{f})\rangle = -\|\tau_\Box(\bar{f})\|^2 \end{aligned} \tag{4.7}$$

for $\bar{f}$. Substituting (4.7) into (4.6) and applying the stability of f, we get

$$\frac{d^2}{ds^2} E_2(\bar{f}_s)|_{s=0} = -8K \int_{\amalg \bar{W}_i} \|\tau_\Box \bar{f}\|^4 \, dz \geq 0 \,.$$

The only possibility is that $\tau_\Box(\bar{f}) = 0$ in each $\bar{W}$, which implies that $f\colon (\Omega, \mathcal{H}) \to (M, \mathcal{F})$ is a transversal wave map. $\Box$

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