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SOME RESULTS ON THE GEOMETRY OF MINKOWSKI PLANE

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Abstract. In this paper we study the geometry of Minkowski plane and obtain some results. We focus on the curve theory in Minkowski plane and prove that the total curvature of any simple closed curve equals to the total Landsberg angle. As the result, the sum of oriented exterior Landsberg angles of any polygon is also equal to the total Landsberg angle, and when the Minkowski plane is reversible, the sum of interior Landsberg angles of any n-gon is $\frac{n-2}{2}$ times of the total Landsberg angle. Our results generalizes the classical results in plane geometry. We also obtain a new characterizations of Euclidean plane among Minkowski planes.

1. Introduction

A *Minkowski norm* on the real n-space $\mathbb{R}^n$ is a function $F\colon \mathbb{R}^n \to \mathbb{R}$ which is positively homogeneous of degree 1, strictly convex and smooth off the origin. The pair $(\mathbb{R}^n, F)$ is called a *Minkowski space*, and it is said to be *reversible* if $F(y) = F(-y)$ for any $y \in \mathbb{R}^n$. When $n = 2, (\mathbb{R}^2, F)$ is called a *Minkowski plane*.

In this paper we study the geometry of Minkowski plane and obtain some results. The main tool we use is the polar expression of $(\mathbb{R}^2, F)$. We first focus on the curve theory and obtain the Frenet type formula for curves in $(\mathbb{R}^2, F)$, and prove that the total curvature of any simple closed curve equals to the total Landsberg angle of $(\mathbb{R}^2, F)$. As the result, the sum of oriented exterior Landsberg angles of any polygon is also equal to the total Landsberg angle, and when F is reversible, the sum of interior Landsberg angles of any n-gon is $\frac{n-2}{2}$ times of the total Landsberg angle. Our results generalizes the classical results in plane geometry. We also characterize the Euclidean plane among Minkowski planes by a function which is essentially defined by Zhongmin Shen.

2. Polar expression of Minkowski plane

Let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^2$, then for any Minkowski norm F on $\mathbb{R}^2$, the function $\phi(y) = \frac{F(y)}{|y|}$ is a positively homogeneous function of degree zero

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on $\mathbb{R}^2\backslash\{0\}$. Let (r,θ) be the standard polar coordinates of $\mathbb{R}^2$, then it is clear that ϕ depends only on θ, and $\phi=\phi(\theta)$ is periodical with period 2π (not necessarily minimal period). Hence we can view ϕ as a function on unit circle $\mathbb{S}$. Now we can express the Minkowski norm F as $F=r\cdot\phi(\theta)$. Recall that the fundament tensor of F is defined by $g_{ij}(y)=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}$, by a direct computation we obtain the following polar expression for $g_{ij}(y)=g_{ij}(\theta)$:

$$g_{11}=\phi^2-2\phi\phi'\cos\theta\sin\theta+((\phi')^2+\phi\phi'')\sin^2\theta\,, \tag{2.1}$$

$$g_{22}=\phi^2+2\phi\phi'\cos\theta\sin\theta+((\phi')^2+\phi\phi'')\cos^2\theta\,, \tag{2.2}$$

$$g_{12}=\phi\phi'\cos 2\theta-((\phi')^2+\phi\phi'')\cos\theta\sin\theta\,, \tag{2.3}$$

$$\det(g_{ij})=g_{11}g_{22}-g_{12}^2=\phi^3(\phi+\phi'')\,. \tag{2.4}$$

By these formulas we clearly have the following

Proposition 2.1. *Let $\phi\colon\mathbb{S}\to\mathbb{R}$ be a positive smooth function, then $F=r\cdot\phi(\theta)$ is a Minkowski norm on $\mathbb{R}^2$ if and only if $\phi+\phi''>0$.*

Let $S=\{y\in\mathbb{R}^2: F(y)=1\}$ be the indicatrix of $\mathbb{R}^2$. The Minkowski norm F defines a Riemannian metric $\hat{g}=g_{ij}(y)dy^i\bigotimes dy^j$ on the punctured plane $\mathbb{R}^2\backslash 0$. We view S as the closed curve in Riemannian surface $(\mathbb{R}^2\backslash 0,\hat{g})$, its arc length $\Lambda=\Lambda(F)$ with respect to $\hat{g}$ is called the *total Landsberg angle* of $(\mathbb{R}^2,F)$. If F is Euclidean, then we certainly have $\Lambda=2\pi$, while for general Minkowski norm, Λ may be smaller or bigger than 2π (see e.g., [1, 2]). On the other hand, if F is reversible, then the inequality $\Lambda\le 2\pi$ always holds, with the equality holds if and only if F is Euclidean [5]. Notice that the equation of S can be written as $y=y(\theta)=\frac{1}{\phi(\theta)}(\cos\theta,\sin\theta)$, by (2.4) the arc element is

$$d\Theta=\sqrt{g_{11}g_{22}-g_{12}^2}(y^1dy^2-y^2dy^1)=\sqrt{1+\frac{\phi''}{\phi}}d\theta\,, \tag{2.5}$$

and consequently,

$$\Lambda=\oint_S d\Theta=\int_0^{2\pi}\sqrt{1+\frac{\phi''}{\phi}}d\theta\,. \tag{2.6}$$

Since $\phi+\phi''>0$, we have $\frac{d\Theta}{d\theta}>0$, and thus Θ and θ can be mutually expressed, and the parameter Θ is also known as the *Landsberg angle*. For two nonzero vectors $y_1,y_2\in\mathbb{R}^2$, the *oriented Landsberg angle* $\measuredangle(y_1,y_2)$ and the *Landsberg angle* $\angle(y_1,y_2)$ between y_1 and y_2 are defined by

$$\measuredangle(y_1,y_2)=\int_{\theta_1}^{\theta_2}\sqrt{1+\frac{\phi''}{\phi}}d\theta=\Theta_2-\Theta_1$$

and

$$\angle(y_1,y_2)=|\measuredangle(y_1,y_2)|\,,$$

respectively. Here θ_1 and θ_2 are polar angles of y_1 and y_2, and Θ_1 and Θ_2 the Landsberg angles of y_1 and y_2, respectively.

3. The curve theory in Minkowski plane

For a given Minkowski plane $(\mathbb{R}^2, F)$, the *fundamental tensor* and the *Cartan tensor* are defined by $g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ and $C_{ijk}(y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$, respectively. For $X = X^i \frac{\partial}{\partial y^i}$, $Y = Y^i \frac{\partial}{\partial y^i}$, $Z = Z^i \frac{\partial}{\partial y^i}$, write $\mathbf{g}_y(X,Y) = g_{ij}(y)X^iY^j$, $\mathbf{C}_y(X,Y,Z) = C_{ijk}(y)X^iY^jZ^k$. Let D be the standard flat connection on $(\mathbb{R}^2, F)$, we have

$$Z \cdot \mathbf{g}_y(X,Y) = \mathbf{g}_y(D_Z X, Y) + \mathbf{g}_y(X, D_Z Y) + 2\mathbf{C}_y(X,Y,Z)\,. \tag{3.1}$$

Recall that $\mathbf{C}_y(y,\cdot,\cdot) = 0$, we call $\mathbf{I}_y = \mathbf{C}_y(u,u,u)$ the *Cartan scalar* of F, where u is a vector satisfying $\mathbf{g}_y(y,u) = 0$, $\mathbf{g}_y(u,u) = 1$.

Now let $c = c(s) = \big(y^1(s), y^2(s)\big)$ be a curve in $(\mathbb{R}^2, F)$ with arc parameter s, thus the tangent vector field $T = \frac{dc}{ds}$ is a unit vector field along the curve, namely, $F^2(T) = \mathbf{g}_T(T,T) = 1$. By (3.1), we get $0 = T \cdot \mathbf{g}_T(T,T) = 2\mathbf{g}_T(\frac{d}{ds}T, T)$, here $\frac{d}{ds}T = D_T T$. Clearly, when $\frac{d}{ds}T \equiv 0$, then $c = c(s)$ is a (part of) straight line; if $\frac{d}{ds}T \neq 0$, let

$$N = \frac{\frac{d}{ds}T}{k} := \frac{\frac{d}{ds}T}{\sqrt{\mathbf{g}_T(\frac{d}{ds}T, \frac{d}{ds}T)}}\,,$$

and write

$$\frac{d}{ds}N = D_T N = aT + bN\,.$$

It is easy to see from (3.1) that

$$\begin{aligned} a &= \mathbf{g}_T\Big(\frac{d}{ds}N, T\Big) = T \cdot \mathbf{g}_T(N,T) - \mathbf{g}_T\Big(N, \frac{d}{ds}T\Big) = -k\,, \\ b &= \mathbf{g}_T\Big(\frac{d}{ds}N, N\Big) = \frac{1}{2}T \cdot \mathbf{g}_T(N,N) - \mathbf{C}_T\Big(\frac{d}{ds}T, N, N\Big) = -k\mathbf{I}_T\,. \end{aligned}$$

Consequently, we get the following *Frenet formulas* for the curve c in $(\mathbb{R}^2, F)$:

$$\frac{d}{ds}T = kN\,, \qquad \frac{d}{ds}N = -k(T + \mathbf{I}_T N)\,. \tag{3.2}$$

Here k is called the *curvature* of c, and T, N the *Frenet frame* of c. If F is Euclidean, then $\mathbf{I}_T = 0$, and (3.2) is reduced to the standard Frenet formulas for curves in Euclidean plane.

Let $F = r \cdot \phi(\theta)$ be the polar expression of F. In polar coordinate we can express the unit tangent vector field T as

$$T = T(\theta) = \frac{1}{\phi(\theta)}(\cos\theta, \sin\theta)\,.$$

A direct computation yields

$$\frac{d}{d\theta}T = \frac{1}{\phi(\theta)}\Big(- \sin\theta - \frac{\phi'}{\phi}\cos\theta, \cos\theta - \frac{\phi'}{\phi}\sin\theta\Big)$$

$$= \frac{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}{\phi^2(\theta)}\,(\cos\overline{\theta}, \sin\overline{\theta})\,, \tag{3.3}$$

where $\overline{\theta} = \theta + \widetilde{\theta}$, and $\widetilde{\theta}$ is determined by

$$\cos\widetilde{\theta} = \frac{-\phi'(\theta)}{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}\,, \qquad \sin\widetilde{\theta} = \frac{\phi(\theta)}{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}} \tag{3.4}$$

Combining (2.1)–(2.3) and (3.4) we get

$$\begin{aligned}&\mathbf{g}_T\big((\cos\overline{\theta}, \sin\overline{\theta}), (\cos\overline{\theta}, \sin\overline{\theta})\big)\\ &\quad= \phi^2(\theta) + (\phi'^2 + \phi\phi'')(\theta)\sin^2(\overline{\theta}-\theta) + \phi(\theta)\phi'(\theta)\sin 2(\overline{\theta}-\theta)\\ &\quad= \frac{\phi^4(\theta)+\phi^3(\theta)\phi''(\theta)}{\phi^2(\theta)+\phi'^2(\theta)}\,,\end{aligned}$$

and thus

$$\mathbf{g}_T\Big(\frac{d}{d\theta}T, \frac{d}{d\theta}T\Big) = 1 + \frac{\phi''(\theta)}{\phi(\theta)}\,,$$

which together with (2.5) yields

$$k = \sqrt{\mathbf{g}_T\Big(\frac{d}{ds}T, \frac{d}{ds}T\Big)} = \sqrt{1+\frac{\phi''}{\phi}}\Big|\frac{d\theta}{ds}\Big| = \Big|\frac{d\Theta}{ds}\Big|\,.$$

As in the Euclidean case, we call

$$k_r = \frac{d\Theta}{ds}$$

the *relative curvature* of curve c. Now let $c = c(s)$ be a smooth simple closed curve in $(\mathbb{R}^2, F)$, and let Ω be the finite domain with boundary c. The positive orientation of c is defined in a way such that Ω is on the left when goes forward along c. As the curve in Euclidean plane $(\mathbb{R}^2, |\cdot|)$, it is well-known that the incremental of the polar angle along c is 2π, namely,

$$\oint_c d\theta = 2\pi\,.$$

Hence, the *total curvature* of c is

$$\oint_c k_r\,ds = \oint_c d\Theta = \int_0^{2\pi}\sqrt{1+\frac{\phi''}{\phi}}d\theta = \Lambda\,,$$

namely, we have proved the following

Theorem 3.1. *The total curvature of any smooth simple closed curve in* $(\mathbb{R}^2, F)$ *equals to the total Landsberg angle of* F.

As in the Euclidean case, Theorem 3.1 can be generalized to the case when c is only piecewise smooth by replacing the curve near the non-smooth points with some smooth curves and then taking the limit.

Corollary 3.2. *Let $c = c(s)\colon [0, L] \to (\mathbb{R}^2, F)$ is a piecewise smooth simple closed curve with non-smooth points $c(s_i)$, $i = 1, \cdots, n$. Let $\alpha_i = \measuredangle(T(s_i - 0), T(s_i + 0))$, $c_i = \{c(s) : s_i \le s \le s_{i+1}\}$, $1 \le i \le n-1$, and $c_n = \{c(s) : s_n \le s \le L\} \cup \{c(s) : 0 \le s \le s_1\}$, then*

$$\sum_{i=1}^{n} \int_{c_i} k_r \, ds + \sum_{i=1}^{n} \alpha_i = \Lambda \, .$$

In particular, the sum of oriented exterior Landsberg angles of any polygon is equal to the total Landsberg angle.

It is clear that when F is reversible, then the Landsberg angle of any straight angle is $\frac{\Lambda}{2}$, and we can prove the following corollary just as in the Euclidean case.

Corollary 3.3. *The sum of interior Landsberg angles of any n-gon in a reversible Minkowski plane is $\frac{n-2}{2}\Lambda$.*

4. A new characterization of Euclidean plane

In this last section we shall provide a new characterization of Euclidean plane among Minkowski planes. Let us first introduce a function defined by Professor Zhongmin Shen. Let $(\mathbb{R}^n, F)$ be a Minkowski n-space. For vector $y \in \mathbb{R}^n \backslash \{0\}$, we obtain a hyperplane

$$W_y = \{u \in \mathbb{R}^n : \mathbf{g}_y(y, u) = 0\} \, .$$

Taking a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ such that $e_1 = y$, and $e_2, \ldots, e_n$ is a basis for W_y. Let

$$B^n = \Big\{ (y^i) \in \mathbb{R}^n : F\Big(\sum_{i=1}^{n} y^i e_i \Big) < 1 \Big\} \, ,$$

$$B_y^{n-1} = \Big\{ (y^a) \in \mathbb{R}^{n-1} : F\Big(\sum_{a=2}^{n} y^a e_a \Big) < 1 \Big\} \, .$$

In [5] Shen defined a function $\tilde{\zeta} = \tilde{\zeta}(y)$ on $(\mathbb{R}^n \backslash \{0\}, F)$ as following:

$$\tilde{\zeta}(y) = \frac{\mathrm{vol}(\mathbb{B}^n)}{\mathrm{vol}(\mathbb{B}^{n-1})} \cdot \frac{\mathrm{vol}(B_y^{n-1})}{F(y)\,\mathrm{vol}(B^n)} \, .$$

Here $\mathbb{B}^k$ denotes the k-dimensional unit ball in Euclidean k-space $\mathbb{R}^k$. Clearly, function $\tilde{\zeta}$ is independent of the choice of basis $e_2, \ldots, e_n$ for W_y, and $\tilde{\zeta} = 1$ when F is Euclidean. Shen asked that whether or not F is Euclidean when $\tilde{\zeta} = 1$. In the following we shall consider the problem of this type for $n = 2$. For this purpose let us first introduce the reversibility $\lambda = \lambda(F)$ of a Minkowski plane $(\mathbb{R}^2, F)$ as following (see [4]):

$$\lambda = \lambda(F) = \max_{y \neq 0} \frac{F(y)}{F(-y)} \, .$$

In term of polar expression, we have

$$\lambda = \max_{\theta\in[0,2\pi]} \frac{\phi(\theta)}{\phi(\theta+\pi)} \tag{4.1}$$

We consider a function $\zeta = \zeta(y)$ on a Minkowski plane $(\mathbb{R}^2\backslash\{0\}, F)$ as following:

$$\zeta(y) = \frac{\Lambda}{4\lambda} \cdot \frac{\mathrm{vol}(B^1_y)}{F(y)\,\mathrm{vol}(B^2)}\,. \tag{4.2}$$

It is clearly that up to a constant, ζ is essentially the same as $\tilde{\zeta}$. We have the following result which provides a new characterization for Euclidean plane among Minkowski planes.

Theorem 4.1. *Let $\zeta = \zeta(y)$ be a function on Minkowski plane $(\mathbb{R}^2\backslash\{0\}, F)$ defined by (4.2). Then F is Euclidean if and only if $\zeta(y) = 1$ for any $y \neq 0$.*

Proof. The necessity is trivial, so we need only to prove the sufficiency. Assume that $\zeta(y) = 1$ for any $y \neq 0$. Let $F = r\cdot\phi(\theta)$ be the polar expression of F, and let $y = (\cos\theta, \sin\theta)$. Taking $e_1 = y$, $e_2 = (\cos\overline{\theta}, \sin\overline{\theta})$, where $\overline{\theta} = \theta + \widetilde{\theta}$, and $\widetilde{\theta}$ is determined by (3.4). Then by (2.1)–(2.3) it can be verified that $e_2 \in W_y$, and consequently,

$$\mathrm{vol}(B^1_y) = \frac{1}{\phi(\overline{\theta})} + \frac{1}{\phi(\overline{\theta}+\pi)}\,. \tag{4.3}$$

In the following we need to compute $\mathrm{vol}(B^2)$. Recall that

$$B^2 = \{(\overline{y^1}, \overline{y^2}) \in \mathbb{R}^2 : F(\overline{y^1}e_1 + \overline{y^2}e_2) < 1\}\,.$$

Let

$$\Omega = \{(y^1, y^2) \in (\mathbb{R}^2, F) : F(y^1, y^2) < 1\}\,,$$

then the following transformation

$$(\overline{y^1}, \overline{y^2}) \mapsto (y^1, y^2) = (\overline{y^1}\cos\theta + \overline{y^2}\cos\overline{\theta}, \overline{y^1}\sin\theta + \overline{y^2}\sin\overline{\theta})$$

maps B^2 onto Ω, and

$$\begin{aligned} dy^1\wedge dy^2 &= (d\overline{y^1}\cos\theta + d\overline{y^2}\cos\overline{\theta})\wedge(d\overline{y^1}\sin\theta + d\overline{y^2}\sin\overline{\theta}) \\ &= \sin(\overline{\theta}-\theta)d\overline{y^1}\wedge d\overline{y^2} = \frac{\phi(\theta)}{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}d\overline{y^1}\wedge d\overline{y^2}\,, \end{aligned}$$

and thus

$$
\begin{aligned}
\mathrm{vol}(B^2) &= \int_{B^2} d\overline{y^1} \wedge d\overline{y^2} = \frac{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}{\phi(\theta)} \int_{\Omega} dy^1 \wedge dy^2 \\
&= \frac{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}{\phi(\theta)} \int_0^{2\pi} d\xi \int_0^{\frac{1}{\phi(\xi)}} r\, dr \\
&= \frac{\sqrt{\phi^2(\theta)+\phi'^2(\theta)}}{2\phi(\theta)} \int_0^{2\pi} \frac{d\xi}{\phi^2(\xi)} \,. \qquad (4.4)
\end{aligned}
$$

From (4.3), (4.4) and the assumption that $\zeta(y) = 1$ for any $y \neq 0$, we have

$$
\frac{\Lambda}{\lambda}\left(\frac{1}{\phi(\bar\theta)} + \frac{1}{\phi(\bar\theta+\pi)}\right) = 2\sqrt{\phi^2(\theta)+\phi'^2(\theta)} \int_0^{2\pi} \frac{d\xi}{\phi^2(\xi)}\,, \tag{4.5}
$$

which together with (4.1) yields

$$
\frac{\Lambda}{\phi(\bar\theta)} \geq \frac{\Lambda}{2\lambda}\left(\frac{1}{\phi(\bar\theta)} + \frac{1}{\phi(\bar\theta+\pi)}\right) = \sqrt{\phi^2(\theta)+\phi'^2(\theta)} \int_0^{2\pi} \frac{d\xi}{\phi^2(\xi)}\,. \tag{4.6}
$$

Note that

$$
d\bar\theta = d\theta + d\widetilde{\theta} = \frac{\phi^2(\theta)+\phi(\theta)\phi''(\theta)}{\phi^2(\theta)+\phi'^2(\theta)} d\theta\,,
$$

and the incremental of $\bar\theta$ is 2π when θ goes from 0 to 2π, it is easy to know from (4.6) that

$$
\Lambda^2 \cdot \int_0^{2\pi} \frac{d\bar\theta}{\phi^2(\bar\theta)} \geq \int_0^{2\pi} \big(\phi^2(\theta)+\phi(\theta)\phi''(\theta)\big)\, d\theta \Big(\int_0^{2\pi} \frac{d\theta}{\phi^2(\theta)}\Big)^2,
$$

which together with the Cauchy-Schwartz inequality implies that

$$
\Lambda^2 \geq \int_0^{2\pi} \big(\phi^2(\theta)+\phi(\theta)\phi''(\theta)\big)\, d\theta \int_0^{2\pi} \frac{d\theta}{\phi^2(\theta)} \geq \Big(\int_0^{2\pi} \sqrt{1+\frac{\phi''}{\phi}}\, d\theta\Big)^2 = \Lambda^2\,.
$$

As the result, above inequality becomes an equality, and

$$
\phi^2(\theta)+\phi(\theta)\phi''(\theta) = \frac{C}{\phi^2(\theta)}
$$

for some positive constant C. It is equivalent to $\det(g_{ij}(y)) = C$ by (2.4), and thus the mean Cartan tensor $I_k = \frac{1}{2}\frac{\partial}{\partial y^k}\log\big(\det(g_{ij}(y))\big) = 0$, and by Deiche's theorem [3], F is Euclidean. So we are done.

$\square$

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